# THE IRRATIONALITY EXPONENT OF SMALL-HEIGHT LACUNARY SERIES

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ABSTRACT. Let $N \geq 2$ be an integer, let $u_1 < u_2 < \cdots$ be positive integers with $\liminf u_{j+1}/u_j \geq 2$, and let $\lambda_j$ be nonzero rational numbers with $\log H(\lambda_j) = o(u_j)$. Then $x = \sum_{j\geq 1} \lambda_j N^{-u_j}$ is irrational and its irrationality exponent is $\limsup u_{j+1}/u_j$, finite or infinite. The proof compares logarithmic gaps between the continued-fraction denominators of consecutive truncations. A perturbation lemma places each gap of the next truncation in at least one of five classes (bounded, inherited, central, short, reflected), each with an explicit error. Reflection moves a large gap far enough to the right to give a uniform bound at the boundary $\liminf u_{j+1}/u_j = 2$. The coefficient condition allows arbitrary signs and cannot be replaced by $\log H(\lambda_j) \leq C_* u_j$ for any fixed $C_* > 0$.

## 1. INTRODUCTION

For an irrational real number $\xi$, its irrationality exponent is

$$\mu(\xi) = \sup\left\{\nu \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < q^{-\nu} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0}\right\}.$$

For a nonzero rational number written in lowest terms, put

$$H(a/b) = \max(|a|, b), \qquad a \in \mathbb{Z} \setminus \{0\}, \quad b \in \mathbb{Z}_{>0}, \quad \gcd(a,b) = 1. \tag{1.1}$$

For $y \in \mathbb{Q}$, let $\mathrm{den}(y)$ denote its positive reduced denominator. All logarithms are natural.

**Theorem 1.** *Let $N \geq 2$ be an integer, let $u_1 < u_2 < \cdots$ be positive integers, and let $\lambda_j \in \mathbb{Q} \setminus \{0\}$ satisfy*

$$\log H(\lambda_j) = o(u_j),$$

*and suppose that*

$$\liminf_{j\to\infty} \frac{u_{j+1}}{u_j} \geq 2.$$

*Then the absolutely convergent series*

$$x = \sum_{j\geq 1} \lambda_j N^{-u_j}$$

*is irrational and*

$$\mu(x) = \limsup_{j\to\infty} \frac{u_{j+1}}{u_j}. \tag{1.2}$$

*Both sides lie in $[2, +\infty]$.*

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2020 *Mathematics Subject Classification.* Primary 11J82; Secondary 11J70, 11B83.

*Key words and phrases.* irrationality exponent, lacunary series, rational height, continued fraction, folding.

The ratios need not converge. For example, integer exponent sequences with repeating ratio patterns $(2, 2, 3)$ and $(2, 5/2)$ give exponents 3 and $5/2$, respectively. Ratios may fall below 2 by a quantity tending to zero. The coefficients may have either sign; any sequence from a finite subset of $\mathbb{Q} \setminus \{0\}$ is allowed. The binary specialization $u_j = 2^{j-1}$ of Theorem 1 is the main theorem of [10], proved there with Hankel determinants and Padé approximants; the present proof uses a different argument based on regular continued fractions, and the sharpness examples of Section 9 follow Section 6 of that paper.

For an integer base $b \geq 2$ and a strictly increasing sequence of positive integers $(n_j)$ with $n_{j+1}/n_j \geq 2$ for all large $j$, the formula $\mu(\sum_j b^{-n_j}) = \limsup n_{j+1}/n_j$ for unit coefficients is a classical consequence of folding, recorded in Bugeaud, Krieger and Shallit [4] with attribution there to Bugeaud [2]. Bugeaud [3, Theorem 7.21] gives a result on prescribed irrationality exponents for a specific family in the middle-third Cantor set. The explicit Cantor-set family in Bugeaud [2, Theorem 2] consists, up to a fixed rational factor, of special cases of the unit-weight part of Theorem 1. Theorem 1 also treats small rational weights and permits ratios tending to 2 from below. Continued-fraction expansions of lacunary series appear in Shallit [8, 9]. The reversal used here is related to the folding identities of Mendès France [6] and van der Poorten and Shallit [7].

Duverney, Kurosawa and Shiokawa [5, Theorem 1] give an irrationality-exponent formula for series $\sum_j \sigma_j/X_j$, where $\sigma_j \in \{-1, 1\}$, $X_j > 1$ are rational, and $X_{j+1} \geq X_j^2$ eventually. Write $T_j$ for their auxiliary sequence of positive integers, defined by

$$T_1 = \operatorname{den}(X_1), \qquad T_{j+1} = T_j^2 \operatorname{den}(X_{j+1}/X_j^2). \tag{1.3}$$

The additional hypothesis is $\log T_{j+1} = o(\log X_j)$. Their conclusion is

$$\mu\left(\sum_j \frac{\sigma_j}{X_j}\right) = \limsup_j \frac{\log X_{j+1}}{\log X_j}. \tag{1.4}$$

For $\lambda_j \in \{-1, 1\}$ and $u_{j+1} \geq 2u_j$ eventually, discard a finite prefix and take $X_j = N^{u_j}$. Then $X_{j+1}/X_j^2$ is an integer and $T_j = 1$, so the DKS formula gives (1.2), including the case of exact ratio 2.

For the following comparison with arbitrary rational weights, assume $\limsup u_{j+1}/u_j < \infty$, so that $u_{j+1} = O(u_j)$. In the subcase $\liminf u_{j+1}/u_j > 2$, discard finitely many terms and take $X_j = N^{u_j}/|\lambda_j|$ and $\sigma_j = \operatorname{sgn}(\lambda_j)$. The height hypothesis makes $\log X_j$ asymptotic to $u_j \log N$, so $X_j > 1$ eventually and the logarithmic ratios have the same limsup as the exponent ratios. The strict lower-ratio margin then gives $X_{j+1} \geq X_j^2$ eventually. The local logarithmic denominator costs are $o(u_j)$: if $\lambda_j = \alpha_j/\beta_j$ in lowest terms, then

$$\operatorname{den}(X_{j+1}/X_j^2) \leq |\alpha_{j+1}|\beta_j^2.$$

Put $h_j = \log H(\lambda_j)/u_j$, so $h_j \to 0$. Since $u_{j+1} \geq (2 + \varepsilon_*)u_j$ eventually for some $\varepsilon_* > 0$, the fixed initial denominator cost is negligible after normalization, and the normalized later costs are bounded by a convergent geometric sum. Their accumulation with weights $2^{j-i}$, $i \leq j$, is therefore still $o(u_j)$, giving $\log T_{j+1} = o(\log X_j)$ as required by DKS. Adding back the omitted rational prefix does not change the irrationality exponent. The case of an infinite limsup follows from the truncations alone (Section 5). Amou and Bugeaud [1, Lemma 3] give a related criterion. In notation local to this paragraph, let $\widehat{A}_n$ be strictly increasing positive

integers, let $\widehat{B}_n$ be nonzero integers, and let $\widehat{\tau} = \limsup \log \widehat{A}_{n+1}/\log \widehat{A}_n$ be finite and at least two. Their local condition is

$$\frac{\widehat{A}_{n+1}/|\widehat{B}_{n+1}|}{(\widehat{A}_n/|\widehat{B}_n|)^2} \geq 2 \quad \text{eventually},$$

where $\widehat{S}_n = (\widehat{A}_{n+1}/|\widehat{B}_{n+1}|)/(\widehat{A}_n/|\widehat{B}_n|)^2$ denotes the left-hand side. Let $\widehat{d}_n = \operatorname{den}(\widehat{S}_n)$, $\widehat{d}_0 = |\widehat{B}_1|$, and

$$\widehat{D}_n = \prod_{i=0}^{n} \widehat{d}_i^{\,2^{n-i}}.$$

Their remaining conditions are $|\widehat{B}_n| = \widehat{A}_n^{o(1)}$ and $\widehat{D}_n = \widehat{A}_n^{o(1)}$. Under these conditions, the irrationality exponent of the series with terms $\widehat{B}_n/\widehat{A}_n$ equals $\widehat{\tau}$. In the proof of their Theorem 2, Amou and Bugeaud [1, Section 5] state that there exists a sequence of ratios greater than 2 with lower limit 2 for which Lemma 3 applies to their construction; the omitted details concern the cumulative denominator condition $\widehat{D}_n = \widehat{A}_n^{o(1)}$.

At the boundary $\liminf u_{j+1}/u_j = 2$, for the choice $X_j = N^{u_j}/|\lambda_j|$ above, the height hypothesis alone does not imply $\log T_{j+1} = o(\log X_j)$: weighting local costs by $2^{j-i}$ can leave a nonnegligible contribution. Also, with this choice, a step $u_{j+1} < 2u_j$ gives $X_{j+1} < X_j^2$ for unit coefficients. Theorem 1 includes arbitrary rational weights of the stated height at this boundary and permits such steps below 2.

The proof uses only regular continued fractions and rational arithmetic. The continued fraction of one truncation need not be a prefix of that of the next, so a perturbation lemma controls each gap separately. The reflection estimate gives a lower bound for the rightward displacement of a large gap in terms of its old width. At the boundary ratio 2 there is no margin, and this shift takes its place. For the infinite continued fraction, each gap gets its own truncation, chosen so that the accumulated finite-stage error stays logarithmic.

## 2. Continued-fraction facts

For a nonintegral rational number we use its canonical finite regular continued fraction, whose last partial quotient is at least 2. For an integer, the canonical expansion consists of that integer alone. For irrational numbers we use infinite regular continued fractions. Convergents are indexed from zero and written with positive denominators; the auxiliary pair with denominator zero is not counted as a convergent. A convergent preceding the terminal rational is called nonterminal.

If $p/q$ is a nonterminal convergent of a real number $\theta$ and the next convergent has denominator $q^+$, then

$$\frac{1}{q(q^+ + q)} \leq \left|\theta - \frac{p}{q}\right| \leq \frac{1}{qq^+}. \tag{2.1}$$

They also hold for rational $\theta$ with its canonical finite expansion. We use Legendre's criterion in both cases: if $p/q$ is reduced and

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2}, \tag{2.2}$$

then $p/q$ is a convergent of $\theta$. If $\theta$ is rational and the error is nonzero, it is nonterminal. For a nonintegral rational $\theta$, let $q_{m-1}$ and $q_m$ be the penultimate and

final denominators in its canonical expansion. The strict inequality rules out the penultimate convergent of the alternative expansion ending in 1: its denominator is $q_m - q_{m-1}$ and its error is $1/(q_m(q_m - q_{m-1}))$, so the strict threshold cannot hold when the canonical last partial quotient is at least 2. For integral $\theta$, a distinct rational with positive denominator has error at least the reciprocal of that denominator, so the strict criterion can hold only at $\theta$ itself. For irrational $\theta$ every convergent is nonterminal.

A *logarithmic gap* is a pair $a = \log q$, $b = \log q^+$ from consecutive convergent denominators; its width is $g = b - a$. A possible initial equality $q = q^+ = 1$ is allowed.

## 3. Reversing a finite denominator profile

Let

$$\alpha = \frac{P}{V} = [a_0; a_1, \dots, a_m], \qquad V \geq 2,$$

be canonical. Write its convergent denominators as $q_{-1} = 0, q_0 = 1, \dots, q_m = V$, and put

$$V' = q_{m-1}, \qquad w_0 = \frac{V'}{V} = [0; a_m, a_{m-1}, \dots, a_1], \qquad M = \log V.$$

**Lemma 3.1.** *For every logarithmic gap $\log s \to \log S$ in the canonical finite continued fraction of $w_0$, there is a logarithmic gap $c \to d$ of $\alpha$ such that*

$$|\log s - (M - d)| \leq \log 2, \qquad |\log S - (M - c)| \leq \log 2. \tag{3.1}$$

*In particular, $0 \leq c \leq d \leq M$.*

*Proof.* Let $r_i$, $0 \leq i \leq m$, be the denominators in the displayed reversed expansion before a possible canonical merge, which is treated at the end. Thus $r_0 = 1$ and $r_m = V$. Continuant symmetry and the splitting identity give

$$V = q_j r_{m-j} + q_{j-1} r_{m-j-1} \qquad (0 \leq j \leq m),$$

where $r_{-1} = 0$. Both denominator sequences are nondecreasing, so

$$\frac{V}{2q_j} \leq r_{m-j} \leq \frac{V}{q_j}. \tag{3.2}$$

Hence the gap $r_i \to r_{i+1}$ corresponds to $q_{m-i-1} \to q_{m-i}$, with the errors in (3.1).

If $a_1 \geq 2$, the reversed expansion is canonical. If $a_1 = 1$, its canonical form merges the final 1 and removes only $r_{m-1}$. The two final gaps merge into $r_{m-2} \to r_m = V$, which corresponds to $q_1 \to q_2$; all earlier gaps retain the preceding correspondence. Since $q_1 = 1$, its upper endpoint is exactly $M - \log q_1$, and its lower endpoint satisfies (3.2) with $j = 2$. The case $m = 1, a_1 = 1$ is excluded by $V \geq 2$. □

## 4. A rational perturbation lemma

Let $\alpha = P/V$ be reduced with $V \geq 2$, and let $\beta = \alpha + \delta$ with $\delta \in \mathbb{Q} \setminus \{0\}$. Put

$$\Delta = |\delta|, \qquad M = \log V, \qquad L = -\log \Delta, \qquad W = \frac{1}{V^2 \Delta} = e^{L-2M}. \tag{4.1}$$

Let $P'/V'$ be the penultimate convergent in the canonical expansion of $\alpha$, and write

$$\sigma = PV' - P'V \in \{1, -1\}, \qquad \zeta = -\frac{\sigma}{V^2 \delta}. \tag{4.2}$$

Put $B = \operatorname{den}(\zeta)$ and $E = \log(4B)$. So $B \geq 1$, $|\zeta| = W$, and $\zeta$ may be negative.

**Lemma 4.1.** *Let $p/q$ be a nonterminal convergent of $\beta$ with next denominator $q^+$. Put*

$$\varepsilon = |\beta - p/q| > 0, \qquad \pi_q = q^+/q, \qquad a = \log q, \quad b = \log q^+, \quad g = b - a.$$

*At least one of the following alternatives holds.*

(1) ***Bounded.*** *$\pi_q \le 6$, so $g \le \log 6$.*

(2) ***Inherited.*** *$p/q \ne \alpha$ is a nonterminal convergent of $\alpha$ whose next denominator $U$ satisfies $U \ge q^+/6$. In particular, the new gap has the same left endpoint $a = \log q$, and*

$$g \le \log U - \log q + \log 6. \tag{4.3}$$

(3) ***Central.*** *$p/q = \alpha$, so $q = V$ and*

$$g \le L - 2M. \tag{4.4}$$

(4) ***Short.*** *One has*

$$g \le \log 12 + 2\log B \le 2E, \qquad a \ge L - M - \log(3/2). \tag{4.5}$$

(5) ***Reflected.*** *There is an old logarithmic gap $c \to d$ of $\alpha$ such that*

$$|a - (L - d)| \le E, \qquad |b - (L - c)| \le E, \tag{4.6}$$

*and*

$$a \ge L - M - \log(3/2). \tag{4.7}$$

*No lower bound on $W$ is assumed.*

*Proof.* Suppose $\pi_q > 6$. If $\varepsilon \ge \Delta/2$, then

$$\left|\alpha - \frac{p}{q}\right| \le \varepsilon + \Delta \le 3\varepsilon \le \frac{3}{qq^+} < \frac{1}{2q^2}. \tag{4.8}$$

If $p/q = \alpha$, then $q = V$ and $\varepsilon = \Delta$. Equation (2.1) gives $q^+ \le 1/(V\Delta) = VW$, which is the central alternative. Otherwise, $p/q$ is reduced and differs from $\alpha$, so Legendre's criterion gives a nonterminal convergent of $\alpha$. Its next denominator $U$ satisfies

$$\frac{1}{q(U+q)} \le \frac{3}{qq^+}, \qquad U \ge q^+/3 - q \ge q^+/6,$$

giving the inherited alternative.

Now suppose $\varepsilon < \Delta/2$. With $w_0 = V'/V$, define the local coordinate $t = \zeta - w_0$. Direct calculation gives

$$\beta = \frac{Pt + P'}{Vt + V'}, \qquad Vt + V' = V\zeta. \tag{4.9}$$

Signs of $\zeta$ and $t$ are arbitrary. For $f(w) = (Pw + P')/(Vw + V')$, the Möbius difference formula is

$$f(t) - f(w) = \frac{\sigma(t - w)}{(Vt + V')(Vw + V')}.$$

Define integers $h, k$ by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} P & P' \\ V & V' \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \tag{4.10}$$

The matrix is unimodular, so $h$ and $k$ are coprime. Here $k \neq 0$: otherwise (4.10) would give $p/q = \alpha$, contrary to the present error assumption. Moreover,

$$k = \sigma(Pq - Vp), \qquad K := |k| = Vq\left|\alpha - \frac{p}{q}\right|.$$

The second row of (4.10) gives $V(h/k) + V' = q/k$. Substituting $w = h/k$ in the Möbius difference formula gives an exact expression for $|t - h/k|$, valid for either sign of $k$ and $\zeta$. The assumption $\varepsilon < \Delta/2$ gives

$$\frac{q}{2VW} \leq K \leq \frac{3q}{2VW}. \tag{4.11}$$

Since $K$ is a positive integer, $K \geq 1$, so

$$q \geq \frac{2}{3}VW. \tag{4.12}$$

This is the bound on $a$ in the short and reflected alternatives.

Write $r = \zeta - h/k = y/s$ in reduced form with $s > 0$. Comparing the denominators of $\zeta$, $h/k$ and $r$ gives

$$K/B \leq s \leq BK. \tag{4.13}$$

The bound $s \leq BK$ comes from the definition of $r$, and $K \leq Bs$ from $h/k = \zeta - r$; signs do not matter. The unimodular difference identity from (4.9)–(4.10) gives the local error $\omega_0$:

$$\omega_0 := |w_0 - r| = \left|t - \frac{h}{k}\right| = \frac{\varepsilon qVW}{K} > 0. \tag{4.14}$$

If $\pi_q \leq 12B^2$, the short alternative holds. Otherwise,

$$\omega_0 s^2 \leq B^2 \varepsilon qVWK \leq \frac{3}{2}B^2\varepsilon q^2 \leq \frac{3B^2}{2\pi_q} < \frac{1}{8}. \tag{4.15}$$

Legendre's criterion makes $r$ a nonterminal convergent of $w_0$. Let $S$ be its next denominator. Equation (2.1) yields

$$\frac{1}{s\omega_0} - s \leq S \leq \frac{1}{s\omega_0}, \qquad \frac{1}{2s\omega_0} \leq S \leq \frac{1}{s\omega_0},$$

where the second lower bound uses $\omega_0 s^2 < 1/8$. Equations (2.1) and (4.11)–(4.14) now give

$$\frac{q}{2BVW} \leq s \leq \frac{2Bq}{VW}, \qquad \frac{q^+}{2BVW} \leq S \leq \frac{2Bq^+}{VW}. \tag{4.16}$$

For the second pair, use

$$\frac{1}{s\omega_0} = \frac{K}{s\varepsilon qVW}, \qquad 1/B \leq K/s \leq B, \qquad q^+ \leq \frac{1}{\varepsilon q} \leq q^+ + q.$$

Since $\log(VW) = L - M$, (4.16) implies

$$\begin{aligned} |\log s - [a - (L - M)]| &\leq \log(2B), \\ |\log S - [b - (L - M)]| &\leq \log(2B). \end{aligned} \tag{4.17}$$

Apply Lemma 3.1 to the gap $s \to S$. The combined endpoint error is at most $\log(2B) + \log 2 = E$, giving (4.6). □

For a reflected gap with old width $g_0 = d - c$, (4.6) gives

$$g \le g_0 + 2E, \qquad a - c \ge (L - 2M) + g_0 - E. \tag{4.18}$$

For the second, use $d \le M$ and $a - c \ge L - d - c - E = L - 2d + g_0 - E$. When $L = 2M + o(M)$, the second inequality bounds the rightward shift $a - c$ from below by the old width $g_0$, up to the error terms.

## 5. Arithmetic of the truncations

Write $\lambda_j = \alpha_j/\beta_j$ in reduced form with $\beta_j > 0$, and put

$$t_n = u_n \log N, \qquad x_n = \sum_{j \le n} \lambda_j N^{-u_j} = \frac{P_n}{V_n} \tag{5.1}$$

in reduced form with $V_n > 0$. The ratio assumption gives a fixed lower ratio $\rho > 1$ for all large indices, for example $\rho = 3/2$. The height hypothesis and geometric growth imply

$$\sum_{j \le n} \log H(\lambda_j) = o(u_n). \tag{5.2}$$

Fix $\eta > 0$. After an index depending on $\eta$, each summand is at most $\eta u_j$, and $u_j \le u_n \rho^{-(n-j)}$ there. Sum this geometric bound and let $\eta$ tend to zero.

Set

$$D_n = \prod_{j \le n} \beta_j, \qquad S_n = \sum_{j \le n} \alpha_j (D_n/\beta_j) N^{u_n - u_j}.$$

Then $x_n = S_n/(N^{u_n} D_n)$ and $\log D_n = o(u_n)$. For every prime $p \mid N$, each summand of $S_n$ before the last is divisible by $p^{v_p(N)(u_n - u_{n-1})}$. The last is $\alpha_n D_{n-1}$, whose valuation is $o(u_n)$ by (5.2). Since $u_n - u_{n-1} \ge (1 - 1/\rho)u_n$ eventually, the last summand has strictly smaller valuation. Hence eventually

$$v_p(S_n) = v_p(\alpha_n D_{n-1}) = o(u_n). \tag{5.3}$$

Since $N$ has only finitely many prime divisors, the valuation estimates hold simultaneously after increasing the initial index. So $S_n \ne 0$. If $G_n = \gcd(S_n, N^{u_n} D_n)$, the part supported outside the primes dividing $N$ is at most $D_n$, and (5.3) bounds the remaining part by $\exp(o(u_n))$. Consequently,

$$\log G_n = o(u_n), \qquad V_n = \frac{N^{u_n} D_n}{G_n}, \qquad M_n := \log V_n = t_n + o(t_n). \tag{5.4}$$

The first omitted term dominates the rest of the tail, giving

$$x - x_n = \lambda_{n+1} N^{-u_{n+1}} (1 + o(1)), \qquad -\log|x - x_n| = t_{n+1} + o(t_{n+1}). \tag{5.5}$$

For this and for absolute convergence, choose $0 < \tau < \log N\,(\rho - 1)/(\rho + 1)$ and use $e^{-\tau u_j} \le |\lambda_j| \le e^{\tau u_j}$ at all late indices. For every $k \ge n + 1$ the ratio of successive absolute terms is at most

$$\exp(-[\rho(\log N - \tau) - (\log N + \tau)]u_k)\,.$$

So the rest of the tail is exponentially small relative to the first omitted term, and the tail is eventually nonzero.

If $x = a'/b'$ were rational, its nonzero difference from $x_n$ would be at least $1/(b'V_n)$, contrary to (5.4)–(5.5) and the eventual lower ratio $\rho > 1$. Thus $x$ is irrational, and its truncations give

$$\mu(x) \geq \limsup_{n\to\infty} \frac{u_{n+1}}{u_n}. \tag{5.6}$$

By (5.4), $V_n \to \infty$, so the truncations give infinitely many distinct rational approximants. Their approximation exponents are $t_{n+1}(1 + o(1))/(t_n(1 + o(1)))$. An infinite limsup gives (1.2) at once.

For the rest of the proof assume

$$F := \limsup_{n\to\infty} \frac{u_{n+1}}{u_n} < \infty. \tag{5.7}$$

Thus $F \geq 2$ and $u_{n+1} = O(u_n)$. Apply Lemma 4.1 to

$$\delta_n = x_{n+1} - x_n = \lambda_{n+1} N^{-u_{n+1}}, \qquad L_n = -\log|\delta_n|.$$

Here the application has $\alpha = x_n$, $\beta = x_{n+1}$, $\delta = \delta_n$, so that $M = M_n$, $L = L_n$, $\zeta = \zeta_n$ and $B = B_n$. The absolute value of the signed coordinate $\zeta_n$ is

$$\frac{1}{V_n^2|\delta_n|} = \frac{\beta_{n+1} G_n^2 N^{u_{n+1}-2u_n}}{|\alpha_{n+1}| D_n^2}.$$

Writing $B_n = \mathrm{den}(\zeta_n)$, reduction can only decrease the denominator, so

$$B_n \leq |\alpha_{n+1}| D_n^2 N^{(2u_n - u_{n+1})_+}. \tag{5.8}$$

The coefficient cost is $o(u_n)$ by (5.7), and $(2u_n - u_{n+1})_+ = o(u_n)$ by the lower ratio hypothesis. Thus

$$\log B_n = o(M_n). \tag{5.9}$$

This includes the steps with $u_{n+1} < 2u_n$. Finally,

$$L_n = t_{n+1} + o(t_n), \qquad \liminf \frac{L_n}{M_n} \geq 2, \qquad \limsup \frac{L_n}{M_n} = F. \tag{5.10}$$

So the negative part of $L_n - 2M_n$ is $o(M_n)$.

## 6. A uniform finite profile bound

In this section $F < \infty$; the case $F = +\infty$ follows from (5.6). Fix

$$\kappa > F - 2. \tag{6.1}$$

In particular, $\kappa > 0$, also when $F = 2$. Put $C = \log 6$. There are $n_0$ and $A \geq \log 6$, depending on $\kappa$ and the series, such that every logarithmic gap $a \to b$ of every $x_n$, $n \geq n_0$, satisfies

$$b - a \leq \kappa a + A + C(n - n_0). \tag{6.2}$$

Put $E_n = \log(4B_n)$. By (5.9)–(5.10) there is a sequence $\eta_n \to 0$ such that

$$\begin{gathered} -\eta_n M_n \leq L_n - 2M_n \leq (F - 2 + \eta_n) M_n, \\ E_n \leq \eta_n M_n, \qquad \log(3/2) \leq \eta_n M_n. \end{gathered} \tag{6.3}$$

Choose $n_0$ so that $V_n \geq 2$ and

$$\eta_n \leq \min\left\{ \frac{\kappa - (F-2)}{2}, \frac{\kappa}{16(1+\kappa)}, \frac{\kappa^2}{32(1+\kappa)} \right\} \tag{6.4}$$

for $n \geq n_0$. Increase $A$, if necessary, to cover the finitely many gaps of $x_{n_0}$.

Induct from $n$ to $n+1$ with Lemma 4.1. A bounded gap has width at most $\log 6$. An inherited gap has the same left endpoint as an old gap and width at most its width plus $\log 6 = C$. Both satisfy (6.2) at index $n+1$. A central gap begins at $a = M_n$ and has width at most

$$L_n - 2M_n \le (F - 2 + \eta_n)M_n \le \kappa a.$$

For the short alternative,

$$g \le 2\eta_n M_n, \qquad a \ge (1 - 2\eta_n)M_n,$$

and (6.4) gives $g \le \kappa a$.

For a reflected gap, let $c \to d$ be its old gap and $g_0 = d - c$. Equations (4.18) and (6.3) give

$$g \le g_0 + 2\eta_n M_n, \qquad a - c \ge g_0 - 2\eta_n M_n, \qquad a \ge (1 - 2\eta_n)M_n. \tag{6.5}$$

If $g_0 < \kappa M_n/4$, then

$$g \le (\kappa/4 + 2\eta_n)M_n \le \kappa(1 - 2\eta_n)M_n \le \kappa a.$$

If $g_0 \ge \kappa M_n/4$, then

$$\begin{aligned} g - \kappa a &\le (g_0 - \kappa c) - \kappa g_0 + (2 + 2\kappa)\eta_n M_n \\ &\le (g_0 - \kappa c) - \frac{\kappa^2}{4}M_n + \frac{\kappa^2}{16}M_n \le g_0 - \kappa c. \end{aligned}$$

Thus the bound for $c \to d$ gives the bound for the new gap without loss, and (6.2) follows.

## 7. Transfer to the infinite continued fraction

Let $p/q$ be a convergent of $x$ with next denominator $q^+$, and put

$$a = \log q, \qquad b = \log q^+, \qquad \varepsilon = |x - p/q|.$$

Let $q \to \infty$. Gaps with $q^+/q \le 6$ are bounded, and their widths divided by the left endpoint tend to zero. Thus suppose $q^+/q > 6$. Choose the least $n \ge n_0$ for which

$$t_n \ge 4(a + b + 1). \tag{7.1}$$

Eventual geometric lower growth gives

$$n = O(\log(a + b + 2)). \tag{7.2}$$

Indeed, $t_n \ge c_0\rho^n$ eventually for some $c_0 > 0$, and minimality gives $t_{n-1} < 4(a + b + 1)$ once $n > n_0$.

The selected $n$ tends to infinity with $q$, since $a + b \to \infty$ and $t_n \ge 4(a + b + 1)$. By (5.4)–(5.5), for all sufficiently large selected indices,

$$V_n \ge e^{t_n/2} > q, \qquad |x - x_n| \le e^{-t_n}. \tag{7.3}$$

On the other hand,

$$\varepsilon \ge \frac{1}{q(q^+ + q)} \ge \tfrac{1}{2}e^{-(a+b)}.$$

Thus $|x - x_n| < \varepsilon/2$. The inequality $V_n > q$ ensures $p/q \ne x_n$, and

$$0 < \left| x_n - \frac{p}{q} \right| \le \frac{3}{2}\varepsilon \le \frac{3}{2qq^+} < \frac{1}{2q^2}. \tag{7.4}$$

Legendre's criterion makes $p/q$ a nonterminal convergent of $x_n$. Let $S$ be the next denominator after $q$ in the continued fraction of $x_n$. Then (2.1) and (7.4) give

$$S \geq \tfrac{2}{3}q^+ - q \geq q^+/6.$$

By (6.2) applied to this finite gap,

$$b - a \leq \log S - \log q + \log 6 \leq \kappa a + A + C(n - n_0) + \log 6.$$

By (7.2),

$$b - a \leq \kappa a + O(\log(a + b + 2)). \tag{7.5}$$

The implied constants depend only on $\kappa$ and the series.

If $b \geq 2(1 + \kappa)a$, then (7.5) gives $b/2 \leq O(\log(2b + 2))$, which fails for large $b$. Hence $b = O(a)$. The term $C(n - n_0)$ in (6.2) is then $O(\log a)$, and the error in (7.5) is $O(\log a) = o(a)$. Consequently,

$$\limsup \frac{b - a}{a} \leq \kappa. \tag{7.6}$$

## 8. Completion of the proof

For an irrational $x$ with convergent denominators $q_m$, (2.1) gives

$$-\frac{\log |x - p_m/q_m|}{\log q_m} = 1 + \frac{\log q_{m+1}}{\log q_m} + o(1).$$

By Legendre's criterion every reduced approximation with $|x - p/q| < 1/(2q^2)$ is a convergent; so is every reduced approximation of fixed exponent greater than 2 once $q$ is large. In the definition of $\mu$, a pair $(p, q)$ need not be reduced. Reducing it does not increase the denominator, so the same inequality holds. A fixed reduced fraction occurs only finitely often among approximations of a fixed positive exponent, because its error is nonzero. In a bounded interval there are only finitely many reduced fractions with bounded denominator. Thus infinitely many such approximations have unbounded reduced denominators, and the preceding Legendre argument applies when the exponent is greater than 2. Therefore

$$\mu(x) = 1 + \limsup_m \frac{\log q_{m+1}}{\log q_m} = 2 + \limsup_m \frac{\log(q_{m+1}/q_m)}{\log q_m}. \tag{8.1}$$

This holds also when the limsup is infinite. Equation (7.6) gives $\mu(x) \leq 2 + \kappa$. Letting $\kappa$ decrease to $F - 2$ and using (5.6) proves (1.2) for finite $F$. The infinite case is in Section 5, which completes the proof of Theorem 1. □

## 9. Height sharpness and scope

The coefficient condition cannot be replaced by $\log H(\lambda_j) \leq C_* u_j$ with any prescribed constant $C_* > 0$. The first construction below shows this. The second shows that, at one fixed linear height scale, the sum can be rational or have infinite irrationality exponent. In this section index the exponents by $j \geq 0$ and take $u_j = 2^j$. Write $\ell = \log N$.

Fix $0 < \theta < 1/3$, and set

$$m_j = \lfloor \theta 2^j \rfloor, \qquad \lambda_j = N^{(-1)^j m_j}, \qquad e_j = 2^j - (-1)^j m_j.$$

These positive rational weights give

$$\log H(\lambda_j) = m_j\ell = \theta\ell 2^j + O(1), \qquad x = \sum_{j\ge 0} N^{-e_j}.$$

The exponents $e_j$ are strictly increasing. For odd $j$ their difference is at least $(1-3\theta)2^j > 0$; for even $j$ it is $2^j + m_j + m_{j+1} > 0$. Their gaps tend to infinity, so the base-$N$ expansion is not eventually periodic and $x$ is irrational. The truncation $p_j/q_j$ through index $j$ has reduced denominator $q_j = N^{e_j}$, since its numerator is 1 modulo $N$, and

$$0 < x - p_j/q_j \le \frac{N^{-e_{j+1}}}{1-N^{-1}} \le 2N^{-e_{j+1}}.$$

Along even indices $e_{j+1}/e_j \to 2(1+\theta)/(1-\theta)$, whence

$$\mu(x) \ge \frac{2(1+\theta)}{1-\theta} > 2. \tag{9.1}$$

Choosing $0 < \theta < \min\{1/3, C_*/\ell\}$ makes $\log H(\lambda_j) \le C_* 2^j$ for every $j$.

At the fixed linear height scale below, positive integer weights can give a rational sum. Set

$$\lambda_0 = N-1, \qquad \lambda_j = N^{2^{j-1}} - 1 \quad (j \ge 1).$$

Then $\lambda_j N^{-2^j} = N^{-2^{j-1}} - N^{-2^j}$ for $j \ge 1$, so

$$\sum_{j=0}^{m} \lambda_j N^{-2^j} = 1 - N^{-2^m} \longrightarrow 1, \qquad \log H(\lambda_j) \sim \tfrac{1}{2}\ell 2^j.$$

At the same scale, replace the later weights by

$$\widetilde{\lambda}_j = N^{2^{j-1}} - 1 + \mathbf{1}_{\{j=r^2 \text{ for some } r\ge 1\}} \quad (j \ge 1), \qquad \widetilde{\lambda}_0 = N-1.$$

They remain positive integers, with logarithmic height asymptotic to $\frac{1}{2}\ell 2^j$. Their sum is

$$\widetilde{x} = 1 + \xi_N, \qquad \xi_N = \sum_{r\ge 1} N^{-2^{r^2}}.$$

The digit expansion shows that $\xi_N$ is irrational. The truncation through $r = J$ has reduced denominator $q_J = N^{2^{J^2}}$, since its numerator is one modulo $N$, and error

$$0 < \xi_N - \sum_{r=1}^{J} N^{-2^{r^2}} \le 2N^{-2^{(J+1)^2}} = 2q_J^{-2^{2J+1}}.$$

Adding one to each approximant preserves its reduced denominator and its approximation error. Thus $\mu(\widetilde{x}) = +\infty$.

The height condition is sufficient but not necessary for a given series. For example, if $b \ge 2$ is an integer, the weights $\lambda_j = b^{-2^j}$ have logarithmic height $(\log b)2^j$, but their sum is $\sum_{j\ge 0}(Nb)^{-2^j}$, which has exponent 2 by Theorem 1 with base $Nb$ and unit weights.

Theorem 1 does not cover $\liminf u_{j+1}/u_j < 2$. The examples above keep the exponent ratios equal to 2 and vary only the coefficient heights. Problem 10.40 in [3, p. 220] asks for the irrationality exponent of binary lacunary sums $\sum_j 2^{-n_j}$ for an arbitrary increasing lacunary exponent sequence $(n_j)$. For $N = 2$, Theorem 1 gives the exponent when $\liminf n_{j+1}/n_j \ge 2$, including ratios tending to 2 from below, and allows small rational weights. The case with eventual ratio at least 2 is recorded in [4] for unit coefficients, with attribution there to Bugeaud's earlier

work, and for coefficients $\pm 1$ it also follows from [5, Theorem 1]. Under the same height hypothesis, the subcase $2 < \liminf n_{j+1}/n_j \le \limsup n_{j+1}/n_j < \infty$ also follows from [5, Theorem 1].

## Use of AI tools

The author devised the main argument. Codex (OpenAI, GPT-6 Astra) carried out exact checks of the continued-fraction identities and searched the literature. Claude Code (Anthropic, Claude Opus 5.5) reviewed the arguments and references. The author takes responsibility for the proof and references.

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