SHIFTED PRIMES IN ARITHMETIC PROGRESSIONS TO LARGE MODULI WITH LINEAR SIEVE REMAINDER WEIGHTS

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Abstract. Let N be a sufficiently large even integer, and fix $B > 0$ and $0 < \varepsilon \leq 10^{-4}$. Take Iwaniec's upper or lower linear-sieve remainder weights. We estimate a signed sum of prime-counting discrepancies on $N/(\log N)^B < p < (1-\varepsilon)N$. Fix $0 < u \leq 1/500$ and a sieve exponent $1/2 < \theta < (5-7/32-3u)/8$. Choose $0 < \sigma \leq \min\{10^{-10}, ((5-7/32-3u)/8-\theta)/200, \sqrt{u/(2\theta)}\}$. Then the complete modulus support lies below $N^{(5-7/32-3u)/8}$. For every fixed saving exponent $A > 0$, the error is $O(N(\log N)^{-A})$. For each of the two signs $\nu \in \{+,-\}$, the result holds uniformly for integers $a \in [N, N\log N/3]$ satisfying $(a,q) = 1$ for every modulus q in the support of the corresponding coefficient w_N^ν. Call these shifts admissible; they form a subset of the interval that includes $a = N$. When $u = 1/500$ and $\theta = 19101/32000 - 10^{-4}$, the ratio $\log D/\log z$ is 298.403125. The proof keeps the smooth-divisor and prime-tuple factors of the linear-sieve weights. It combines dispersion with a Heath–Brown decomposition. The boundary, Type-I and Type-II terms are estimated separately.

1. Introduction

The Bombieri–Vinogradov theorem [2, 3] gives level $1/2$ for primes in arithmetic progressions on average. Related early weighted work includes Fouvry–Iwaniec [19, 20]. Consider weighted sums with a fixed residue. Bombieri–Friedlander–Iwaniec reached level $4/7$ with well-factorable weights [6, Theorem 10]. With a fixed residue, Maynard [7, Theorems 1.1–1.2] proved level $3/5$ for triply well-factorable weights and level $7/12$ for the upper linear-sieve weights. Parts I–III address fixed residue classes, well-factorable estimates, and uniform residue classes, respectively [21, 7, 22]. Lichtman [1, Theorem 1.7, (1.6)] obtained level $66/107$ for triply well-factorable weights with a fixed residue. For such weights he also obtained level $153/256-\delta$, uniformly over residues $|a| < N^{1+\delta}$ for each fixed $\delta > 0$ [1, Theorem 1.7, (1.7)]. His Proposition 6.6 treats Iwaniec's modified upper and lower linear-sieve weights. There the largest prime factors are restricted to prescribed short intervals. Its admissible level is $\vartheta \leq \boldsymbol{\vartheta}(t_1) - \delta$ with $\boldsymbol{\vartheta}(t_1)$ defined in [1, (6.2)]. Take at least three prescribed factors and t_1 below the threshold in that proposition. Then the level becomes $\vartheta \leq \boldsymbol{\vartheta}(t_1,t_2,t_3) - \delta$, with the function of [1, (6.4)] and fixed $\delta > 0$. The error constant in that proposition depends on the fixed residue a [1, Proposition 6.6]. Lichtman's modified linear-sieve weights reach level $10/17$ [25].

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Pascadi [8, Theorem 1.3] gives levels approaching 5/8 for triply well-factorable weights, with a fixed residue. For suitable upper linear-sieve weights his levels approach 3/5. Pascadi's Theorem 5.4 also treats both signs of the restricted well-factorable linear-sieve weights and refines Proposition 6.6 [8, Theorem 5.4]. His Lemma 5.6 records a correction to a small error in Section 6 of [1] [8, Lemma 5.6].

Here $z = N^u$ with $u \le 1/500$ and $\theta > 1/2$, so $s = \theta/u > 250$, in the fundamental-lemma range. The cited linear-sieve levels use sifting ranges with $s \approx 1$–2 and are not directly comparable.

The congruence $p \equiv a \pmod q$ says that q divides the shifted prime $a-p$. So the sums below count shifted primes $a-p$ in the class 0 modulo q. The modulus coefficients are Iwaniec's explicit lower and upper remainder weights [10, Theorem 4]. We retain their smooth-divisor factors, prime-tuple factors, and distinctness conditions. We estimate the complete signed sum for the admissible shifts in the stated interval. Theorem 1.2 covers the fixed parameters $u, \theta, A, B, \varepsilon$ below. Its support exponent tends to 153/256 as $u \downarrow 0$. This numerical endpoint matches Lichtman's moving-residue level for triply well-factorable weights [1, Theorem 1.7, (1.7)]; the residue ranges differ. At $u = 1/500$ the exponent is smaller. The coefficient family here is the full remainder in [10, Theorem 4]. It includes the smooth-only term and the distinctness masks. Lichtman's Proposition 6.6 estimates restricted factor-pattern components of different modified sieve weights. Its implied constant depends on the fixed residue. Here uniformity is only over the support-coprime shifts specified below, an admissible subset of $[N, N\log N/3]$ that includes $a = N$; no uniformity over the full interval is asserted.

The local statement uses the hypotheses of Section 3. Let B be a Heath–Brown box at scale X. Write $w_B(n)$ for its finite source coefficient and $C_B(q)$ for its modulus coefficient. Set

$$\Delta_B(q;a) = \sum_n w_B(n)\left(\mathbf{1}_{n\equiv a\,(q)} - \frac{\mathbf{1}_{(n,q)=1}}{\varphi(q)}\right).$$

In Section 3, w_B is written as a sum over the variables of B; this is the same discrepancy.

Proposition (Dispersion for a source box)**.** *Fix $0 < \sigma \le 10^{-10}$ and constants $K_{\mathrm{scale}} > 0$ and $C_{\mathrm{box}} > 0$. Here K_{scale} bounds the logarithmic difference between the prime scale X and N. The source boxes have relative width $(\log X)^{-C_{\mathrm{box}}}$. Let N be sufficiently large and even, $N/(\log N)^{K_{\mathrm{scale}}} \le X \le N$, and put $L = \log X$ and $A_0 = X^{1+\sigma/20}$. Let a be a positive integer with $a \le X(\log N)^{K_{\mathrm{scale}}} \le A_0$. Suppose every supported modulus is coprime to $2a$. Suppose B and C_B satisfy (H1)–(H5) of Section 3. These include the support condition $q \le X^{\vartheta_0}$, where*

$$\eta_L = 10^{-4}, \quad \vartheta_0 = \frac{19101}{32000} - \eta_L = 0.59680625, \quad \ell_0 = \frac{53}{412} - \frac{\eta_L}{2}, \quad c_\sigma = \frac{\ell_0\sigma^2}{4} > 0.$$

For each fixed $A' > 0$ there is a finite $C = C(\sigma, K_{\mathrm{scale}}, C_{\mathrm{box}}, A', \mathcal{C}_{\mathrm{src}})$ such that

$$\left|\sum_{q\le X^{\vartheta_0}} C_B(q)\Delta_B(q;a)\right| \ll_{\sigma,K_{\mathrm{scale}},C_{\mathrm{box}},A',\mathcal{C}_{\mathrm{src}}} XL^{-A'} + X^{1-c_\sigma}L^C. \tag{1.1}$$

Here $\mathcal{C}_{\mathrm{src}}$ denotes the fixed uniform constants in (H1)–(H5). The estimate is uniform in B, N, and all a as above. The implied constant and the threshold for N are ineffective.

Theorem 3.1 in Section 3 is the abstract dispersion estimate under the support and packing hypotheses in (H3). Theorem 3.2 is the dispersion engine used for Theorem 1.2. It allows support up to $X^{0.6}$.

Theorem 1.1 (Signed shifted-prime mean value at $u = 1/500$)**.** *Fix $A, B > 0$, $0 < \varepsilon \le 10^{-4}$, and*

$$\frac{1}{2} < \theta < \Theta_* := \frac{19101}{32000}.$$

Set

$$u = \frac{1}{500}, \qquad \eta_\Theta = \Theta_* - \theta, \qquad 0 < \sigma \le \min\left\{10^{-10}, \frac{\eta_\Theta}{200}, \sqrt{\frac{1}{1000\theta}}\right\},$$

$$\tau = \sigma^9, \quad D = N^\theta, \quad z = N^{1/500}, \quad Y = D^{\sigma^2}, \quad K_{\mathrm{scale}} = B + 1,$$

and choose an integer $C_{\mathrm{box}} \ge \lceil 2A+32 \rceil$; constants may depend on this choice, as in Theorem 1.2. Let N be sufficiently large and even. For either sign $\nu \in \{+,-\}$, let $w_N^\nu(q)$ be the complete Iwaniec remainder coefficient of formula (W) *in Section 5. Its prime set is $\mathcal{P}_N = \{p \text{ prime} : p \nmid 2N\}$, with the grid and distinctness conditions there. Let a be an integer with $N \le a \le N \log N/3$ and $(a,q) = 1$ for every $q \in \operatorname{supp} w_N^\nu$. Call these shifts admissible; they form a subset of the interval that includes $a = N$. Put $I_N = (N/(\log N)^B, (1-\varepsilon)N)$. Let π_{I_N} count its primes, and $\pi_{I_N}(q,a)$ those congruent to a modulo q. Then*

$$\operatorname{supp} w_N^\nu \subseteq \{q < D^{1+\sigma+\tau} : (q, 2N) = 1\}, \qquad D^{1+\sigma+\tau} < N^{\Theta_*}, \qquad |w_N^\nu(q)| \ll_{\sigma,\theta} 1,$$

and

$$\left| \sum_q w_N^\nu(q) \left\{ \pi_{I_N}(q,a) - \frac{\pi_{I_N}}{\varphi(q)} \right\} \right| \ll_{\sigma,A,\theta,B,\varepsilon,C_{\mathrm{box}}} N(\log N)^{-A}.$$

The sum covers the complete support of formula (W)*. For the admissible choice $\theta = 19101/32000 - 10^{-4}$, the sieve ratio $\log D/\log z = 500\theta$ equals* 298.403125*. The implied constant and the threshold for N are ineffective.*

Remark (parameter conventions). In Theorem 1.1 one takes $A_{\mathrm{agg}} = A$ and $C_{\mathrm{box}} \ge \lceil 2A+32 \rceil$. The shifted choice $A_{\mathrm{agg}} = A+3$ and the corresponding $C_{\mathrm{box}} \ge \lceil 2A+38 \rceil$ occur only in the auxiliary source-wide estimate (3.95), which is not used in the proofs of Theorems 1.1–1.2 or Corollary 1.3, nor in Section 5.

Theorem 1.2 (Signed shifted-prime mean value)**.** *Fix $A > 0$, $B > 0$, $0 < \varepsilon \le 10^{-4}$, $0 < u \le 1/500$, and*

$$\frac{1}{2} < \theta < \Theta(u) := \frac{5 - 7/32 - 3u}{8}.$$

Put

$$\eta_\Theta = \Theta(u) - \theta,$$

$$0 < \sigma \le \min\{10^{-10}, \eta_\Theta/200, \sqrt{u/(2\theta)}\},$$

$$\tau = \sigma^9, \quad D = N^\theta, \quad z = N^u, \quad Y = D^{\sigma^2}.$$

Set $K_{\mathrm{scale}} = B + 1$. Choose an integer box-width exponent

$$C_{\mathrm{box}} \ge \lceil 2A + 32 \rceil.$$

The implied constant and the threshold for N below may also depend on the chosen C_{box}. For the canonical choice $C_{\mathrm{box}} = \lceil 2A + 32 \rceil$ this is a dependence on A alone.

Equivalently $C_{\mathrm{box}} \geq 2(A + c_R)$, with the boundary exponent $c_R = 16$ of (3.24) and (B.19). This also gives the logarithm-freezing condition $C_{\mathrm{box}} \geq A+18$ from (B.18). Let N be sufficiently large and even. For each sign $\nu \in \{+, -\}$, let $w_N^\nu(q)$ be the complete Iwaniec remainder coefficient defined by formula (W) *in Section 5. Its prime set is $\mathcal{P}_N = \{p \text{ prime} : p \nmid 2N\}$, with the grid and distinctness conditions specified there. Let a be any integer satisfying*

$$N \leq a \leq N \log N/3, \qquad (a, q) = 1 \quad \text{for every } q \in \operatorname{supp} w_N^\nu.$$

Call these shifts admissible; they form a subset of the interval that includes $a = N$. Put

$$I_N = \left(\frac{N}{(\log N)^B}, (1 - \varepsilon)N \right).$$

Write π_{I_N} for the number of primes in I_N. Write $\pi_{I_N}(q, a)$ for the number of these primes with $p \equiv a \pmod q$. For every fixed σ in the displayed range, these coefficients satisfy

$$\operatorname{supp} w_N^\nu \subseteq \{q < D^{1+\sigma+\tau} : (q, 2N) = 1\},$$
$$D^{1+\sigma+\tau} < N^{\Theta(u)}, \qquad |w_N^\nu(q)| \ll_{\sigma,u,\theta} 1,$$

and

$$\left| \sum_q w_N^\nu(q) \left\{ \pi_{I_N}(q, a) - \frac{\pi_{I_N}}{\varphi(q)} \right\} \right| \ll_{\sigma,A,u,\theta,B,\varepsilon,C_{\mathrm{box}}} N(\log N)^{-A}. \tag{1.2}$$

The sum includes the entire support of formula (W). *The sieve ratio is $\log D/\log z = \theta/u$. The condition on σ gives $Y = D^{\sigma^2} = N^{\theta\sigma^2} \leq N^{u/2} < z$, as required for the rough-factor bounds. The implied constant and the threshold for N are ineffective. The proof uses the Siegel–Walfisz theorem.*

Proof of Theorem 1.1. Apply Theorem 1.2 with $u = 1/500$. Then $\Theta(u) = 19101/32000$, and its condition on σ becomes the displayed condition above. All the support, coefficient, and discrepancy bounds specialize as stated. The numerical sieve ratio follows from $\log D/\log z = \theta/u$. At $\theta = 19101/32000 - 10^{-4}$ the proof uses Theorem 3.2 through Theorem 1.2. That theorem allows support up to $X^{0.6}$. It does not use the introductory proposition. At $\theta = \vartheta_0$ the support bound $D^{1+\sigma+\tau}$ of (W) exceeds X^{ϑ_0}. □

In Theorem 1.2, $D = N^\theta$ and $D^{1+\sigma+\tau} < N^{\Theta(u)}$ by the choice of σ. The introductory source-box estimate has the fixed local level $\vartheta_0 = 19101/32000 - 10^{-4}$. It assumes its support hypothesis. The packing ceiling for Theorem 1.2 is $\Theta(u)$. The latter applies the $X^{0.6}$ dispersion theorem 3.2 together with the source estimates below. Here $\Theta(u) < 153/256 < 0.6$. Fix $B > 0$. Each source shell has $X > N/(2(\log N)^B)$. So, for sufficiently large N,

$$N^{\Theta(u)} \leq \left(\frac{N}{2(\log N)^B} \right)^{0.6} < X^{0.6}.$$

Thus the full support of Iwaniec's remainder weights lies in the range of Theorem 3.2. Section 4 proves the modulus packing. Appendix B estimates the Heath–Brown, boundary, and other source terms. Section 5 assembles these estimates.

Corollary 1.3 (Logarithmic-integral comparator)**.** *Under the hypotheses of Theorem 1.2, for each $\nu \in \{+,-\}$,*

$$\left|\sum_q w_N^\nu(q)\left\{\pi_{I_N}(q,a) - \frac{\substack{\mathrm{Li}((1-\varepsilon)N)\\ -\mathrm{Li}(N/(\log N)^B)}}{\varphi(q)}\right\}\right| \ll_{\sigma,A,u,\theta,B,\varepsilon,C_{\mathrm{box}}} N(\log N)^{-A}.$$

The implied constant and the threshold for N are ineffective.

The proof uses the spectral large sieves of Deshouillers and Iwaniec [5]. It uses the Kim–Sarnak exponent 7/64 [13]. It also uses Deligne's bound for holomorphic coefficients [14]. Section 2 states these inputs. The dispersion argument and its application are in Sections 3–5.

2. Classical inputs and Theorem 2.8

Put $e(t) = \exp(2\pi i t)$. Write $n \sim N$ for $N < n \le 2N$. Unspecified arithmetic indices are positive integers. Norms of sequences are counting-measure ℓ^2 norms. A star denotes reduced residue classes in residue sums. In character sums it denotes primitive characters. These two uses are distinguished explicitly below. Constants are uniform within each sequence family. Put

$$\theta_{\mathrm{KS}} = \frac{7}{32}. \tag{2.1}$$

Thus $\theta_{\mathrm{KS}} = 2 \cdot 7/64$, where 7/64 is the unramified Ramanujan exponent. This symbol is distinct from the level parameter θ and the exceptional parameters ϑ_j.

2.1. Spectral normalization and large sieves.

The spectral large sieves are due to Deshouillers and Iwaniec [5].

Let $q \ge 1$. Use Petersson-orthonormal bases at level $\Gamma_0(q)$. The measure is $dx\,dy/y^2$. Take $\sigma_\infty = I$. For a Maass cusp form,

$$u_j(z) = \sum_{n\neq 0} \rho_{j\infty}(n)\sqrt{y}\,K_{i\kappa_j}(2\pi|n|y)e(nx), \qquad \lambda_j = \tfrac{1}{4} + \kappa_j^2. \tag{2.2}$$

This is the Fourier expansion of [5, (1.15)], as printed there. It has no factor $2\sqrt{|n|}$; that factor enters only through the dictionary (2.11). For a holomorphic form f of even weight $k \ge 2$, write

$$f(z) = \sum_{n\ge 1} \psi_f(\infty,n)e(nz). \tag{2.3}$$

The Eisenstein coefficients $\varphi_{\mathfrak{c}\infty n}(s)$ have the normalization of [5, (1.17)]. Their relation to the Whittaker coefficients ρ^{Dr} of Drappeau, whose full dictionary is (2.11) in §2.2 below, is

$$\rho^{\mathrm{Dr}}_{\mathfrak{c}\infty}(n,t) = \frac{\pi^{1/2+it}|n|^{it-1/2}}{\Gamma(1/2+it)}\varphi_{\mathfrak{c}\infty n}(1/2+it). \tag{2.4}$$

Let $\mathfrak{c}$ run over inequivalent cusps. Use width-one scaling matrices. The continuous-spectrum measure in the DI large sieve is plain dt.

Theorem 2.1 (consequence of [5, Theorem 2]**).** Let $\epsilon > 0$, $q \geq 1$, $K \geq 1$ and $N \geq \frac{1}{2}$. Let $a_n \in \mathbb{C}$ be supported on $N < n \leq 2N$. Then each of

$$H(K,a) = \sum_{\substack{2\leq k\leq K\\ k\in\text{even}}} \frac{(k-1)!}{(4\pi)^{k-1}} \sum_{f\in\mathcal{B}_k(q)} \left|\sum_{n\sim N} a_n n^{-(k-1)/2}\psi_f(\infty,n)\right|^2,$$

$$M(K,a) = \sum_{|\kappa_j|\leq K} \frac{1}{\cosh(\pi\kappa_j)} \left|\sum_{n\sim N} a_n\rho_{j\infty}(n)\right|^2, \tag{2.5}$$

$$E(K,a) = \sum_{\mathfrak{c}} \int_{-K}^{K} \left|\sum_{n\sim N} a_n n^{it}\varphi_{\mathfrak{c}\infty n}(1/2+it)\right|^2 dt$$

is at most

$$C_\epsilon \left(K^2 + \frac{N^{1+\epsilon}}{q}\right) \sum_{n\sim N} |a_n|^2, \tag{2.6}$$

uniformly in q, K, N, a. By [5, Theorem 2, (1.28)–(1.30)], each expression is bounded by $C_\epsilon\{K^2 + \mu(\mathfrak{a})N^{1+\epsilon}\}\|a_N\|_2^2$, with the constant depending only on ϵ. At $\mathfrak{a} = \infty$, $\mu(\infty) = 1/q$, giving (2.6); the source includes the endpoint $N \geq \frac{1}{2}$. The third line of (2.5) has no factor $1/\cosh(\pi t)$. The cusp transport for Theorem 2.8 is in §2.7.3.

For an exceptional eigenvalue $0 < \lambda_j < \frac{1}{4}$, choose $i\kappa_j > 0$, and put

$$\vartheta_j = 2i\kappa_j = 2\sqrt{\tfrac{1}{4} - \lambda_j}. \tag{2.7}$$

Theorem 2.2 ([5, Theorem 5]**, the specialization used here).** Let $\epsilon > 0$, $q \geq 1$, $N \geq \frac{1}{2}$ and $Y \geq 1$. Let $a_n \in \mathbb{C}$ be supported on $n \sim N$. Then

$$\sum_{j \text{ exceptional}} Y^{\vartheta_j} \left|\sum_{n\sim N} a_n\rho_{j\infty}(n)\right|^2 \ll_\epsilon \left(1 + \sqrt{\frac{NY}{q}}\right)\left(1 + \sqrt{\frac{N^{1+\epsilon}}{q}}\right)\|a\|_2^2. \tag{2.8}$$

This is [5, Theorem 5]. It uses the notation of Theorem 2, so $N \geq \frac{1}{2}$ is allowed. The implied constant depends on ϵ alone.

Theorem 2.3 (from [5, proof of Theorem 7, (8.17)]**).** Let $\epsilon > 0$. Let a_n be the characteristic sequence of an integer interval $I \subset (N, 2N]$. Then, for some c_ϵ and uniformly in $Q, N, Y \geq 1$ and I,

$$\sum_{2\leq q\leq Q} \sideset{}{^{(q)}}\sum_{j \text{ exceptional}} Y^{\vartheta_j} \Big|\sum_{n\sim N} a_n\rho_{j\infty}(n)\Big|^2 \leq c_\epsilon (QN)^\epsilon \big(Q + N + \sqrt{NY}\big)N. \tag{2.9}$$

The range starts at $q = 2$ because Theorem 2.8 uses level $q = rs$ with $r > R \geq 1$, so $q \geq 2$ there.

Proof. By [5, (8.17) and (8.4)],

$$\sum_{Q_0<q\leq 16Q_0} \sum_{\text{exc}} Y^{\vartheta_j} \left|\sum_{N<n\leq 2N} a_n\rho_{j\infty}(n)\right|^2 \leq c_{\epsilon'}^{\mathrm{DI}} (Q_0N)^{5\epsilon'}(Q_0 + N + \sqrt{NY})N$$

for $Q_0 \geq 1$. In [5] the interval is printed as $[N, N_1]$, $N \leq N_1 \leq 2N$. The sum runs over $N < n \leq 2N$, so a_n is the characteristic sequence of the initial interval $(N, N_1] \cap \mathbb{Z}$. The source estimate (8.17) is uniform for $Q_0, N, Y \geq 1$ and every real $N_1 \in [N, 2N]$; its final statement imposes no relation between Y and Q_0, N. In the proof, the induction first treats $1 \leq N \leq Q_0$ at $Y_* = Q_0^{2-2\epsilon'}/N$, while (8.18) handles the remaining large-N range, and the final comparison extends the result to every $Y \geq 1$. We write all intervals in this half-open form.

(i) *q-blocks.* Let $I_Q = \max\{i \geq 0 : 16^{-i}Q \geq 1\}$. The blocks $(16^{-i}Q, 16^{1-i}Q]$, $1 \leq i \leq I_Q$, together with $(1, 16]$, cover $2 \leq q \leq Q$. There are at most $2 + \log_{16} Q$ of them. Every summand is nonnegative. So each block may be enlarged to its full DI range. Each block has $Q_0 \leq Q$.

(ii) *Intervals.* Write $I = (u, v] \cap \mathbb{Z}$ with $N \leq u \leq v \leq 2N$. Then $\mathbf{1}_I = \mathbf{1}_{(N,v]} - \mathbf{1}_{(N,u]}$, both initial intervals in DI's sense, and $|A - B|^2 \leq 2|A|^2 + 2|B|^2$. This costs a factor 4.

Hence the left side of (2.9) is at most $4c_{\epsilon'}^{\mathrm{DI}}(2+\log_{16} Q)(QN)^{5\epsilon'}(Q+N+\sqrt{NY})N$. Take $\epsilon' = \epsilon/10$ and $c_\epsilon := 4c_{\epsilon/10}^{\mathrm{DI}} \sup_{Q\geq 1}(2 + \log_{16} Q)Q^{-\epsilon/2} < \infty$. □

The interval hypothesis is the one in [5, (8.17)], and the induction closes in the proof of Theorem 7. Theorem 2.3 comes from this step of that proof. [5, Theorem 7] treats $a_n = 1$ on $n \leq N$. Theorem 2.3 needs $N \geq 1$. The one-integer block $N = \frac{1}{2}$ is treated in §2.7.5.

2.2. **Hecke exponents and the adopted trace normalization. Kim–Sarnak input.** The bound 7/64 at an unramified place is [13, Appendix 2, Proposition 2]; the Satake normalization is stated in [13, p. 175]. Independently, the archimedean input $\lambda_j \geq 975/4096$ is the lower bound in [13, Appendix 2]; it is not a consequence of the finite-place estimate below. Let f be a weight-zero Maass newform of congruence level. Each exceptional parameter of f satisfies $0 \leq i\kappa \leq 7/64$. For a prime p not dividing the level,

$$|\alpha_p|, |\beta_p| \leq p^{7/64}, \qquad |\lambda_f(p^\nu)| \leq (\nu+1)p^{7\nu/64} \quad (\nu \geq 0). \tag{2.10}$$

Thus the preceding archimedean input gives $\vartheta_j \leq \theta_{\mathrm{KS}}$, equivalently $\lambda_j \geq 975/4096$ by (2.7). At the finite places, (2.10) gives $|\lambda_f(k)| \leq \tau(k)k^{\theta_{\mathrm{KS}}/2}$ for k coprime to the level. We use these bounds for newforms of level dividing q. These newforms diagonalize the unramified Hecke operators on the full level-q space. For holomorphic newforms we use Deligne's $|\lambda_f(k)| \leq \tau(k)$, again for k coprime to the level. Both factorizations use primes coprime to the level.

Deligne input. The holomorphic bound is [14, Théorème 8.2]. Both inputs enter at primes coprime to the level, through the prime-power recurrences and multiplicativity above.

Bruggeman and Kuznetsov independently introduced the relevant trace formulas; see [5, p. 220] and their original papers [27, 26]. For the normalization and formulas used here, see [9, (4.7), (4.9)–(4.19), Lemma 4.5], in the PLMS version. The coefficient dictionary is

$$\begin{aligned} \rho_{f\mathfrak{a}}^{\mathrm{Dr}}(n) &= \frac{\rho_{f\mathfrak{a}}^{\mathrm{DI}}(n)}{2\sqrt{|n|}} && \text{(Maass)}, \\ \rho_{f\mathfrak{a}}^{\mathrm{Dr}}(n) &= (4\pi n)^{-k/2}\psi_f(\mathfrak{a}, n) && \text{(holomorphic)}, \end{aligned} \tag{2.11}$$

with (2.4) for Eisenstein series. The Maass identity follows from $W_{0,it}(4\pi|n|y) = 2\sqrt{|n|y}K_{it}(2\pi|n|y)$.

For $\phi \in C_c^\infty((0,\infty))$, define the DI transforms

$$\begin{aligned}\widehat{\phi}(t) &= \frac{\pi}{\sinh \pi t}\int_0^\infty \frac{J_{2it}(x) - J_{-2it}(x)}{2i}\phi(x)\frac{dx}{x},\\ \check{\phi}(t) &= \frac{4}{\pi}\cosh \pi t \int_0^\infty K_{2it}(x)\phi(x)\frac{dx}{x},\\ \widetilde{\phi}(k-1) &= \int_0^\infty J_{k-1}(x)\phi(x)\frac{dx}{x}.\end{aligned} \tag{2.12}$$

Values at removable singularities are limits. In Drappeau's notation the corresponding transforms are $-4\widehat{\phi}$, $2\pi\check{\phi}$, and $4i^k\widetilde{\phi}(k-1)$.

Theorem 2.8 and Section 3 use the following identity. Let $q = rs$, $(r,s) = 1$, and $r > 1$; take

$$\sigma_{1/s} = \begin{pmatrix}\sqrt{r} & 0\\ s\sqrt{r} & 1/\sqrt{r}\end{pmatrix}, \qquad \mathcal{C}(\infty, 1/s) = \{\gamma = cs\sqrt{r} : c \geq 1, (c,r) = 1\}. \tag{2.13}$$

The scaling matrix and admissible moduli in (2.13) are also given in [5, (1.6) and (2.4)] and [9, Lemma 4.3]. Since $r > 1$, the cusps ∞ and $1/s$ are inequivalent. Hence the same-sign identity has no diagonal term. If a matrix in $\Gamma_0(rs)$ maps ∞ to the reduced fraction $1/s$, its lower-left entry is $\pm s$. Then $rs \mid s$, so $r = 1$. Every application here has $r > R \geq 1$. For $m, n \geq 1$, put $\mathcal{K}_\pm(m,n) = \sum_{\gamma\in\mathcal{C}}\gamma^{-1}S_{\infty,1/s}(m,\pm n;\gamma)\phi(4\pi\sqrt{mn}/\gamma)$. In the DI coefficients, Drappeau's formulas read

$$\begin{aligned}\mathcal{K}_+(m,n) = \frac{1}{\pi}&\sum_{\substack{k\geq 2\\ k \text{ even}}} i^k(k-1)!(4\pi\sqrt{mn})^{1-k}\widetilde{\phi}(k-1)\sum_f \overline{\psi_f(\infty,m)}\psi_f(1/s,n)\\ &- \sum_j \frac{\widehat{\phi}(\kappa_j)}{\cosh \pi\kappa_j}\overline{\rho_{j\infty}(m)}\rho_{j,1/s}(n)\\ &- \frac{1}{\pi}\sum_{\mathfrak{c}}\int_{\mathbb{R}}\widehat{\phi}(t)(n/m)^{it}\overline{\varphi_{\mathfrak{c}\infty m}(1/2+it)}\varphi_{\mathfrak{c},1/s,n}(1/2+it)\,dt,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\mathcal{K}_-(m,n) = \frac{\pi}{2}&\sum_j \frac{\check{\phi}(\kappa_j)}{\cosh \pi\kappa_j}\overline{\rho_{j\infty}(m)}\rho_{j,1/s}(-n)\\ &+ \frac{1}{2}\sum_{\mathfrak{c}}\int_{\mathbb{R}}\check{\phi}(t)(n/m)^{it}\overline{\varphi_{\mathfrak{c}\infty m}(1/2+it)}\varphi_{\mathfrak{c},1/s,-n}(1/2+it)\,dt.\end{aligned}$$

The proof uses only the identities (2.7.T16)–(2.7.T18), derived in §2.7.2. The continuous integrals carry plain dt. Let α_m, β_n be finite sequences. Multiply (2.14) by $\alpha_m\beta_n$, sum in (m,n), then take spectral absolute values and apply Cauchy–Schwarz. The first functional is $L_1(u) = \sum_m \overline{\alpha_m}\rho_{u\infty}(m)$. The spectral product is $\overline{L_1(u)}\sum_n \beta_n\rho_{u,1/s}(\pm n)$.

2.3. **BFI and the classical progression estimates.** For a finitely supported sequence f, define

$$\Delta_f(q,a) = \sum_{n\equiv a\ (q)} f(n) - \frac{1}{\varphi(q)}\sum_{(n,q)=1} f(n), \qquad (a,q) = 1. \tag{2.15}$$

The notation of [6, (1.1)] writes $\Delta_f(x;q,a)$ for (2.15) with both sums cut at $n \le x$.

The progression estimates are from Bombieri, Friedlander and Iwaniec [6]. We use [6, (A2), Theorem 0(a), and proof of Theorem 0(a)].

BFI hypothesis (A2) [6, (A2)]**.** Fix $B_\beta > 0$ for the sequence family. Let β be in the family, with support parameter $N \ge 2$. For every $d, k \ge 1$ and integer $\ell \ne 0$ with $(\ell, k) = 1$, and for every $A > 0$, there is C_A with

$$\left| \sum_{\substack{n \equiv \ell \ (k) \\ (n,d)=1}} \beta_n - \frac{1}{\varphi(k)} \sum_{(n,dk)=1} \beta_n \right| \le C_A \|\beta\|_2 N^{1/2} \tau(d)^{B_\beta} (\log(2N))^{-A}. \tag{2.16}$$

The C_A are independent of N, d, k, ℓ and of β in the family. As in [6], d and k run over all positive integers. If $\|\beta\|_2^2 \ge N(\log 2N)^{-B'}$ for a fixed B', then a bound $C_A N \tau(d)^{B_\beta} (\log 2N)^{-(A+B'/2)}$ implies (2.16), after adjusting the constant.

Theorem 2.4 ([6, Theorem 0(a), (2.2)]**).** Let β_n be supported on $n \le N$, $N \ge 2$, and satisfy (2.16). For every $A > 0$, there is $B_1 > 0$ such that, uniformly for $Q \le N(\log N)^{-B_1}$,

$$\sum_{q \le Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} |\Delta_\beta(q,a)|^2 \ll \|\beta\|_2^2 N (\log N)^{-A}. \tag{2.17}$$

The constants depend on A, B_β and the constants in (2.16). For a sequence on $N < n \le 2N$, replace N by $2N$ in (2.17).

[6, Theorem 0(b)] is not used; its discrepancy has the cutoff $n \le x$ of [6, (1.1)], while (2.15) is a full-support discrepancy. The switched bilinear form uses the full-support bound (3.49), proved from (3.25), (3.48) and Theorem 2.4.

That bound uses [6, (1.6)]. Let $c_h \in \mathbb{C}$ be supported on $1 \le h \le H$. The large sieve gives

$$\sum_{q \le Q} \sum_{\chi \bmod q}^{\text{primitive}} \left| \sum_h c_h \chi(h) \right|^2 \ll (Q^2 + H) \|c\|_2^2. \tag{2.19}$$

We use this initial-interval form with H the upper support bound.

Bombieri–Vinogradov input. Write $\psi(y;q,a) = \sum_{n \le y, n \equiv a \ (q)} \Lambda(n)$. For every $A > 0$ there is $B = B(A) > 0$ such that, for $y \ge 3$ and $Q \le y^{1/2} (\log y)^{-B}$,

$$\sum_{q \le Q} \max_{(a,q)=1} \left| \psi(y;q,a) - \frac{y}{\varphi(q)} \right| \ll_A y (\log y)^{-A}. \tag{2.20}$$

This is BFI's formulation of the Bombieri–Vinogradov theorem [6, (1.4)]; compare the original results [2, 3]. The maximum allows the varying reduced residue $a = N \bmod q$. We apply (2.20) at each endpoint separately, with its own level inequality. Thus (2.20) is used at fixed y only.

Siegel–Walfisz input. We use the power-log form of [17, Corollary 5.29]. Remove prime powers and sum by parts. Let $C > 0$ and $A > 0$ be fixed. For $y \ge 3$, $q \le (\log y)^C$ and $(a,q) = 1$,

$$\pi(y;q,a) = \frac{\mathrm{Li}(y)}{\varphi(q)} + O_{A,C}\big(y (\log y)^{-A}\big). \tag{2.21}$$

The constant is uniform in reduced a and is ineffective. Subtraction at two comparable endpoints keeps any fixed logarithmic saving. We use (2.21) with this

power-log error. Three further forms need their own proofs. The first saves a power of the interval length. The second keeps the tensor and coprimality structure of the structured sequences. The third is localized to one dyadic box of the Heath–Brown decomposition. Section 3 proves the short-interval estimates and those for its structured sequences.

2.4. **The linear sieve and its structured remainder.** Let $\mathcal{A}$ be a finite integer sequence with nonnegative multiplicities. Let $\mathcal{P}$ be a prime set. Put

$$P(z)=\prod_{\substack{p<z\\ p\in\mathcal{P}}}p,\quad S(\mathcal{A},\mathcal{P},z)=\sum_{a\in\mathcal{A}}1_{(a,P(z))=1},\quad |\mathcal{A}_d|=\frac{\omega(d)}{d}X+r(\mathcal{A},d). \tag{2.22}$$

Here $X>0$ and ω is multiplicative. Require $0<\omega(p)<p$ for $p\in\mathcal{P}$. Set $\omega(p)=0$ for excluded primes. Set $V(z)=\prod_{p<z,p\in\mathcal{P}}(1-\omega(p)/p)$. Use the continuous linear-sieve solutions

$$\begin{aligned} sF(s)&=2e^{\gamma}\quad(0<s\le 3), & sf(s)&=0\quad(0<s\le 2),\\ (sF(s))'&=f(s-1)\quad(s>3), & (sf(s))'&=F(s-1)\quad(s>2). \end{aligned} \tag{2.23}$$

On $s\ge 1$, F decreases to 1. Also f increases to 1, with $f=0$ through $s=2$. In particular $F(1)=2e^{\gamma}$. We use this endpoint when the level is at least the sifting limit.

Let $K_{\mathrm{M}}\ge 2$ be one fixed absolute constant such that, for all $z>w\ge 2$,

$$\prod_{w\le p<z}\left(1-\frac{1}{p}\right)^{-1}\le\frac{\log z}{\log w}\left(1+\frac{K_{\mathrm{M}}}{\log w}\right).$$

Such a constant exists by Mertens' product estimate, after increasing it to cover bounded w.

Theorem 2.5 ([11, Theorem 1]**, dimension one).** Suppose, for some fixed $K\ge 2$ and every $z>w\ge 2$,

$$\prod_{\substack{w\le p<z\\ p\in\mathcal{P}}}(1-\omega(p)/p)^{-1}\le\frac{\log z}{\log w}\left(1+\frac{K}{\log w}\right). \tag{2.24}$$

Let $y\ge z\ge 2$ and $s=\log y/\log z$. Then

$$\begin{aligned} S(\mathcal{A},\mathcal{P},z)&\le XV(z)\{F(s)+e^{\sqrt{K}}Q_1(s)(\log y)^{-1/3}\}+\sum_{\substack{d<y\\ d\mid P(z)}}|r(\mathcal{A},d)|,\\ S(\mathcal{A},\mathcal{P},z)&\ge XV(z)\{f(s)-e^{\sqrt{K}}Q_1(s)(\log y)^{-1/3}\}-\sum_{\substack{d<y\\ d\mid P(z)}}|r(\mathcal{A},d)|, \end{aligned} \tag{2.25}$$

where, for $s\ge 1$,

$$Q_1(s)\le\exp\{-s\log s+s\log\log(3s)+O(s)\}. \tag{2.26}$$

See [11]. The hypotheses are (1.1)–(1.3), and Theorem 1 is (1.4)–(1.6). The dimension-one constants $\beta=2$ and $A=2e^{\gamma}$ appear in [11, p. 176]; the source states $B=0$ in [11, p. 173]. For $s\le 2$ the lower main function in (2.25) vanishes. Hence its lower main term is nonpositive.

For the later prime set $\mathcal{P}_N=\{p:p\nmid 2N\}$, take the multiplicative function with $\omega(p)=1$ on $\mathcal{P}_N$, $\omega(p)=0$ off it, and $\omega(p^a)=0$ for $a\ge 2$. Its product in (2.24) is

over a subset of all primes in $[w,z)$, so (2.24) holds uniformly in N with the fixed value $K=K_{\mathrm{M}}$. The same data satisfy (2.27) with $L_\omega=2$. Thus, whenever (2.25) is specialized to this set, its factor is $e^{\sqrt{K_{\mathrm{M}}}}$. This is fixed; combining it with (2.26) changes only the absolute constant in the $O(s)$ term there. This ω is a nominal sieve density: by Theorem 2.6 the constructed coefficients are independent of $\mathcal{A}$, X, and ω.

The structured remainder is from [10, Theorem 4].

Assume (2.24) and nonnegativity of ω on prime powers. Fix $L_\omega>1$. Assume, for all $z>w\ge 2$,

$$\sum_{\substack{w\le p<z\\ p\in\mathcal{P}}}\sum_{a\ge 2}\frac{\omega(p^a)}{p^a}\le\frac{L_\omega}{\log(3w)}. \tag{2.27}$$

These are [10, (1)–(2)] in dimension one, with $L=L_\omega$.

Fix $0<\epsilon_s<1/3$ and $D\ge 2$. Put $\delta_s=\epsilon_s^9$ and

$$\mathcal{G}(D)=\{D^{\epsilon_s^2(1+\delta_s)^n}:n\ge 0\}. \tag{2.28}$$

Let $\mathcal{H}(D)$ consist of tuples $\mathbf{D}=(D_1,\dots,D_\rho)$ from this grid with $D_\rho\le\cdots\le D_1<D^{1/2}$. Define

$$\begin{aligned}\mathcal{D}^+(D)&=\{\mathbf{D}\in\mathcal{H}(D):D_1\cdots D_{2l}D_{2l+1}^3<D\quad(0\le l\le(\rho-1)/2)\},\\ \mathcal{D}^-(D)&=\{\mathbf{D}\in\mathcal{H}(D):D_1\cdots D_{2l-1}D_{2l}^3<D\quad(1\le l\le\rho/2)\}.\end{aligned} \tag{2.29}$$

[10, Theorem 4] gives $|\mathcal{D}^+|,|\mathcal{D}^-|<\exp(8\epsilon_s^{-3})$. This bound follows the definitions of $\mathcal{D}^+$ and $\mathcal{D}^-$. The grid is built from the relative level of each application.

Theorem 2.6 (from [10, Theorem 4]**, with** $R=1$**).** Assume (2.22), (2.24), (2.27), with the parameters above. Then there are coefficients $\phi_d^\pm$ and $\lambda_d^\pm(\mathbf{D})$ of modulus at most 1. They are independent of $\mathcal{A},X,\omega$. They satisfy

$$\begin{aligned}S(\mathcal{A},\mathcal{P},z)&\le XV(z)(F(s)+E_s)+R_1^++R^+\quad(2\le z\le D),\\ S(\mathcal{A},\mathcal{P},z)&\ge XV(z)(f(s)-E_s)-R_1^--R^-\quad(2\le z\le D^{1/2}),\end{aligned} \tag{2.30}$$

where $s=\log D/\log z$ and

$$E_s\ll\epsilon_s+\epsilon_s^{-8}e^{K+L_\omega}(\log D)^{-1/3}, \tag{2.31}$$

with an absolute implied constant. [10, Theorem 4] contains e^{K+L}. For the coefficient construction used here, the prime set and multiplicative function just specified give $K=K_{\mathrm{M}}$ and $L_\omega=2$. Hence (2.31) has the fixed factor $e^{K_{\mathrm{M}}+2}$; it is retained in the formula and may be absorbed into the implied constant. The prime-power dependence comes from its (2), namely (2.27). In its estimate (31), put $u=D^{\epsilon_s^2}$. Then

$$U=\sum_{\substack{u\le p<z\\ p\in\mathcal{P}}}\sum_{a\ge 2}\frac{\omega(p^a)}{p^a}\le L_\omega,\qquad\prod_{\substack{u\le p<z\\ p\in\mathcal{P}}}\left\{\left(1-\frac{\omega(p)}{p}\right)\sum_{a\ge 0}\frac{\omega(p^a)}{p^a}\right\}\le e^U.$$

For the first bound use $\max\{2,u\}$ in (2.27). An empty prime interval contributes zero. The second bound follows factor by factor. Thus $e^U\le e^{L_\omega}$ bounds the extra prime-power factors in the nonsquarefree terms. Formula (2.31) keeps the factor

e^{K+L_ω} of the source. The signed remainders are

$$R_1^\nu = \sum_{\substack{d<D^{\epsilon_s} \\ d\mid P(D^{\epsilon_s^2})}} \phi_d^\nu r(\mathcal{A}, d), \tag{2.32}$$

$$R^\nu = \sum_{\mathbf{D}\in\mathcal{D}^\nu(D)} \sum_{\substack{d<D^{\epsilon_s} \\ d\mid P(D^{\epsilon_s^2})}} \lambda_d^\nu(\mathbf{D}) \sum_{\substack{D_i\le p_i<D_i^{1+\delta_s} \\ p_i\mid P(z) \\ p_i\ne p_j\ \ ((i,j)\in I(\mathbf{D}))}} r(\mathcal{A}, dp_1\cdots p_\rho). \tag{2.33}$$

The sets $I(\mathbf{D})$ impose distinctness on equal-label tail positions. Unequal grid intervals are disjoint. Coefficients may be zero on unused tuples. They depend only on $\mathcal{P}, D, z, \epsilon_s, \mathbf{D}$ and the distinctness conditions. In particular they are independent of $\mathcal{A}, X, \omega$. They may change with the level or the cutoff.

[10, Theorem 4] holds for general $R \ge 1$, $0 < \epsilon_s < 3^{-R}$, with the respective ranges in (2.30). Its preceding combinatorial identities give $S \le W_R^+$ and $S \ge W_R^-$. The coefficients come from [10, (23)–(25), (26), and Lemma 4]; these formulas also show their independence of the sequence.

Relative-level grid. Choose $0 < \sigma \le 10^{-10}$ first, and put $\tau = \sigma^9$. For any level $L_c \ge 2$, substitute

$$\epsilon_s = \sigma, \qquad D = L_c, \qquad \delta_s = \epsilon_s^9 = \sigma^9 = \tau. \tag{2.33a}$$

Then, term for term,

$$\mathcal{G}(D) = \{L_c^{\sigma^2(1+\tau)^n} : n \ge 0\}, \qquad D^{\epsilon_s^2} = L_c^{\sigma^2}, \qquad D^{\epsilon_s} = L_c^{\sigma}. \tag{2.33b}$$

Thus the labels, half-open prime intervals, smooth-prime cutoff, and smooth-divisor support agree term by term with (2.32)–(2.33). These are the Iwaniec parameters at level L_c, and the tuple conditions (2.29) are unchanged. With this τ, Theorem 2.6 applies as stated. In the application to (W), Section 5 takes $L_c = D = N^\theta$, so the grid there is exactly the global grid in Theorem 1.2. The coefficients are the values supplied by Iwaniec's construction for $(\mathcal{P}, L_c, \sigma, \mathbf{D})$ at the cutoff z of the application, here $z = N^u$. The cutoff also restricts the prime variables in (2.33). The proof uses only the bound $|\cdot| \le 1$ and the support of these coefficients. It never compares coefficient values at different cutoffs.

[10, Lemma 3] gives (F, f) with error $O(e^{\sqrt{K}-s}(\log D)^{-1/3})$. In its use here the sieve constant is $K = K_{\mathrm{M}}$, fixed independently of N, D, z. Therefore $e^{\sqrt{K_{\mathrm{M}}}}$ is absorbed into the implied constant, and the error is $O(e^{-s}(\log D)^{-1/3})$ with that fixed constant. Likewise, in the profile (2.26), this multiplier changes only its absolute $O(s)$ constant. The lemma cites [11, Lemmas 18 and 20]. Theorem 4 includes the closed endpoints $z = D$ and $z = D^{1/2}$ of (2.30), although [11, Lemma 20] has strict inequalities.

2.5. **The fundamental lemma, Weil bound, and Heath–Brown identity. Fundamental lemma (cf.** [6, Lemma 4]**; the prime-support condition is explicit in** [18, Lemma 5]**).** Let $D \ge 2$, $s \ge 3$ and $z = D^{1/s}$. Let $P(z)$ be the product of all primes below z. There are $\lambda_d^\pm$, supported on $d \le D$, $d \mid P(z)$, with $\lambda_1^\pm = 1$, $|\lambda_d^\pm| \le 1$ and

$$(\lambda^- * 1)(n) = (\lambda^+ * 1)(n) = 1 \quad \text{if } (n, P(z)) = 1,$$
$$(\lambda^- * 1)(n) \le 0 \le (\lambda^+ * 1)(n) \quad \text{otherwise},$$

$$\sum_{d\le D}\frac{\lambda_d^{\pm}}{d}=\prod_{p<z}(1-1/p)\times\{1+O(e^{-s})\}. \tag{2.34}$$

The support $d \mid P(z)$ means that every supported d is squarefree and uses only primes below z. In (2.34), this makes $(\lambda^{\pm} * 1)(n) = \sum_{d|(n,P(z))} \lambda_d^{\pm}$ a sum over exactly the sieving divisors: if $(n, P(z)) = 1$, only $d = 1$ contributes, and otherwise FI78's divisor-sum signs apply to $g = (n, P(z))$. The support is also used in the remainder sum in (2.25). The explicit Bonferroni weights in (3.102)–(3.104) use the same support to sieve only primes in $\mathcal{P}$ below z and to obtain the stated divisor count. For the AA37 weights in (W), the corresponding condition $d \mid P_N(Y)$ separates the smooth divisor, whose primes are below Y, from tail primes $p_i \ge D_i \ge Y$. This is used in the unique recovery of d in the proof of the slot identity (4.13) and in Appendix B, Lemma B.1.

The error $O(e^{-s})$ suffices below. Removing the primes of an auxiliary modulus changes the local Euler product and needs its own argument. The rough-integer estimates below use explicit Brun–Bonferroni weights; see also [30]. The Iwaniec endpoint $s = 1$ follows from Theorem 2.6.

Weil input. For every $c \ge 1$ and $u, v \in \mathbb{Z}$, define

$$S(u,v;c)=\sum_{x \bmod c,(x,c)=1} e((ux+v\bar{x})/c).$$

The required bound is

$$|S(u,v;c)|\le\tau(c)(u,v,c)^{1/2}c^{1/2}. \tag{2.35}$$

This is the Weil–Estermann bound with full gcd. Its gcd factor covers nonprimitive indices; see [17, Corollary 11.12]. In (2.14), (2.35) serves for absolute convergence. Each quantitative application keeps the gcd and modulus factors in its sums. For prime moduli this is Weil's bound; prime powers use Estermann's bound, and the general modulus follows by the Chinese remainder theorem.

Lemma 2.7 (the exact Heath–Brown identity used here). Let $X \ge 2$, $K \ge 1$ be an integer, and let $z \ge 1$ satisfy $z^K > X$. Then for every integer $1 \le n \le X$,

$$\Lambda(n)=\sum_{j=1}^{K}(-1)^{j-1}\binom{K}{j}\sum_{\substack{m_1\cdots m_j n_1\cdots n_j=n\\ m_i\le z}}\mu(m_1)\cdots\mu(m_j)\log n_1. \tag{2.36}$$

The identity is [12, Lemma 1, (6)].

Equation (2.36) is algebraic. For $\Re s > 1$, set $M_z(s) = \sum_{m\le z} \mu(m)m^{-s}$, $Y(s) = \zeta(s)M_z(s)$. The coefficients of $Y - 1$ vanish at indices $1 \le n \le z$. The polynomial identity $1-(1-Y)^K=\sum_{j=1}^{K}(-1)^{j-1}\binom{K}{j}Y^j$ gives

$$-\frac{\zeta'}{\zeta}=\sum_{j=1}^{K}(-1)^{j-1}\binom{K}{j}(-\zeta')\zeta^{j-1}M_z^j+\left(-\frac{\zeta'}{\zeta}\right)(1-Y)^K. \tag{2.37}$$

Each nonzero factor from $Y - 1$ has index $> z$. Hence the last term has no coefficient of index $n \le X$. Coefficient extraction proves (2.36). For the order-three decomposition on $n \le 2X_m$, take $K = 3$, $z = 2X_m^{1/3}$. Then $z^3 = 8X_m > 2X_m$. The Möbius variables have support $m_i \le 2X_m^{1/3}$. Dyadic restrictions are imposed after

this identity. Coefficient extraction preserves both product constraints and prime ordering.

2.6. **Theorem 2.8.** Theorem 2.8 belongs to the line of incomplete-Kloosterman estimates of [5, Theorem 12] and [9, Theorem 2.1]; Drappeau states that his theorem extends DI82 Theorem 12. The closest refined quintilinear estimate is [1, Theorem 1.8, (1.8)–(1.9)]. Lichtman identifies [23, 24] as antecedents for the exceptional-spectrum and residue-dependence refinements. The statements are not identical: the hypotheses (2.38)–(2.40) and bound (2.42) are the variant proved here from (2.14) in §2.7. Section 3 uses the full bound (2.42) in (3.14)–(3.15).

Theorem 2.8. Let $C, D, N, R, S \geq 1$, $A \geq 1$, and $0 < \epsilon_0 < 1/100$. Let $b_{n,r,s} \in \mathbb{C}$ be supported on

$$0 < n \leq N, \qquad R < r \leq 2R, \qquad S < s \leq 2S. \tag{2.38}$$

Let $g : \mathbb{R}^5_{>0} \longrightarrow \mathbb{C}$ be smooth, supported in

$$(C, 2C] \times (D, 2D] \times \mathbb{R}^3_{>0}.$$

For each multi-index $\nu = (\nu_1, \dots, \nu_5) \in \mathbb{Z}^5_{\geq 0}$, assume

$$|\partial^\nu g(c, d, n, r, s)| \leq G_\nu \big(c^{-\nu_1} d^{-\nu_2} n^{-\nu_3} r^{-\nu_4} s^{-\nu_5}\big)^{1-\epsilon_0}. \tag{2.39}$$

The G_ν are uniform in the family. Let $a \in \mathbb{Z}$ satisfy $0 < |a| \leq A$, and require

$$b_{n,r,s} = 0 \quad \text{unless } (a, rs) = 1. \tag{2.40}$$

Set $\mathcal{X} = 2 + ACDNRS$. For every $\epsilon > 0$,

$$\begin{aligned} &\left| \sum_{(r,s)=1} \sum_{0<n\leq N} b_{n,r,s} \sum_{\substack{c,d\geq 1 \\ (rd,sc)=1}} g(c,d,n,r,s) e\Big(\frac{an\overline{rd}}{sc}\Big) \right| \\ &\qquad \ll_{\epsilon,\epsilon_0,\{G_\nu\}} \mathcal{X}^{\epsilon+18\epsilon_0} J_*(a), \end{aligned} \tag{2.41}$$

where the inverse is modulo sc, and

$$\begin{aligned} J_*(a)^2 = &\sum_{\substack{n'' \mid a^\infty \\ n'' \leq 2N}} A^{\theta_{\mathrm{KS}}} (n'')^{\theta_{\mathrm{KS}}} \\ &\times \Big[CS\Big(\frac{N}{n''} + RS\Big)(C + DR) + |a|NRS\Big] \|\widetilde{b}(n'')\|_2^2 \\ &+ \Big[CS(CD\sqrt{R})^{2\theta_{\mathrm{KS}}} (N + RS)^{1-\theta_{\mathrm{KS}}} \\ &\quad \times (C + DR)^{1-2\theta_{\mathrm{KS}}} + D^2 NR\Big] \|b\|_2^2. \end{aligned} \tag{2.42}$$

Here $n'' \mid a^\infty$ if each prime factor of n'' divides a. This includes $n'' = 1$. Also

$$\|b\|_2^2 = \sum_{n,r,s} |b_{n,r,s}|^2, \qquad \|\widetilde{b}(n'')\|_2^2 = \sum_{\substack{n',r,s \\ (n',a)=1}} |b_{n'n'',r,s}|^2, \tag{2.43}$$

with b extended by zero outside (2.38). When $a = \pm 1$, only $n'' = 1$ occurs in (2.42). The implied constant depends only on the displayed parameters and the seminorm bounds G_ν. It is uniform in (a, A, C, D, N, R, S, b).

For $(r,s)=1$, the condition $(rd,sc)=1$ means $(c,r)=1$ and $(d,sc)=1$. Pairs with $(c,s)>1$ are included, as are both signs of a. The sequences and weights may be complex.

2.7. **Proof of Theorem 2.8.** The proof uses the large-sieve bounds above, the Hecke bounds of Kim–Sarnak and Deligne, and the trace formula (2.14). We derive the opposite-sign trace formula below.

In this subsection C, D, N_T, R_T, S_T, A are the parameters of Theorem 2.8. Write R, S, N_1 for R_T, S_T, N_T; Section 3 uses R, S for other scales. Put

$$\begin{aligned} \mathcal{X} = 2 + ACDN_1RS, \quad A_a = |a_T|, \quad q = rs, \\ (r,s)=1, \quad \theta_{\mathrm{KS}} = 7/32, \quad \vartheta_j = 2i\kappa_j \in (0, \theta_{\mathrm{KS}}] \end{aligned} \tag{2.7.T1}$$

on an exceptional eigenspace. All norms below use counting measure. Spectral bases are Petersson-orthonormal.

2.7.1. *Completion and separation.* Fix n, r, s, c, put $k = sc$, and define $\widehat{g}_c(\xi) = \int g(c,t,n,r,s)e(-\xi t)\,dt$. Group d in reduced classes modulo k and apply Poisson summation. This gives the exact identity

$$\sum_{(d,k)=1} g(c,d,n,r,s)e(a_T n\overline{rd}/k) = k^{-1}\sum_{h\in\mathbb{Z}} \widehat{g}_c(h/k)S(h\bar{r}, a_T n; k). \tag{2.7.T2}$$

The zero frequency is bounded using $|c_k(a_T n)| \le (a_T n, s)(a_T n, c)$. Since $\sum_{c\le 2C}(u,c) \le 2C\tau(u)$ and $\sum_{s\le 2S}(u,s)^2 \le 4S^2\tau(u)$, Cauchy–Schwarz gives its absolute contribution

$$\ll_\epsilon \mathcal{X}^\epsilon D(N_1R)^{1/2}\|b\|_2. \tag{2.7.T3}$$

This is the fourth term of (2.42).

Choose a small auxiliary exponent $\epsilon_1 > 0$. Set

$$H = 4SCD^{\epsilon_0-1}\mathcal{X}^{\epsilon_1}, \qquad \epsilon_2 = \epsilon_1 + \epsilon_0. \tag{2.7.T4}$$

Integrate by parts in d. Use (2.39) in d on $d \in (D, 2D]$. For every $\xi \neq 0$ and $j \ge 0$, this gives $|\widehat{g}_c(\xi)| \ll_j D(|\xi|D^{1-\epsilon_0})^{-j}$. Every $h \neq 0$ has $|h|/k \ge |h|/(4SC)$, because $k = sc \le 4SC$. By (2.7.T4) this gives $(|h|/k)D^{1-\epsilon_0} \ge (|h|/H)\mathcal{X}^{\epsilon_1}$. Hence, for every h with $|h| > H$,

$$|\widehat{g}_c(h/k)| \ll_j D\,\mathcal{X}^{-j\epsilon_1}(H/|h|)^j. \tag{2.7.T5}$$

For $j \ge 2$ and every $H > 0$, $\sum_{|h|>H}(H/|h|)^j \le 2 + 2H$. If $H < 1$, the sum runs over all $h \neq 0$ and is at most $2\zeta(2)H^2 \le 2 + 2H$. If $H \ge 1$ it is at most $2(1 + H/(j-1))$. With $|S(h\bar{r}, a_T n; k)| \le k$, the tail $|h| > H$ of (2.7.T2) is $\ll_j (1+H)D\mathcal{X}^{-j\epsilon_1}$ for each (c,n,r,s). Summing against $|b_{n,r,s}|$ over $c \le 2C$ gives $\ll_j \mathcal{X}^{7-j\epsilon_1}\|b\|_2$, using $\sum|b| \le (N_1RS)^{1/2}\|b\|_2$, $C, D, N_1, R, S \le \mathcal{X}$ and $1 + H \le 5\mathcal{X}^{1+\epsilon_1}$. Taking $j > 20/\epsilon_1$ makes it smaller than (2.7.T3). If $H < 1$, the retained range $1 \le |h| \le \lfloor H \rfloor$ is empty. Every nonzero frequency lies in this tail, including $h = \pm 1$. Thus (2.7.T2) contains only the zero frequency and this tail. Equation (2.7.T3) completes this case.

For $1 \le |h| \le \lfloor H \rfloor$, write $h = \varsigma_h m$ and $\varsigma = \varsigma_h \operatorname{sgn}(a_T)$, with signs $\varsigma_h, \varsigma \in \{\pm 1\}$. The identity $S(-u, v; k) = S(u, -v; k)$ makes the second Kloosterman argument $\varsigma A_a n$. Partition the integers m into

$$\mathcal{I}_j = (M_j, \min(2M_j, \lfloor H \rfloor)] \cap \mathbb{Z}, \qquad M_j = 2^j/2. \tag{2.7.T6}$$

Thus the first nonempty block is exactly $\{1\}$, with $M = 1/2$. Partition n into $(N', 2N']$, $N' \ge 1/2$. Shortened final blocks remain integer intervals. There are

$O(\log^2 \mathcal{X})$ pairs of blocks; each restricted coefficient norm is bounded by the original norm. For the current block define

$$b^\flat(n'')_{n',r,s} = b_{n'n'',r,s}\, 1_{\{N'<n'n''\le 2N'\}}\, 1_{\{(n',a_T)=1\}}.$$

The slice satisfies $\|b^\flat(n'')\|_2 \le \|\widetilde{b}(n'')\|_2$. Restrict b to the current block. By support, $b^\flat(n'') = \widetilde{b}(n'')$ term by term. Thus the block estimate is exactly (2.42)–(2.43).

For one block write

$$\frac{\widehat{g}_c(\varsigma_h m/(sc))}{sc} = \frac{D}{SC} W_0(c,m,n,r,s), \qquad \gamma = cs\sqrt{r}, \qquad W_1 = W_0\gamma/(CS\sqrt{R}). \tag{2.7.T7}$$

Differentiate under the integral and use (2.7.T4). Then

$$|\partial^\nu W_1| \ll_\nu \mathcal{X}^{|\nu|\epsilon_2} c^{-\nu_1} m^{-\nu_2} n^{-\nu_3} r^{-\nu_4} s^{-\nu_5}. \tag{2.7.T8}$$

The block sum is $D\sqrt{R}$ times a sum with weight $\gamma^{-1}W_1 S(m\bar{r}, \varsigma A_a n; sc)$. Put

$$Z = \frac{4\pi\sqrt{A_a MN'}}{CS\sqrt{R}}, \qquad x = \frac{4\pi\sqrt{A_a mn}}{cs\sqrt{r}}. \tag{2.7.T9}$$

Invert $c = 4\pi\sqrt{A_a mn}/(xs\sqrt{r})$ and multiply by fixed smooth cutoffs in m, n, r, s, each equal to 1 on that variable's range $(y, 2y]$ $(y = M, N', R, S)$ and supported in $[y/2, 3y]$. Since $x/Z = \sqrt{(m/M)(n/N')(R/r)}\,(C/c)(S/s)$ and $C < c \le 2C$, the resulting smooth function $F(x,m,n,r,s)$ is supported on $1/(12\sqrt{3}) \le x/Z \le 6\sqrt{2}$, hence on $Z/21 \le x \le 9Z$. Mellin inversion in its four log coordinates gives

$$W_1 = (2\pi)^{-4} \int_{\mathbb{R}^4} \phi_{\mathbf{t}}(x) m^{it_1} n^{it_2} r^{it_3} s^{it_4}\, d\mathbf{t}, \tag{2.7.T10}$$

$$\max_{j\le 5} \sup_x |x^j \phi_{\mathbf{t}}^{(j)}(x)| \ll \mathcal{X}^{17\epsilon_2} \prod_{\nu=1}^{4} (1+|t_\nu|)^{-3} =: \mathfrak{B}(\mathbf{t}). \tag{2.7.T11}$$

For (2.7.T11), integrate three times by parts in each Mellin variable. There are four variables. Allow five x-derivatives. This uses at most 17 derivatives in (2.7.T8). The chain rule uses $\partial_x c = -c/x, \partial_m c = c/(2m), \partial_n c = c/(2n), \partial_r c = -c/(2r), \partial_s c = -c/s$. Thus (2.7.T10) and all ensuing integral interchanges are absolutely convergent. For fixed $\mathbf{t}$ set $\alpha_m = m^{it_1} 1_{\mathcal{I}_j}(m)$ and $\beta_{n,r,s} = b_{n,r,s} n^{it_2} r^{it_3} s^{it_4}$. Estimate the following sum.

$$\mathcal{L}_{\mathbf{t}} = \sum_{r,s,m,n} \alpha_m \beta_{n,r,s} \sum_{(c,r)=1} \gamma^{-1} \phi_{\mathbf{t}}(4\pi\sqrt{mA_a n}/\gamma) S(m\bar{r}, \varsigma A_a n; sc). \tag{2.7.T12}$$

2.7.2. *The cusp pair, both signs, and transform bounds.* Use $\sigma_\infty = I$ and

$$\sigma_{1/s} = \begin{pmatrix} \sqrt{r} & 0 \\ s\sqrt{r} & 1/\sqrt{r} \end{pmatrix}. \tag{2.7.T13}$$

Multiply $\begin{pmatrix} a & b \\ rsc_0 & d \end{pmatrix}$ by (2.7.T13). With $c = rc_0 + d$, the positive cusp moduli are $\gamma = cs\sqrt{r}$, $(c,r) = 1$, as in (2.13). Conversely choose $d \equiv c\ (r)$, $(d, sc) = 1$, and $c_0 = (c-d)/r$; this supplies every allowed modulus. Double cosets correspond to

the reduced residues modulo sc. Insert their top and bottom entries in the cusp Kloosterman definition. This gives

$$S_{\infty,1/s}(m,\nu;\gamma) = e(\nu\bar{s}/r)S(m\bar{r},\nu;sc), \qquad \nu \neq 0. \tag{2.7.T14}$$

The numerator modulo r is νc. Modulo sc it is $m\bar{r}\,u+\nu\bar{u}$, with $u = r\bar{d}$. This proves (2.7.T14) for either sign and for complex Fourier coefficients.

We use the trace formula in DI's coefficient normalization, with the transforms $\widehat{\phi}, \check{\phi}, \widetilde{\phi}$ of (2.12). Let $m, \nu > 0$ and $r > R \geq 1$. Use the same-sign Drappeau formula of Section 2. Apply the exact dictionary below.

$$\rho^{\mathrm{Dr}}_{u,\mathfrak{a}}(n) = \frac{\rho^{\mathrm{DI}}_{u,\mathfrak{a}}(n)}{2\sqrt{|n|}}, \quad \rho^{\mathrm{Dr}}_{\mathfrak{c}\mathfrak{a}}(n,t) = \frac{\pi^{1/2+it}|n|^{it-1/2}}{\Gamma(1/2+it)}\varphi_{\mathfrak{c}\mathfrak{a}n}(1/2+it). \tag{2.7.T15}$$

The holomorphic dictionary is $\rho^{\mathrm{Dr}}_{f,\mathfrak{a}}(n) = (4\pi n)^{-\ell/2}\psi_f(\mathfrak{a},n)$. For signs $+,-$, the Maass weights are as follows.

$$h_+(\kappa) = -\widehat{\phi}(\kappa)/\cosh\pi\kappa, \quad h_-(\kappa) = \frac{\pi}{2}\check{\phi}(\kappa)/\cosh\pi\kappa; \tag{2.7.T16}$$

For measure dt, the continuous weights are as follows.

$$e_+(t) = -\widehat{\phi}(t)/\pi, \qquad e_-(t) = \check{\phi}(t)/2. \tag{2.7.T17}$$

Only sign + has a holomorphic coefficient. It is

$$\frac{i^\ell(\ell-1)!}{\pi(4\pi)^{\ell-1}}\,\widetilde{\phi}(\ell-1). \tag{2.7.T18}$$

There is no $1/\cosh(\pi t)$ in (2.7.T17).

For the opposite sign the normalization follows from the following derivation. Take a width-one scaling matrix $\sigma_{\mathfrak{a}}$. Let $\Gamma_{\mathfrak{a}}$ stabilize $\mathfrak{a}$ in $\Gamma = \Gamma_0(q)$, including $-I$. Then $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \{\pm T^k : k \in \mathbb{Z}\}$. Define, for $m > 0$,

$$U_{\mathfrak{a}m}(z,s) = \sum_{\delta\in\Gamma_{\mathfrak{a}}\backslash\Gamma} (\Im\sigma_{\mathfrak{a}}^{-1}\delta z)^s e(m\sigma_{\mathfrak{a}}^{-1}\delta z), \qquad \langle F,G\rangle = \int_{\Gamma\backslash\mathbb{H}} F(z)\overline{G(z)}\,\frac{dx\,dy}{y^2}.$$

Each coset enters once, with weight one. The series and unfolding converge absolutely for $\Re s > 1$. Unfold the first series and decompose the second into cusp double cosets of positive lower-left entry γ. Translation in the real variable leaves the Kloosterman factor $S_{\mathfrak{a}\mathfrak{b}}(m,-\nu;\gamma)$, with the cusp convention of (2.7.T14). Explicitly, a matrix $\sigma_{\mathfrak{b}}^{-1}\delta\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} a & b \\ \gamma & d\end{smallmatrix}\right)$ gives phase $e((\nu a - md)/\gamma)$; its negative inverse has entries $(-d,b;\gamma,-a)$, giving the same phase in $S_{\mathfrak{a}\mathfrak{b}}(m,-\nu;\gamma)$. The remaining integral is

$$I_\gamma = \gamma^{-2s_2}\int_{\mathbb{H}} y^{s_1+s_2-2}|z|^{-2s_2}\exp\bigl(2\pi i\{mz-\nu/(\gamma^2 z)\}\bigr)\,dx\,dy.$$

The zero-lower-left coset, if present for equivalent cusps, vanishes by the integral of $e((m+\nu)x)$ over $[0,1]$. Put $s_- = s_1 - s_2$, $s_+ = s_1 + s_2$, $z = i\varrho e^{i\alpha}$, $-\pi/2 < \alpha < \pi/2$, and $\varrho = (\sqrt{\nu/m}/\gamma)e^u$. Then

$$I_\gamma = \gamma^{-s_+}(\nu/m)^{s_-/2}\int_{-\pi/2}^{\pi/2}\cos^{s_+-2}\alpha\int_{\mathbb{R}} e^{s_- u}e^{-(4\pi\sqrt{m\nu}/\gamma)\cosh(u+i\alpha)}\,du\,d\alpha.$$

For each interior α, shift $u + i\alpha$ to the real line. The vertical sides vanish exponentially, and the inner integral is $2e^{-is_-\alpha}K_{s_-}(4\pi\sqrt{m\nu}/\gamma)$. Let $\Re s_i > 1$. At either angular endpoint the radial growth is of order $\cos^{-|\Re s_-|}\alpha$. For $\Re s_- = 0$, a logarithm

may also occur. The remaining exponent is $\Re s_+ - 2 - |\Re s_-| = 2\min \Re s_i - 2 > 0$. Hence the original absolute integral is finite. Hence Fubini and the endpoint limits are legitimate. Applying (2.7.T20) gives the following identity for $\Re s_1, \Re s_2 > 1$:

$$\begin{aligned}\langle U_{\mathfrak{a}m}(s_1), \overline{U_{\mathfrak{b}\nu}(s_2)}\rangle &= \pi 2^{3-s_1-s_2}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1)\Gamma(s_2)}(\nu/m)^{(s_1-s_2)/2} \\ &\quad\times \sum_{\gamma}\gamma^{-s_1-s_2}S_{\mathfrak{a}\mathfrak{b}}(m,-\nu;\gamma)K_{s_1-s_2}(4\pi\sqrt{m\nu}/\gamma).\end{aligned} \tag{2.7.T19}$$

The polar substitution and contour shift above give the factor $2K_{s_1-s_2}$. The remaining angular integral is

$$\int_{-\pi/2}^{\pi/2}\cos^{s_1+s_2-2}\alpha\, e^{-i(s_1-s_2)\alpha}\,d\alpha = \pi 2^{2-s_1-s_2}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1)\Gamma(s_2)}. \tag{2.7.T20}$$

For an independent normalization, unfold once and use (2.2). This gives

$$\langle U_{\mathfrak{a}m}(s), u_j\rangle = \overline{\rho_{j\mathfrak{a}}(m)}\frac{\sqrt{\pi}(4\pi m)^{1/2-s}}{\Gamma(s)}\Gamma(s-\tfrac{1}{2}-i\kappa_j)\Gamma(s-\tfrac{1}{2}+i\kappa_j).$$

This is the integral of $y^{s-3/2}e^{-2\pi my}K_{i\kappa_j}(2\pi my)$. Use the standard integral representation of K and a beta integral. The second factor $\langle u_j, \overline{U_{\mathfrak{b}\nu}(s_2)}\rangle$ is the same expression with $\rho_{j\mathfrak{b}}(-\nu)$, ν, s_2 in place of $\overline{\rho_{j\mathfrak{a}}(m)}$, m, s. There is no conjugation of s_2 in this bilinear pairing. Multiplication gives exactly (2.7.T21). Applying the same unfolding to the Eisenstein expansion (2.4), with its standard Parseval measure, gives (2.7.T22). Thus Parseval gives the spectral side with

$$\begin{aligned}P &= \frac{\pi m^{1/2-s_1}\nu^{1/2-s_2}(4\pi)^{1-s_1-s_2}}{\Gamma(s_1)\Gamma(s_2)}, \\ \Lambda(s_1,s_2;t) &= \prod_{j=1}^{2}\Gamma(s_j-\tfrac{1}{2}-it)\Gamma(s_j-\tfrac{1}{2}+it).\end{aligned} \tag{2.7.T21}$$

Its Maass summand is $P\overline{\rho_{j\mathfrak{a}}(m)}\rho_{j\mathfrak{b}}(-\nu)\Lambda(s_1,s_2;\kappa_j)$. Its continuous integral is

$$\begin{aligned}P\sum_{\mathfrak{c}}\int (\nu/m)^{it}&\overline{\varphi_{\mathfrak{c}\mathfrak{a}m}(1/2+it)}\varphi_{\mathfrak{c}\mathfrak{b},-\nu}(1/2+it) \\ &\times \Lambda(s_1,s_2;t)|\Gamma(1/2+it)|^{-2}\,dt.\end{aligned} \tag{2.7.T22}$$

The factor at the second cusp has index $-\nu$. It arises because the second Eisenstein parameter in Parseval is conjugated: $\overline{E(z,1/2+it)} = E(z,1/2-it)$. So the continuous product pairs a conjugated coefficient at $1/2+it$ with an unconjugated one at $1/2+it$.

For the unscaled Eisenstein series the continuous Parseval measure is $dt/(4\pi)$. Its coefficient of $\sqrt{y}K_{it}(2\pi|n|y)e(nx)$ is

$$b_{(\mathfrak{c},\mathfrak{a})}(n,t) = 2\pi^{1/2+it}|n|^{it}\ \frac{\varphi_{\mathfrak{c},\mathfrak{a},n}(1/2+it)}{\Gamma(1/2+it)}.$$

Hence $(4\pi)^{-1}\overline{b_{(\mathfrak{c},\mathfrak{a})}(m,t)}b_{(\mathfrak{c},\mathfrak{b})}(-\nu,t)$ equals

$$(\nu/m)^{it}\overline{\varphi_{\mathfrak{c},\mathfrak{a},m}(1/2+it)}\varphi_{\mathfrak{c},\mathfrak{b},-\nu}(1/2+it)|\Gamma(1/2+it)|^{-2}.$$

The coefficient product contributes 4π, canceling the Parseval denominator. Thus (2.7.T22) has plain dt and no hyperbolic-cosine denominator. The second φ is unconjugated at the same t.

Write $\sigma_i = \Re s_i > 1$, $\delta = |\sigma_1 - \sigma_2|$, $\sigma_* = \min(\sigma_1, \sigma_2)$, and $x = 4\pi\sqrt{m\nu}/\gamma$, with fixed $m, \nu \geq 1$. Replace both Poincaré summands by their moduli. For one double coset with positive lower-left entry, the absolute integral is

$$\ll_{\sigma_1,\sigma_2,m,\nu} \gamma^{-\sigma_1-\sigma_2}\big(1 + \gamma^{\delta} + \log(2+\gamma)\big). \tag{2.7.T23}$$

The radial integral is a fixed positive factor times $K_\delta(x\cos\alpha)$. For $\delta > 0$, its small-argument bound is $O_\delta((x\cos\alpha)^{-\delta})$; for $\delta = 0$, it is $O(1 + |\log(x\cos\alpha)|)$. The angular weight is $\cos^{\sigma_1+\sigma_2-2}\alpha$. Thus the power endpoint has exponent $2\sigma_* - 2 > 0$, while the logarithmic endpoint is integrable and contributes $O(1+\log(2+\gamma))$ after angular integration. The implied constant depends on the fixed spectral parameters and on m, ν, and is uniform in γ.

Consequently the controlling absolute-sum majorant is

$$\sum_{\gamma} N_{\mathfrak{ab}}(\gamma)\gamma^{-2\sigma_*}\big(1 + \log(2+\gamma)\big) < \infty. \tag{2.7.T24}$$

For the cusp pair $(\infty, 1/s)$, (2.7.T14) gives $\gamma = cs\sqrt{r}$, $(c, r) = 1$, and $N(\gamma) = \varphi(sc) \leq sc$. The sum is bounded by a constant times $\sum_{c\geq 1} c^{1-2\sigma_*}(1+\log(2+c))$, which converges because $\sigma_* > 1$. For a general fixed cusp pair, the cited cumulative count $\sum_{\gamma\leq T} N_{\mathfrak{ab}}(\gamma) \ll_{\mathfrak{a},\mathfrak{b}} T^2$ and dyadic decomposition bound the tail by $\sum_{k\geq 0}(1+k)2^{-2(\sigma_*-1)k} < \infty$. Any bounded initial range is finite.

The logarithm in (2.7.T23) is attained: for $\sigma_1 = \sigma_2 = 2$, $m = \nu = 1$ and the admissible moduli $\gamma = \sqrt{2}c$, c odd, the absolute cell integral is $\pi\gamma^{-4}\log\gamma + O(\gamma^{-4})$. The summable majorant (2.7.T24) supplies the uniform step beyond the fixed-γ angular endpoint estimate.

Both sides continue to the connected domain $\Re s_i > 3/4$, $|\Re(s_1 - s_2)| < 1/2$. For the geometric side, Weil's bound and $K_\mu(x) \ll x^{-|\Re\mu|}(1 + |\log x|)$ give locally uniform absolute convergence. For the spectral side, apply Stirling and DI's positive large sieves. These give locally uniform absolute convergence. Since $i\kappa_j \leq 7/64$, the exceptional gamma factors have no poles here. Equality therefore holds at $s_1 = 1 + iu, s_2 = 1 - iu$. With

$$\mathcal{H}(t,u) = \frac{\cosh\pi t}{\cosh\pi(t-u)\cosh\pi(t+u)}, \tag{2.7.T25}$$

this gives

$$\begin{aligned}\sum_{\gamma} \frac{4\pi\sqrt{m\nu}}{\gamma^2} S(m,-\nu;\gamma) K_{2iu}(4\pi\sqrt{m\nu}/\gamma) \\ = \frac{\pi^2}{2}\sum_j \frac{\bar\rho_j(m)\rho_j(-\nu)\mathcal{H}(\kappa_j,u)}{\cosh\pi\kappa_j} + \frac{\pi}{2}\sum_{\mathfrak{c}}\int E_{\mathfrak{c}}(t)\mathcal{H}(t,u)\,dt,\end{aligned} \tag{2.7.T26}$$

where

$$E_{\mathfrak{c}}(t) := (\nu/m)^{it}\overline{\varphi_{\mathfrak{ca}m}(1/2+it)}\varphi_{\mathfrak{cb},-\nu}(1/2+it).$$

The factor $|\Gamma(1/2+it)|^{-2}$ of (2.7.T22) is part of the spectral kernel; $E_{\mathfrak{c}}$ excludes it. The constant eigenfunction contributes 0 to the sum over j: its Fourier coefficients vanish at the nonzero frequencies m and $-\nu$.

To integrate (2.7.T26) against a test function, set

$$\mathcal{R}(u)=\int_0^\infty K_{2iu}(x)\phi(x)x^{-2}\,dx,\qquad k(u)=\frac{4}{\pi^2}u\sinh(2\pi u)\mathcal{R}(u). \tag{2.7.T27}$$

Kontorovich–Lebedev inversion gives $\int_{\mathbb{R}} k(u)K_{2iu}(x)\,du = \phi(x)/x$; see [29, §10.43(v), (10.43.30)–(10.43.31)]. Use the following kernel identity.

$$\int_{\mathbb{R}} u\sinh(2\pi u)\mathcal{H}(t,u)K_{2iu}(x)\,du=\cosh(\pi t)xK_{2it}(x). \tag{2.7.T28}$$

For $|\Re\mu|+|\Re\nu|<1$, the absolutely convergent product integral is

$$\int_0^\infty K_\mu(x)K_\nu(x)\,dx=\frac{\pi^2}{4\cos(\pi(\mu+\nu)/2)\cos(\pi(\mu-\nu)/2)}. \tag{2.7.T29}$$

Use the Bessel product representation $K_\mu K_\nu(x)=2\int_0^\infty K_{\mu+\nu}(2x\cosh v)\cosh((\mu-\nu)v)\,dv$. Integrate in x. For $|\Re b|<1$, use $\int_0^\infty \cosh(bv)/\cosh v\,dv=(\pi/2)\sec(\pi b/2)$. The latter follows by expanding $1/\cosh v$ and the partial fractions of secant. In the subsequent KL application, $(\mu,\nu)=(2it,2iu)$ with $u\in\mathbb{R}$; here $t\in\mathbb{R}$ on the regular and continuous spectrum, while an exceptional parameter satisfies $|\Im t|\le 7/64$. Thus $|\Re\mu|+|\Re\nu|\le 7/32<1$ and $|\Re(\mu-\nu)|<1$. Apply KL inversion to $xK_{2it}(x)$, whose KL transform is, by (2.7.T29), $(\pi^2/4)\mathcal{H}(t,u)/\cosh\pi t$. This proves (2.7.T28).

All interchanges here are justified as follows. Split the integral representation of K_{iv} at $\sinh U=\max(1,1/x)$. Integrate by parts on its tail. This gives

$$|K_{iv}(x)|\ll e^{-\pi|v|/2}(1+|v|+\log(2+1/x)). \tag{2.7.T30}$$

Apply the self-adjoint operator $(x\partial_x)^2-x^2$ repeatedly to $\phi(x)/x$. Then $\mathcal{R}(u)\ll_{\phi,j} e^{-\pi|u|}(1+|u|)^{1-2j}$. On real t, $|\mathcal{H}(t,u)|\le 4\cosh(\pi t)e^{-2\pi\max\{|t|,|u|\}}$; exceptional parameters give an exponentially decaying function of u as well. DI's positive quadratic estimates imply convergence after the remaining $(1+|t|)^{-4}$ bound. The geometric side is absolutely summable by Weil. These bounds justify Fubini on both sides of (2.7.T26). Equations (2.7.T27)–(2.7.T28) now give exactly (2.7.T16)–(2.7.T17) for sign $-$. For sign $+$, let $r>R\ge 1$. Insert (2.7.T15) in the classical same-sign formula. This gives the minus signs in (2.7.T16)–(2.7.T17) and the coefficient (2.7.T18). These are the normalizations of Section 2, (2.14).

Let ϕ be supported in $[Z/21,9Z]$. With $\mathfrak{B}=\max_{j\le 5}\sup x^j|\phi^{(j)}(x)|$, the transform bounds are

$$|\widehat{\phi}(t)|+|\check{\phi}(t)|\ll\mathfrak{B}(1+|\log Z|)(1+Z)^{-1}\min\{1,((1+Z)/|t|)^3\}\quad(t\in\mathbb{R}), \tag{2.7.T31}$$

$$|\widehat{\phi}(i\nu)|+|\check{\phi}(i\nu)|\ll\mathfrak{B}(1+|\log Z|)(1+Z^{-2\nu})/(1+Z),\quad 0<\nu\le 1/4, \tag{2.7.T32}$$

and the same bound as (2.7.T31) for $\widetilde{\phi}(\ell)$ with $|t|$ replaced by ℓ. To prove them, put $\Phi(y)=\int\cos(xy)\phi(x)\,dx/x$ and $\Phi_c(y)=\int e^{-ixy}\phi(x)\,dx/x$. Five integrations by parts give $|\Phi^{(j)}(y)|+|\Phi_c^{(j)}(y)|\ll\mathfrak{B}Z^j\min(1,(Z|y|)^{-5})$, $j\le 3$. The exact Bessel representations are

$$\begin{aligned}\check{\phi}(t)&=\frac{4}{\pi}\int_0^\infty\cos(2tu)\Phi(\sinh u)\,du,\\ \widehat{\phi}(t)&=-2\int_0^\infty\cos(2tu)\Phi(\cosh u)\,du,\\ \widetilde{\phi}(\ell)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i\ell u}\Phi_c(\sin u)\,du.\end{aligned}$$

Their absolute bounds follow by splitting at $\sinh u = 1/Z$; three integrations by parts give the cubic decay. The periodic holomorphic integral has no boundary terms. Replacing cosine by $\cosh(2\nu u)$ and splitting at the same point gives (2.7.T32). Truncate the oscillatory representations first. For real t, the tails are $O((1+|t|)/(x\sinh U))$. For $t = i\nu$, they are $O((1+\nu)e^{2\nu U}/(x\sinh U))$. Since $2\nu \le 1/2$, these tails tend to zero and justify the interchanges.

2.7.3. *Cusp transport and all Hecke factors.* Choose integers w, y with $rw - sy = 1$, and let

$$W = r^{-1/2}\begin{pmatrix} r & y \\ rs & rw \end{pmatrix}. \tag{2.7.T33}$$

Then $\sigma_{1/s}^{-1} W = \begin{pmatrix} 1 & y/r \\ 0 & 1 \end{pmatrix}$, with $y/r \equiv -\bar{s}/r(1)$. Also $W\Gamma_0(rs)W^{-1} = \Gamma_0(rs)$. Here is an integer computation of the latter fact. For $A = r^{-1/2}\begin{pmatrix} r\alpha & \beta \\ rs\gamma & r\delta \end{pmatrix}$, $r\alpha\delta - s\beta\gamma = 1$, conjugating $\begin{pmatrix} a & b \\ rsc & d \end{pmatrix}$ gives entries

$$\begin{pmatrix} a\alpha\delta r - \alpha b\gamma rs + \beta c\delta rs - \beta d\gamma s & -a\alpha\beta + \alpha^2 br + \alpha\beta d - \beta^2 cs \\ rs(a\delta\gamma - b\gamma^2 s + c\delta^2 r - d\delta\gamma) & -a\beta\gamma s + \alpha b\gamma rs + \alpha d\delta r - \beta c\delta rs \end{pmatrix}. \tag{2.7.T34}$$

Both $W = A(1, y, 1, w)$ and $W^{-1} = A(w, -y, -1, 1)$ have this form, proving both inclusions. Thus composition by W is unitary, commutes with the Laplacian, and preserves each eigenspace. The weight-ℓ slash action has the same properties.

Comparison of Fourier series after the translation y/r gives

$$\rho_{F,1/s}(\nu) = e(\nu\bar{s}/r)c_{F\circ W}(\nu), \quad \psi_F(1/s,\nu) = e(\nu\bar{s}/r)\psi_{F|_\ell W}(\infty,\nu). \tag{2.7.T35}$$

For Eisenstein series, substitute $W\gamma W^{-1}$ in the defining coset series for $\Re z > 1$. It gives $E_{\mathfrak{c}}(Wz,u) = E_{W^{-1}\mathfrak{c}}(z,u)$, using $W^{-1}\sigma_{\mathfrak{c}}$ as a width-one scaling matrix. Changing the source-cusp representative or translating its scaling matrix does not change the Fourier coefficient at the fixed observation cusp ∞. Meromorphic continuation therefore gives

$$\varphi_{\mathfrak{c},1/s,\nu}(u) = e(\nu\bar{s}/r)\varphi_{W^{-1}\mathfrak{c},\infty,\nu}(u). \tag{2.7.T36}$$

In (2.7.T12), (2.7.T14) contributes $e(-\varsigma A_a n\bar{s}/r)$. Equations (2.7.T35)–(2.7.T36) cancel this phase exactly for both signs, in every spectrum.

For $(k,q) = 1$, define the normalized Hecke operator below.

$$T_k F(z) = k^{-1/2}\sum_{ad=k}\sum_{b \bmod d} F((az+b)/d). \tag{2.7.T37}$$

Sum each Fourier term, of index ν', over b. This forces $d \mid \nu'$, and the term moves to index $\nu = a\nu'/d$. The coefficient at $\nu \neq 0$ is

$$c_{T_k F}(\nu) = \sum_{a|(k,\nu)} c_F(k\nu/a^2). \tag{2.7.T38}$$

The prefactor is 1 because, in the normalization (2.2) of [5, (1.15)], the factor $\sqrt{y}$ contributes $\sqrt{a/d}$, and $k^{-1/2}d\sqrt{a/d} = 1$. For weight ℓ, use $T_k = k^{(\ell-1)/2}\sum_{ad=k} d^{-\ell}\sum_b(\cdot)$; the normalized coefficients $a_F(\nu) = \nu^{-(\ell-1)/2}\psi_F(\infty,\nu)$ satisfy the same identity at coprime (k,ν).

Split $n = n'n''$, $n'' \mid a_T^\infty$, $(n', a_T) = 1$, and put $k = A_a n''$. Then $(k, q) = (k, n') = 1$. The number of pieces is

$$\mathcal{D} \le (2N_1)^\epsilon \prod_{p \mid a_T} (1 - p^{-\epsilon})^{-1} \ll_\epsilon \mathcal{X}^{2\epsilon}, \quad \tau(k)^2 \ll_\epsilon \mathcal{X}^\epsilon. \tag{2.7.T39}$$

The product bound is uniform in a_T. Choose p_ϵ so that $(1 - p^{-\epsilon})^{-1} \le p^\epsilon$ for $p > p_\epsilon$. The smaller primes contribute a constant depending only on ϵ. The remaining product is at most $|a_T|^\epsilon$. On a Maass eigenspace V, the transported n''-functional is $Y \circ T_k \circ W$, where $Y(v) = \sum_{n'} \beta_{n'n''} c_v(\varsigma n')$. By Riesz representation and unitarity,

$$\sum_{u_j \in V} |\ell_{2,n''}(u_j)|^2 \le \tau(k)^2 k^{\theta_{\mathrm{KS}}} \sum_{u_j \in V} |Y(u_j)|^2. \tag{2.7.T40}$$

The self-adjoint unramified T_k diagonalizes on the full level space. By Kim–Sarnak its norm is at most $\tau(k)k^{\theta_{\mathrm{KS}}/2}$. For holomorphic forms replace $k^{\theta_{\mathrm{KS}}}$ by 1, by Deligne. For negative index, $c_v(-n') = \overline{c_{\bar v}(n')}$. Conjugation permutes an orthonormal basis. Hence the positive-index quadratic bound holds with conjugated coefficients. The argument allows complex coefficients and any orthonormal basis of each eigenspace.

The continuous spectrum requires its own proof. Represent a cusp by u/w, $w \mid q$, put $w' = (w, q/w)$. The orbit conditions in [5, Lemma 3.6, (3.9)–(3.11)] $(c_1, d_1) = 1, (c_1, q) = w, c_1 d_1 \equiv uw \pmod{ww'}$, with $c_1 = w\gamma$, give exactly

$$\begin{aligned} \varphi_{u/w,\infty,n}(z) = (w'/(wq))^z \sum_{\gamma \ge 1, (\gamma, q/w) = 1} \gamma^{-2z} \\ \sum_{\substack{d_1 \bmod \gamma w \\ (d_1, \gamma w) = 1,\ \gamma d_1 \equiv u (w')}} e(-nd_1/(\gamma w)). \end{aligned} \tag{2.7.T41}$$

For a character $\chi \bmod w'$, let $\mathcal{V}_\chi(n; z) = \sum^*_{u \bmod w'} \chi(u) \varphi_{u/w,\infty,n}(z)$. For $(k, q) = (k, n') = 1$, either sign of n', the identity is

$$\mathcal{V}_\chi(kn'; z) = \bar\chi(k) \left(\sum_{d \mid k} \chi^2(d) d^{1-2z} \right) \mathcal{V}_\chi(n'; z). \tag{2.7.T42}$$

To prove it, sum (2.7.T41) over u. Write $c_1 = c_1' c_1''$, where c_1'' is supported on primes of q and $(c_1', q) = 1$. CRT factors the inner sum as $\chi(c_1') G_\chi(-n; c_1'') c_{c_1'}(n)$. The outer character supplies another $\chi(c_1')$. Replacing the Gauss-sum variable by $\bar k d_1$ gives $\bar\chi(k)$ on c_1''. The other factor is

$$L^{(q)}(2z, \chi^2)^{-1} \sum_{d \mid\mid n, (d, q) = 1} \chi^2(d) d^{1-2z},$$

by $c_{c_1'}(n) = \sum_{d \mid (c_1', n)} d\mu(c_1'/d)$; this divisor sum factors at kn'. All manipulations converge absolutely for $\Re z > 1$. Both sides are meromorphic Fourier coefficients, so (2.7.T42) holds meromorphically, hence at $z = 1/2 + it$ outside a discrete set.

There are finitely many cusp characters. Orthogonality in $u \bmod w'$ gives

$$\begin{aligned}\sum_{\mathfrak{c}} \left| \sum_{n'} v_{n'} |kn'|^{it} \varphi_{\mathfrak{c},\infty,kn'}(1/2+it) \right|^2 \\ \leq \tau(k)^2 \sum_{\mathfrak{c}} \left| \sum_{n'} v_{n'} |n'|^{it} \varphi_{\mathfrak{c},\infty,n'}(1/2+it) \right|^2 .\end{aligned} \tag{2.7.T43}$$

Here the character eigenvalue is $\bar{\chi}(k) \sum_{d|k} \chi^2(d)(k/d^2)^{it}$, of modulus at most $\tau(k)$. Transport (2.7.T36) permutes the complete cusp set. For negative indices, $\varphi_{\mathfrak{c}\infty,-n'}(1/2+it) = \overline{\varphi_{\mathfrak{c}\infty,n'}(1/2-it)}$. Changing t to $-t$ and conjugating coefficients converts each symmetric integration window to DI's positive-index continuous quadratic form. The discrete exceptional set is null for the dt-integral.

2.7.4. *Regular, continuous, and holomorphic estimates.* Put $Q_T = RS$, $L_{n''} = N'/n'' \geq 1/2$. Include $|t| < 1$ among the spectral windows K. By (2.7.T31), the undivided transform satisfies the following bound.

$$\mathfrak{h}(K) \ll \mathfrak{B}(\mathbf{t}) \mathcal{X}^{o(1)} (1+Z)^{-1} \min\{1, ((1+Z)/K)^3\}. \tag{2.7.T44}$$

For Maass forms use the positive DI weights $1/\cosh \pi\kappa_j$. For holomorphic forms use the positive weights $(\ell-1)!/(4\pi)^{\ell-1}$. These weights are split equally between the two Cauchy–Schwarz factors. For the continuous spectrum use plain dt. Take the supremum of $e_{\varsigma}(t)$ outside the positive quadratic form. By (2.7.T17), $|e_{\varsigma}| \leq |\Phi_{\varsigma}|/2$, with $\Phi_+ = \widehat{\phi}, \Phi_- = \check{\phi}$. Thus every norm uses a positive measure.

[5, Theorem 2] at ∞ gives the m-side

$$\sum_{q \asymp Q_T} \tau(q)(K^2 + M^{1+\epsilon}/q)M \ll \mathcal{X}^{3\epsilon}(Q_T K^2 + M)M. \tag{2.7.T45}$$

For the n-side, Cauchy–Schwarz over n'' costs $\mathcal{D}$. Apply (2.7.T40) or (2.7.T43). Then use [5, Theorem 2] at scale $L_{n''}$. The bound is

$$\ll \mathcal{D} \sum_{n''} \tau(k)^2 k^{\theta_{\mathrm{KS}}} (K^2 + L_{n''}^{1+\epsilon}/Q_T) \|b^\flat(n'')\|_2^2. \tag{2.7.T46}$$

On continuous and holomorphic parts (2.7.T46) holds with $k^{\theta_{\mathrm{KS}}}$ replaced by 1. [5, Theorem 2] allows the endpoint $M = 1/2$ in (2.7.T45).

Use $K^2 + U \leq ((1+Z)^2 + U)\max(1, K/(1+Z))^2$. The sum over dyadic K of the latter factor times the cubic decay in (2.7.T44) is $O(1+\log(2+Z))$. The cubic decay also absorbs a factor K^{ϵ}: with it the sum is $O((1+Z)^{\epsilon}\log(2+Z))$. Thus the bound (2.6) in the form $(K^2 + N/q)(KN)^{\epsilon}$ gives the same estimate. Cauchy–Schwarz and (2.7.T39) therefore yield

$$\begin{aligned}|\mathcal{L}_{\mathbf{t}}^{\mathrm{reg}}| \ll \mathcal{X}^{4\epsilon} \mathfrak{B}(\mathbf{t}) (1+Z)^{-1} [(Q_T(1+Z)^2 + M)M]^{1/2} \\ \times \left[\sum_{n''} (A_a n'')^{\theta_{\mathrm{KS}}} ((1+Z)^2 + L_{n''}/Q_T) \|b^\flat(n'')\|_2^2 \right]^{1/2} .\end{aligned} \tag{2.7.T47}$$

The same calculation with plain dt proves (2.7.T47) for the continuous part. It uses the symmetry shown after (2.7.T43). The same cubic summation proves absolute convergence of its integral, justifying the window partition. Thus the continuous part uses the measure dt throughout.

2.7.5. *Exceptional spectrum and the one-integer block.* Write $L = L_{n''}$, $\Xi = 1/Z$, $\Omega = 1 + Q_T/L$. Since $0 < \vartheta_j \le \theta_{\mathrm{KS}}$,

$$1 + \Xi^{\vartheta_j} \le 2(1 + \Xi/\sqrt{\Omega})^{\vartheta_j}\Omega^{\vartheta_j/2}. \tag{2.7.T48}$$

Fix n''. Apply Cauchy–Schwarz, assigning $Y = (1 + \Xi/\sqrt{\Omega})^2$ to the m-side and Ω to the n-side. With $\ell_{2,n''}$ as in (2.7.T40), the two exceptional quadratic forms are

$$\mathcal{M}_{n''} = \sum_{r,s}\sideset{}{^{(rs)}_{\mathrm{exc}}}\sum Y^{\vartheta_j}\Big|\sum_m \alpha_m\rho_{j\infty}(m)\Big|^2,$$

$$\mathcal{N}_{n''} = \sum_{r,s}\sideset{}{^{(rs)}_{\mathrm{exc}}}\sum \Omega^{\vartheta_j}|\ell_{2,n''}(u_j)|^2,$$

with r, s over the support of b. Let $L \ge 1/2$. Apply (2.7.T40) and Theorem 2.2 at level $rs > Q_T$, length L and $Y = \Omega$. Since $L\Omega/(rs) \le 1 + L/Q_T$, the factor in (2.8) is $\ll \max(1, L)^{\epsilon}(1 + L/Q_T)$. Summing over r, s gives

$$\mathcal{N}_{n''} \ll \mathcal{X}^{2\epsilon}(A_a n'')^{\theta_{\mathrm{KS}}}(1 + L/Q_T)\|b^\flat(n'')\|_2^2. \tag{2.7.T49}$$

Put $Q' = 4Q_T$, $Y_0(M) = (Q' + M)^2/M$, and

$$Z_{n''} = \frac{C^2S^2R}{(Q_T + M)^2(L + Q_T)}. \tag{2.7.T50}$$

The identity $\Xi^2/\Omega = (16\pi^2A_a n'')^{-1}C^2S^2R/[M(L + Q_T)]$ gives

$$\max(1, Y/Y_0(M))^{\theta_{\mathrm{KS}}} \le 1 + (A_a n'')^{-\theta_{\mathrm{KS}}}Z_{n''}^{\theta_{\mathrm{KS}}}. \tag{2.7.T51}$$

Let $M \ge 1$. Then (2.9) applies to every integer interval inside $(M, 2M]$. Use the positive Hilbert space with weights Y^{ϑ_j}. Apply Abel summation to m^{it_1}. The norm is at most $1 + |t_1|\log 2$ times the maximum interval norm. Take the maximum after the norm. It remains outside the spectral sum. Apply (2.9) at $Y_0(M)$. Then use monotonicity

$$Y^{\vartheta_j} \le \max(1, Y/Y_0)^{\theta_{\mathrm{KS}}}Y_0^{\vartheta_j}, \tag{2.7.T52}$$

For fixed q, the coefficient α_m is independent of the factorization $q = rs$. Put $w(q) = \#\{(r, s)$ in the current support : $rs = q\}$ and $\Delta = \max_{2\le q\le Q'} w(q)$, and let $\omega_0(q)$ count the distinct prime factors of q. Since $(r, s) = 1$, $w(q) \le 2^{\omega_0(q)} \le \tau(q) \ll_\eta q^\eta$. In the Hilbert space over all levels $2 \le q \le Q'$ and their exceptional forms, vector-valued Abel summation gives $\mathcal{M}_{n''}^{1/2} \le \Delta^{1/2}(1 + |t_1|\log 2)\sup_{I\subset(M,2M]} F_Y(I)^{1/2}$, where $F_Y(I) = \sum_{2\le q\le Q'}\sum_{\mathrm{exc}}^{(q)} Y^{\vartheta_j}|\sum_{m\in I}\rho_{j\infty}(m)|^2$ and I ranges over integer intervals. Thus the supremum is outside the whole quadratic form. Apply (2.9) with exponent $\eta = \epsilon/2$ at $Y_0(M)$. By (2.7.T4), $M \le H \le 4\mathcal{X}^2$ when $\epsilon_1 \le 1$; also $Q' = 4Q_T \le 4\mathcal{X}$. Consequently $\Delta(Q'M)^\eta \ll_\epsilon \mathcal{X}^{2\epsilon}$. Together with (2.7.T51)–(2.7.T52), this is absorbed in $\mathcal{X}^{3\epsilon}$ and gives

$$\mathcal{M}_{n''} \ll \mathcal{X}^{3\epsilon}(1 + t_1^2)[1 + (A_a n'')^{-\theta_{\mathrm{KS}}}Z_{n''}^{\theta_{\mathrm{KS}}}](Q_T + M)M. \tag{2.7.T53}$$

[5, (8.17)] excludes $M = 1/2$. We prove that endpoint here. Let $p(q)$ be the least prime not dividing q. Every smaller prime divides q. The classical Chebyshev lower bound $\Theta(x) := \sum_{\ell\le x}\log\ell \gg x$ for $x \ge 2$ (see [17]) gives

$$p(q) \ll 1 + \log q. \tag{2.7.T54}$$

Here $\Theta(p(q)-1) \le \log q$, because the product of all smaller primes divides q; the case $p(q)=2$ is immediate. The implied constant in (2.7.T54) is absolute and independent of a_T.

On an exceptional eigenspace V, let v represent $u \mapsto c_u(1)$. Self-adjointness and (2.7.T38) show that $T_p v, T_{p^2} v$ represent $c_u(p), c_u(p^2)$. The Hecke identity $T_p^2 = T_{p^2} + I$, with $\|T_p\| \le 2p^{\theta_{\mathrm{KS}}/2}$, gives

$$\sum_{u_j \in V} |c_{u_j}(1)|^2 \le 8p^{\theta_{\mathrm{KS}}} \sum_{u_j \in V} |c_{u_j}(p)|^2 + 2 \sum_{u_j \in V} |c_{u_j}(p^2)|^2. \tag{2.7.T55}$$

Multiply by Y^{ϑ_j}, which is constant on V. Group levels by $p(q) \le P := C_0(1 + \log(4Q_T))$, where C_0 is an absolute constant supplied by (2.7.T54). Positivity permits enlarging each group to all $2 \le q \le Q'$. For $k = p, p^2$, the singleton $\{k\}$ is an integer interval in $(k/2, k]$, with $k/2 \ge 1$, so (2.9) applies at $N = k/2$. In DI's terms, subtract the initial intervals $(k/2, k]$ and $(k/2, k-1]$. The resulting factor 4 is included in c_ϵ.

For the endpoint, the same factorization count multiplies the level-q quadratic form by at most $\tau(q)$. The following bound holds for any positive exponent. In its use in (2.7.T57), take $\eta = \min(\epsilon/2, 1/4)$ and use $\tau(q) \ll_\epsilon Q_T^{\epsilon/2}$ for $Q_T < q \le 4Q_T$. This keeps their combined loss within the displayed ϵ-power after adjustment of its constant. At $Y_k = (Q' + k/2)^2/(k/2)$ it gives

$$\sum_{2 \le q \le Q'} \sideset{}{^{(q)}_{\mathrm{exc}}}\sum Y^{\vartheta_j} |c_{u_j}(k)|^2 \ll_\epsilon (Q'k)^\epsilon (Q' + k/2) k \max(1, Y/Y_k)^{\theta_{\mathrm{KS}}}. \tag{2.7.T56}$$

Since $Y_0(1/2)/Y_k = k(Q'+1/2)^2/(Q'+k/2)^2 \le k \le 9k/4$, insert (2.7.T56), with its exponent taken as η, in (2.7.T55). Sum over the at most P possible primes:

$$\begin{aligned} &\sum_{Q_T < q \le 4Q_T} \tau(q) \sideset{}{^{(q)}_{\mathrm{exc}}}\sum Y^{\vartheta_j} |c_{u_j}(1)|^2 \\ &\qquad \ll_\epsilon Q_T^\epsilon P^6 (Q_T + \tfrac12)^{\frac12} \max(1, Y/Y_0(1/2))^{\theta_{\mathrm{KS}}}, \end{aligned} \tag{2.7.T57}$$

for $0 < \epsilon \le 1/4$, with the implied constant including the divisor bound and the change from ϵ to η. Before simplification, the sum is at most a constant times

$$c_\epsilon (9/4)^{\theta_{\mathrm{KS}}} Q'^{1+\epsilon} \sum_{p \le P} (8p^{2+2\theta_{\mathrm{KS}}+\epsilon} + 2p^{4+2\theta_{\mathrm{KS}}+2\epsilon});$$

this is $\ll_\epsilon Q_T^{1+\epsilon} P^6$, because $5 + 2\theta_{\mathrm{KS}} + 2\epsilon \le 6$. The loss P^6 is logarithmic and uniform in a_T. In particular $P^6 \ll_\epsilon \mathcal{X}^\epsilon$; together with the factorization loss already included in (2.7.T57), this fits the $\mathcal{X}^{3\epsilon}$ allowance of (2.7.T53). Together with (2.7.T51), this proves (2.7.T53) also for $M = 1/2$. Since $\alpha_1 = 1$, Abel summation costs nothing here.

Apply the triangle inequality over n'' before Cauchy–Schwarz. Then (2.7.T49), (2.7.T53), and (2.7.T32) give

$$\begin{aligned} |\mathcal{L}_{\mathbf{t}}^{\mathrm{exc}}| \ll{}& \mathcal{X}^{6\epsilon} \mathfrak{B}(\mathbf{t}) (1 + |t_1|)(1+Z)^{-1} \sqrt{M(Q_T + M)} \\ &\times \sum_{n''} \big[(A_a n'')^{\theta_{\mathrm{KS}}/2} + Z_{n''}^{\theta_{\mathrm{KS}}/2} \big] ((L + Q_T)/Q_T)^{1/2} \|b^\flat(n'')\|_2. \end{aligned} \tag{2.7.T58}$$

There is no exceptional continuous or holomorphic spectrum.

2.7.6. *The four terms and completion of the proof.* Put

$$\mathfrak{T} = A_a M N'/(C^2 S), \quad M_0 = CS/D, \quad \varpi = 4\mathcal{X}^{\epsilon_2}. \tag{2.7.T59}$$

The quantity $\mathfrak{T}$ is unrelated to the cell endpoint T of Section 3. Then $M \le \varpi M_0$, $Z^2 Q_T = 16\pi^2 \mathfrak{T}$, so $(1+Z)^{-2} Q_T^{-1} \asymp (Q_T + \mathfrak{T})^{-1}$. Square (2.7.T47) and multiply by $D^2 R$. Apart from the stated auxiliary powers, sum the following over n''.

$$(A_a n'')^{\theta_{\mathrm{KS}}} D^2 R \frac{M(M+Q_T+\mathfrak{T})(L+Q_T+\mathfrak{T})}{Q_T+\mathfrak{T}} \|b^\flat(n'')\|_2^2. \tag{2.7.T60}$$

The first, noncancelling branch of (2.7.T58) has the same bound. Termwise, use

$$(Q_T+M)(L+Q_T)/Q_T \le [Q_T(1+Z)^2 + M][(1+Z)^2 + L/Q_T]. \tag{2.7.T61}$$

To obtain (2.42), use the following identity.

$$\frac{Q_T(M+Q_T+\mathfrak{T})(L+Q_T+\mathfrak{T})}{Q_T+\mathfrak{T}} = (L+Q_T)(M+Q_T) + Q_T\mathfrak{T} - \frac{LM\mathfrak{T}}{Q_T+\mathfrak{T}}. \tag{2.7.T62}$$

Multiply (2.7.T62) by M/Q_T, then drop the last (nonpositive) term:

$$\frac{M(M+Q_T+\mathfrak{T})(L+Q_T+\mathfrak{T})}{Q_T+\mathfrak{T}} \le \frac{M(L+Q_T)(M+Q_T)}{Q_T} + M\mathfrak{T}.$$

Since $D^2 M/S \le \varpi CD$ and $M + Q_T \le \varpi S(C+DR)/D$, it follows that

$$\begin{aligned} &D^2 R \frac{M(M+Q_T+\mathfrak{T})(L+Q_T+\mathfrak{T})}{Q_T+\mathfrak{T}} \\ &\qquad \le \varpi^2 \{CS(L+RS)(C+DR) + A_a N' RS\}. \end{aligned} \tag{2.7.T63}$$

The last term follows from $D^2 RM\mathfrak{T} = A_a N' RS(DM/(CS))^2$. Using $L \le N_1/n''$, $N' \le N_1$, $A_a \le A$ proves the first two terms of (2.42), with $A^{\theta_{\mathrm{KS}}}(n'')^{\theta_{\mathrm{KS}}}$ kept explicit.

For the cancelling branch of (2.7.T58), use

$$Z_{n''}^{\theta_{\mathrm{KS}}}(L+Q_T) \le (C^2 S^2 R)^{\theta_{\mathrm{KS}}} (Q_T+M)^{-2\theta_{\mathrm{KS}}} (N_1+Q_T)^{1-\theta_{\mathrm{KS}}}, \tag{2.7.T64}$$

and $\sum_{n''} \|b^\flat(n'')\|_2 \le \mathcal{D}^{1/2}\|b\|_2$. Since $M \mapsto M(Q_T+M)^{1-2\theta_{\mathrm{KS}}}$ is increasing, substitute M_0, paying at most $\varpi^{2-2\theta_{\mathrm{KS}}}$. The resulting identity is

$$\begin{aligned} &\frac{D^2 R M_0 (Q_T+M_0)^{1-2\theta_{\mathrm{KS}}} (C^2S^2R)^{\theta_{\mathrm{KS}}} (N_1+Q_T)^{1-\theta_{\mathrm{KS}}}}{Q_T} \\ &\qquad = CS(CD\sqrt{R})^{2\theta_{\mathrm{KS}}} (N_1+RS)^{1-\theta_{\mathrm{KS}}} (C+DR)^{1-2\theta_{\mathrm{KS}}}. \end{aligned} \tag{2.7.T65}$$

This is the third term of (2.42). The fourth was (2.7.T3). The truncated tail is smaller than it.

To account for all losses, reconstruct the block sum as $D\sqrt{R}(2\pi)^{-4}\int \mathcal{L}_{\mathbf{t}}\, d\mathbf{t}$. Equation (2.7.T11) gives $\int \mathfrak{B}(\mathbf{t})(1+|t_1|)\, d\mathbf{t} \ll \mathcal{X}^{17\epsilon_2}$. The amplitude's ϖ loss in (2.7.T63) is at most $\mathcal{X}^{\epsilon_2}$ after taking its square root. The log partitions and divisor factors cost $\mathcal{X}^{O(\epsilon_1)}$. This includes the DI small ϵ losses, $\mathcal{D}$, and the one-integer P^6 loss. The bound is uniform in a_T. Consequently the final loss is

$$\mathcal{X}^{O(\epsilon_1)+18\epsilon_0}. \tag{2.7.T66}$$

Choose the freely available ϵ_1 small enough relative to the requested ϵ. Equations (2.7.T63), (2.7.T65), and (2.7.T3) then prove (2.41)–(2.43), with loss $\mathcal{X}^{\epsilon+18\epsilon_0}$, for both signs and including the $M = 1/2$ block. This proves Theorem 2.8 in the form used in Section 3, (3.14)–(3.15).

3. Dispersion for abstract sources

This section proves a local implication for an abstract source. It uses Linnik's dispersion method [28]. The second-dispersion setup has close precedents in Maynard [7] and Lichtman [1, §5]; the proof below tracks its own coefficient masks, restoration terms, and moving residue. The local hypotheses are (H1)–(H5). The source-wide estimate (H6) and the box-family coefficient bound (H7) are used only in the global summation in Section 3.10. Each application verifies the required conditions for its own measures. The geometric and source partition results appear below. Theorem 2.8 is stated in Section 2, (2.38)–(2.43), and derived from the classical inputs in §2.7. We keep its full moving-frequency bracket. The local estimate has three terms. The restoration error E_{GR} is bounded apart from the spectral term.

3.1. **Notation and source hypotheses.** *Local notation.* Put $L = \log X$; §3.10 writes $\log N$ explicitly. T is the lower endpoint of the selected Type-II cell, as in Lemma B.3 for the prime source. Here $\vartheta_{\mathrm{sp}} = 7/32$. The zero-mode functional is $\mathcal{Z}(V)$ of (3.31); Z is the scale of z. B is a Heath–Brown box. D and C are the scales of b and c in (3.62), as in Theorem 2.8. $B_{z,y}$ is the count (3.70). S is the scale of γ_2. S_B^{PSDT} denotes the signed box sum in (3.12). Type I-1 and I-2 denote boxes with one and two large variables (§3.11). Since $\log X = \log N + O(\log\log N)$, fixed powers of $\log X$ and $\log N$ are comparable. I_T is the selected integer interval of a Type-II cell; I_N is reserved for the source interval of Theorem 1.2. Some letters have local meanings, fixed at each use. The letter C is a fixed logarithmic exponent in (3.13), (3.92) and similar bounds, and the scale of c from (3.62) on. Similarly, ε is the source cut of Theorem 1.2, ϵ is a small fixed exponent, and $\varepsilon_y \in \{\pm 1\}$ is a sign. The letter Y is a summation length in (3.16)–(3.19) and (3.35). It is the auxiliary scale H_*SF in the spectral step near (3.70). In Theorem 1.2 it is the smooth-prime cutoff D^{σ^2}. $\mathcal{T}$ is the tuple set of §3; it is unrelated to T.

Let N be a sufficiently large even integer and let a be a positive integer. Fix $0 < K_{\mathrm{scale}} < \infty$ and

$$0 < \sigma \le 10^{-10}, \quad \tau = \sigma^9, \quad \delta = 30\sigma, \quad \vartheta_{\mathrm{sp}} = 7/32, \quad \eta = \sigma/1000,$$

$$\eta_L = 10^{-4}, \qquad \ell_0 = \frac{53}{412} - \frac{\eta_L}{2} = \frac{264897}{2060000}, \qquad c_\sigma = \ell_0\sigma^2/4. \tag{3.1}$$

The local product scale is $N/(\log N)^{K_{\mathrm{scale}}} \le X \le N$. All implied constants in this section may depend on the fixed K_{scale} and the fixed source bounds. They may also depend on the parameters displayed at the point of use. Put

$$L = \log X, \quad z_0 = X^{1/(\log\log X)^3}, \quad u_0 = N^{\ell_0\sigma^2}, \\ p_0 = \min\{p \text{ prime} : p \nmid 2aN\}. \tag{3.2}$$

In particular, $N/u_0 = N^{1-\ell_0\sigma^2}$.

Thus $p_0 \ge 3$ and $p_0 \nmid aN$. Every prime below p_0 divides $2aN$. Chebyshev's bound $\sum_{p<p_0} \log p \gg p_0$ gives $p_0 \ll \log(2aN)$. By C4 below, $a \le X(\log N)^{K_{\mathrm{scale}}}$, so $p_0 \ll \log N < z_0 \le u_0$ eventually. Write

$$\mathcal{S}_t(q;a) = 1_{t\equiv a\ (q)} - \frac{1_{(t,q)=1}}{\varphi(q)}. \tag{3.3}$$

A **Class-II box** consists of the following data. These are source hypotheses.

C1. Exact unrestricted coefficient. Fix a source term, a prefix cell, a sign of the well-factorable remainder in Iwaniec's theorem [10, Theorem 4], and a tail-label vector b. Its coefficient is exactly

$$C_b(q) = \sum_{d,(p_j)} \lambda_d 1_{q=d\prod_{j\in J} p_j}, \qquad |\lambda_d| \le 1, \tag{3.4}$$

where each p_j ranges independently over its fixed prime set and $p_j \ge u_L$. Here $u_L \ge u_0$ is the source's lower cutoff for labelled tail primes, and $P(u_L;N) = \prod_{p<u_L,\, p\nmid 2N} p$. The variable $d \mid P(u_L;N)$ is squarefree with the cap of that theorem. The full modulus is at most $X^{0.6}$. Every supported full modulus is prime to $2N$. Selectors are constant on the block. Exact identities remove the equal-tail, prefix-diagonal, natural-comparator, cut, and selector-transfer corrections. Each is a separate contribution outside (3.4).

C2. Exact factors. There is a labelled partition $J = J_1 \sqcup J_2 \sqcup J_3$, with d attached to J_1, such that

$$C_b = C_{\mathrm{slot}}(\gamma_1 * \gamma_2 * \gamma_3), \qquad C_{\mathrm{slot}} = \prod_{i=1}^{3} |J_i|! = O_\sigma(1), \qquad |\gamma_i| \le 1. \tag{3.5}$$

The last two factors are u_0-rough. If $J_s = \varnothing$ for $s = 2, 3$, then $\gamma_s(m) = \mathbf{1}_{m=1}$. The factor γ_1 always contains the smooth-divisor sum of Iwaniec's theorem. If $J_1 = \varnothing$, then $\gamma_1(m) = \sum_d \lambda_d \mathbf{1}_{m=d}$. Equality (3.5) holds on all supported moduli, with repeated-prime multiplicities. The source's labelled-tuple map proves it. It covers moduli with repeated primes as well as squarefree ones.

C3. Packing. Localize dyadically the first factor γ_1 at $\asymp Q$, γ_3 at $\asymp R$, and γ_2 at $\asymp S$. Here T is the lower endpoint of the selected Type-II cell; N_I is its integer length. Enlarge the sufficiently-large threshold for N to absorb fixed dyadic constants. Then

$$Q < TX^{-29\sigma}, \qquad QRS < M_I X^{-89\sigma}, \qquad N_I \ge QX^{\sigma/3}, \quad M_I \ge X^{1/2}. \tag{3.6}$$

Here Q is the scale of the first factor γ_1 (the outer slot scale, written Q_{slot} in Appendix B). It is not the scale of the full modulus q of (3.4), which is QRS. Thus (3.6) gives the first inequality in (3.7). We list the four spectral packing inequalities together.

$$Q < TX^{-\sigma}, \quad T^2R^2S < X^{1-8\sigma}, \quad A_0^{\vartheta_{\mathrm{sp}}} TQR^2S^5 < X^{2-15\sigma},$$

$$(TR/S)^{2\vartheta_{\mathrm{sp}}} TQR^2S^5 < X^{2-15\sigma}, \qquad A_0 = X^{1+\sigma/20}. \tag{3.7}$$

Each application must verify the stated packing and interval conditions for its own source. Appendix B does so for the prime source in Theorem 1.2.

Removed squareful and other restricted terms. A source may remove forbidden tails, forbidden prefixes or retained allowed repeats. Their contributions are bounded separately. One admissible bound is $O_\sigma(N^{1-\ell_0\sigma^2}L^4)$, after all source-label and representation multiplicities. Theorem 3.2 concerns the unrestricted coefficient (3.4). The removed terms need their own estimates.

The prime source in Theorem 1.2 is $\mathbf{1}_{\mathbb{P}}(n)\mathbf{1}_{N/(\log N)^B<n<(1-\varepsilon)N}$. Appendix B verifies its coefficient, interval, cutoff, support, and boundary conditions. The hypotheses above also state the local estimate for any other source that satisfies them.

Apply the finite HB expansion below. Then split into disjoint short product boxes of relative width $L^{-C_{\mathrm{box}}}$. Boundary boxes contribute separately. On each interior Class-II box select an HB variable in the window. Maximize its lower-endpoint exponent $e_i = \log_N V_i$. Break ties by the fixed variable index, as in the classification in Appendix B. Call it n, and the product of the other variables m.

In (3.8), allow a roughness cutoff $\zeta \ge z_0$, where $z_0 = z_0(X)$ is the baseline in (3.2). All estimates below are uniform in ζ. Upper bounds use only $P^-(n) \ge z_0$. The SW proof (3.25) for the selected factor uses the cutoff ζ. It remains uniform under stronger roughness.

$$\alpha_n \in \left\{\mu(n), 1, \frac{\log n}{L}\right\} 1_{P^-(n)\ge\zeta} 1_{I_T}(n), \quad |\alpha_n| \le 1, \qquad |\beta_m| \ll \tau_{23}(m). \tag{3.8}$$

The fixed HB binomial coefficients and tags may be kept as exterior scalars. $I_T \subset (T, 2T]$ has length $N_I \asymp TL^{-C_{\mathrm{box}}}$. The m-interval is majorized by

$$\psi_I(m) = \phi_0((m - m_0)/M_I), \qquad M_I \asymp (X/T)L^{-C_{\mathrm{box}}}, \tag{3.9}$$

Fix $\phi_0 \in C_c^\infty([-1, 1])$, with $0 \le \phi_0 \le 1$ and $\phi_0 = 1$ on $[-1/2, 1/2]$. It is independent of N, X, the source and the box. It is also independent of all shell and tuple variables. For every $j \ge 0$, the seminorms $\|\phi_0^{(j)}\|_1$ and $\|\phi_0^{(j)}\|_\infty$ are fixed constants. The support of $\psi_I(m) = \phi_0((m - m_0)/M_I)$ lies in the doubled interval. The majorant depends on the box only through m_0 and M_I. On that entire support the cut geometry gives

$$1 \le |mn - a| \le X^3. \tag{3.10}$$

C4. Moving frequency. All original moduli are prime to a, and $0 < a \le X(\log N)^{K_{\mathrm{scale}}} \le A_0$. This is the moving frequency in the spectral theorem. It may vary with N.

C5. Window. The source selection gives

$$N^{60\sigma} \le T \le N^{1/3+14\sigma}, \qquad X^{60\sigma} \le T \le X^{1/3+15\sigma}. \tag{3.11}$$

The second assertion follows from the first and $\log(N/X)/\log N = o(1)$, with N_0 chosen after fixed σ. **Claim P (nonresonance):** the doubled support satisfies (3.10). The source assumptions for the non-II extension are in §3.11.

3.2. **Statement and theorem inputs. Length hypothesis for the selected integer interval (LEN).** Uniformly over the selected boxes, the interval has length $N_I \asymp TL^{-C_{\mathrm{box}}}$ and satisfies $N_I \ge (2T)^{1-2^{-126}}$. The integer interval used in (3.17) has this same length, with the endpoint convention specified by the source. Lemma B.3, (B.22)–(B.27), proves this for the prime source. Each application proves it for its own source.

For the coefficient of one Heath–Brown box define

$$\Delta_B(q;a)=\sum_{\mathbf{v}\in B} w(\mathbf{v})\mathcal{S}_{\prod v_i}(q;a), \qquad S_B^{\mathrm{PSDT}}=\sum_q C_B(q)\Delta_B(q;a). \tag{3.12}$$

3.3. **An abstract dispersion estimate.** The local argument is source-independent. We state it before the verification for specific sources. Let $\mathcal{B}$ be a finite family of dyadic boxes at scale X, and let a be a moving frequency. For $B\in\mathcal{B}$ write $C_B(q)$ for its signed modulus coefficient. Define $\Delta_B(q;a)$ by (3.12).

The following hypotheses define the class used in this paper.

(H1) Exact coefficient:
On every box, $C_B(q)$ has the exact form (3.4), with its fixed labelled prime sets, squarefree smooth-divisor variable d, and repeated-prime multiplicities. Every supported modulus is coprime to $2a$.

(H2) Three-factor decomposition:
The labelled variables split into three groups satisfying (3.5). The last two factors are u_0-rough, as in C2. The identity holds on all supported moduli, including repeated-prime multiplicities.

(H3) Geometric packing and box data:
After dyadic localization, the scales Q,R,S and the test scale T satisfy (3.6)–(3.7). On each box, the Heath–Brown coefficients factor as $\alpha_n\beta_m$ with (3.8). The smooth majorant has the form (3.9). Its center satisfies $|m_0|\le C_{\mathrm{ctr}}X/T$, where C_{ctr} is a fixed source-class constant, uniform over all boxes, shells, tuple variables, and permitted N,X,a. Its full doubled support satisfies (3.10). The selected integer interval satisfies LEN, including its strong length bound and integer endpoint convention. For this abstract dispersion estimate, assume modulus support in $q\le X^{\vartheta_0}$, with ϑ_0 as in (3.12a).

(H4) Moving frequency:
The frequency is the positive integer a, with N even, $0<a\le X(\log N)^{K_{\mathrm{scale}}}\le A_0$, and all original moduli are coprime to a.

(H5) Selected window:
The tested scale obeys (3.11), with all parameters fixed after $\sigma>0$ is fixed.

(H6) Source-wide remainders and boundary:
Let a source supply a logarithm-freezing estimate and boundary estimate for the full box family. After summing over all source labels, shells, boxes, and representation multiplicities, these estimates must give

$$E_{\log}\ll N(\log N)^{c_{\log}-C_{\mathrm{box}}}, \qquad E_{\mathrm{bdry}}\ll N(\log N)^{c_R-C_{\mathrm{box}}/2},$$

with fixed $c_{\log}$ and c_R. This condition is used in the global summation, not in the local theorems below.

(H7) Box-family coefficient bound:
The finite box family $\mathcal{B}=\{B_j\}$ carries outer coefficients c_j, coming from the source labels and the Heath–Brown identity. At fixed σ they satisfy (3.91): $\sum_j|c_j|\le C_{\sigma,C_{\mathrm{box}},\mathcal{C}_{\mathrm{src}}}(\log N)^{K_0}$ with a fixed $K_0\le 24C_{\mathrm{box}}+26$. This condition is used only for the aggregate bound (3.92), not for the per-box estimates.

Write $\mathcal{C}_{\mathrm{src}}$ for the fixed source constants in (H1)–(H5), such as C_{slot}, C_{ctr} and the implied constants in (3.8)–(3.11).

The Siegel–Walfisz bound (3.25) for the selected factor follows in §3.4 from the classical theorem [17, Corollary 5.29]. BFI [6] and Theorem 2.8, proved from classical inputs in §2.7, are theorem inputs, not source hypotheses. The source-wide

boundary bound (3.24) is hypothesis (H6). It is used in the source estimates, not in the per-box theorem (3.13).

Put

$$\Theta_0 = \frac{19101}{32000}, \qquad \vartheta_0 = \Theta_0 - \eta_L > \frac{1}{2}. \tag{3.12a}$$

The number Θ_0 is the small-atom empty-prefix packing level of Proposition 4.6 in §4. The exponent ϑ_0 is a support convention in the abstract dispersion estimate. The actual support level comes from the packing inequalities (3.6)–(3.7) after all Iwaniec factors are restored. In Theorem 1.2, (B.2)–(B.4) give $q < N^{e_Q}$ with $e_Q < \Theta(u) \le \Theta(0) = 153/256 < 0.6$; hence $q \le X^{0.6}$ on every source shell for sufficiently large N. The fixed exponent ϑ_0 belongs to the abstract support hypothesis of the introductory source-box estimate. The variable ceiling $\Theta(u) = (5 - 7/32 - 3u)/8$ in Theorem 1.2 controls the Iwaniec coefficient family as u varies. At $u = 1/500$, $\Theta(u) = 19101/32000 = \Theta_0$, whereas $\vartheta_0 = \Theta_0 - 10^{-4}$. The introductory estimate applies to its stated abstract support class; the Iwaniec result at this endpoint uses the $X^{0.6}$ theorem below. Theorem 3.1 is the abstract dispersion estimate under the support and packing hypotheses in (H3), including the stronger support $q \le X^{\vartheta_0}$. Theorem 3.2 is the dispersion engine used for the actual source support. Theorem 1.2 in the application uses that engine together with the source estimates of Appendix B. At $u = 1/500$ it gives the limiting ceiling $\Theta(1/500)$ for this specified signed mean value.

Theorem 3.1 (Dispersion estimate under the (H3) packing hypotheses)**.** (a) *Assume (H1)–(H5). Then, for every fixed $A' > 0$ and every box B,*

$$\left| \sum_{q \le X^{\vartheta_0}} C_B(q) \Delta_B(q;a) \right| \ll_{\sigma, K_{\mathrm{scale}}, C_{\mathrm{box}}, A', \mathcal{C}_{\mathrm{src}}} XL^{-A'} + X^{1-c_\sigma} L^C, \tag{3.12b}$$

where $c_\sigma = \ell_0 \sigma^2/4 > 0$ and $C = C(\sigma, C_{\mathrm{box}}, A', K_{\mathrm{scale}}, \mathcal{C}_{\mathrm{src}})$ is the exponent of (3.13). The implied constant and the threshold for X are uniform in the box, the even integer N, and all moving frequencies a permitted by (H1)–(H5). They may depend on the fixed source bounds and on σ. They are ineffective, since the proof uses the Siegel–Walfisz theorem in (3.25). Under (H3) every supported modulus satisfies $q \le X^{\vartheta_0}$, so the truncation in (3.12b) is vacuous and its left side is $|S_B^{\mathrm{PSDT}}|$.

(b) *Assume in addition (H7). Fix $A > 0$ and take $A' = A + 24C_{\mathrm{box}} + 31$ in* (a)*. Then the weighted sum $E_{\mathrm{II}} = \sum_j |c_j| |S_{B_j}^{\mathrm{PSDT}}|$ over the box family satisfies (3.92), that is, $E_{\mathrm{II}} \ll_{\sigma, A, C_{\mathrm{box}}, K_{\mathrm{scale}}, \mathcal{C}_{\mathrm{src}}} N(\log N)^{-A}$. This bound is uniform in the scales X of the boxes. For $N/(\log N)^{K_{\mathrm{scale}}} \le X \le N$ one has $\log X \ge \log N - K_{\mathrm{scale}} \log\log N \ge \frac{1}{2}\log N$ for large N, so $X(\log X)^{-A'} \le 2^{A'} N(\log N)^{-A'}$ and $X^{1-c_\sigma}(\log X)^C \le N^{1-c_\sigma}(\log N)^C$. Part* (b) *holds for any fixed $C_{\mathrm{box}} > 0$. The condition $C_{\mathrm{box}} \ge \lceil 2A + 32 \rceil$ of Theorem 1.2 serves only the source-wide errors of (H6).*

Under the support hypothesis in (H3), the theorem gives arbitrary logarithmic saving for these signed sums through $q \le X^{\vartheta_0} = X^{0.59680625}$, where $\vartheta_0 > 1/2$.

Proof. The support bound assumed in (H3) makes the sum in (3.12b) the full sum S_B^{PSDT} of (3.12). Hypotheses (H1)–(H5) imply those of Theorem 3.2. Hence Theorem 3.2 applies to B and gives (3.13), which is (3.12b). This proves (a). Part (b) follows from (a) and (H7) by the summation (3.92) in §3.10. □

Theorem 3.2 (Dispersion theorem)**.** *Assume (H1)–(H5), where in (H3) the support bound $q \le X^{\vartheta_0}$ may be relaxed to $q \le X^{0.6}$. Thus each box satisfies (3.8)–(3.10) and LEN. Then, for each fixed $A' > 0$,*

$$|S_B^{\mathrm{PSDT}}| \ll_{\sigma, K_{\mathrm{scale}}, C_{\mathrm{box}}, A', \mathcal{C}_{\mathrm{src}}} XL^{-A'} + X^{1-c_\sigma} L^C. \tag{3.13}$$

Here $C = C(\sigma, C_{\mathrm{box}}, A', K_{\mathrm{scale}}, \mathcal{C}_{\mathrm{src}})$. It multiplies only power-saving terms. It enters only the final choice of N_0. The exponent C is fixed after the proof parameters. It is uniform in the box, each class in the random partition, the residue, and the moving shift. Constants and N_0 may depend on the fixed σ. All bounds are for fixed $\sigma > 0$ as $N \to \infty$. The implied constants and the threshold N_0 are ineffective, since the proof uses the Siegel–Walfisz theorem in (3.25). The estimate (3.13) is per box and does not use (H6) or (H7). If (H7) also holds and $A' = A + 24C_{\mathrm{box}} + 31$, then the weighted sum over the box family satisfies (3.92).

Hypotheses (H1)–(H5) give the coefficient, factor identity, packing, box, frequency and window conditions, including LEN and the integer endpoint convention. BFI [6, Theorem 0(a)] supplies (3.48), Theorem 2.8 of §2.7 supplies (3.14)–(3.15), and (3.25) supplies SW. The source-wide condition (H6) enters the source error terms, including (B.29) in the proof of Theorem 1.2 and the auxiliary bound (3.95). It is not used in the per-box Theorem 3.2. The box-family bound (H7) enters only (3.92). The spectral part gives a term $X^{1-1.725\sigma}L^C$ (§3.9). It is absorbed in $X^{1-c_\sigma}L^C$, since $c_\sigma < \sigma/2 < 1.725\sigma$.

The dependencies have the following order:

$$\begin{gathered}\text{source identities, box geometry and source estimates} \implies \text{(H1)–(H7)},\\ \text{Section 2 and (3.25)} \implies \text{BFI, Theorem 2.8 and SW},\\ \text{(H1)–(H5)} + \text{theorem inputs} \implies \text{(3.13)},\\ \text{(3.13)} + \text{(H7)} \implies \text{(3.92)}.\end{gathered}$$

The hypotheses are stated independently of any particular source. Appendix B verifies them for Theorem 1.2. Source comparison and the estimates for removed squareful terms are separate steps. Aggregate logarithm-freezing estimates enter in §3.10. Theorem 3.2 uses (H1)–(H5) and the stated theorem inputs.

The inputs are the following theorems, printed in Section 2. The Siegel–Walfisz theorem (SW) [17, Corollary 5.29] gives prime distribution in reduced progressions to moduli at most a fixed power of the logarithm. [6, Theorem 0(a), (2.2)], with [6, (A2)], gives the BDH quadratic estimate. It applies to sequences satisfying its norm-normalized SW condition. On the domain $(rd, sc) = 1$, Theorem 2.8 gives

$$\begin{aligned}&\left|\sum_{(r,s)=1}\sum_{0<n\le N_T} b_{n,r,s}\sum_{c,d:(rd,sc)=1} g(c,d,n,r,s)e\left(\frac{a_T n\overline{rd}}{sc}\right)\right|\\ &\ll (2 + A_T C D N_T R_T S_T)^{\epsilon+18\epsilon_0} J_*(a_T).\end{aligned} \tag{3.14}$$

Here $0 < |a_T| \le A_T$, where A_T is the frequency bound A of Theorem 2.8, and $b = 0$ unless $(a_T, rs) = 1$. The positive scales meet the endpoint conditions of Section 2. The bracket is

$$
\begin{aligned}
J_*^2 = \sum_{n''\mid a_T^\infty} A_T^{\vartheta_{\mathrm{sp}}}(n'')^{\vartheta_{\mathrm{sp}}} \big[& CS_T(N_T/n'' + R_TS_T)(C + DR_T) \\
& + |a_T|N_TR_TS_T\big] \|b^\flat(n'')\|_2^2 \\
+ \big[& CS_T(CD\sqrt{R_T})^{2\vartheta_{\mathrm{sp}}}(N_T + R_TS_T)^{1-\vartheta_{\mathrm{sp}}} \\
& \times (C + DR_T)^{1-2\vartheta_{\mathrm{sp}}} + D^2N_TR_T\big] \|b\|_2^2.
\end{aligned} \tag{3.15}
$$

Here $b^\flat(n'')$ is the dyadic-block slice of $\tilde{b}(n'')$ with $(n', a) = 1$. The slice satisfies $\|b^\flat(n'')\|_2 \le \|\tilde{b}(n'')\|_2$. Since b is supported on the present dyadic block, $b^\flat(n'') = \tilde{b}(n'')$ term by term. Hence (3.15) is exactly (2.42)–(2.43).

The smooth derivatives in (3.14) obey the variable-based hypothesis (2.39), with the exponent $1 - \epsilon_0$ displayed there: each derivative is scaled by the corresponding positive argument c, d, n, r, s itself. In particular, the initial interval $0 < n \le N_T$ is not treated as a dyadic block of scale N_T. Theorem 2.8 is used in the form of Section 2, (2.38)–(2.43). It is proved in §2.7. Its spectral identity, including the continuous spectral measure, is exactly (2.14). The I-2 estimate (Lemma 3.5) also uses the composite-modulus Weil–Estermann bound [15, 16] in §3.11.

The sum in (3.13) is a signed modulus sum; triangle inequalities over source labels and boxes are applied outside it.

3.4. **Elementary preparation, moments, and SW. Divisor moments.** For integers $K, k \ge 1$,

$$
\tau(n)^K \le \tau_{2^K}(n), \qquad \sum_{n\le Y} \frac{\tau_k(n)}{n} \le (1 + \log Y)^k. \tag{3.16}
$$

At p^v, the first inequality is $(v + 1)^K \le \binom{v+2^K-1}{v}$. The ratio of successive right/left quotients is at least 1. This holds since $(1 + t)^K \le 1 + (2^K - 1)t$ for $0 \le t \le 1$, by convexity. The second inequality follows by summing over the k factors. Also $\tau_j(n) \le \tau(n)^{j-1}$.

Suppose $Y^{1-1/k} \le H \le Y$. Select a largest factor in a k-factorization. Then

$$
\sum_{Y<n\le Y+H} \tau_k(n) \le k \sum_{d\le(2Y)^{1-1/k}} \tau_{k-1}(d)(H/d + 1) \le 3kH(1 + \log(2Y))^{k-1}. \tag{3.17}
$$

Let $1 \le u \le Y$ and $0 < \varpi \le 1$. Split off prime factors larger than $Y^{\varpi/2}$. Greedily group the rest into factors at most Y^{ϖ}. There are at most $\lfloor 2/\varpi \rfloor + 1$ groups. The large-prime part has at most $2/\varpi$ prime factors with multiplicity. Hence

$$
\tau(u)^K \le 2^{2K/\varpi} \sum_{d\mid u,\ d\le Y^{\varpi}} \tau(d)^{K(\lfloor 2/\varpi\rfloor+1)}. \tag{3.18}
$$

For $u = |mn - a| \le X^3$, take $Y = X^3, \varpi = 1/12$. Put $h = (d, n)$. If $h \nmid a$, the congruence $d \mid mn - a$ has no solutions. Otherwise it has at most $|J|h/d + 1$ solutions on J. Sum (3.18) and apply (3.16). For $|J| \le 3M_I$ this gives

$$
\sum_{m\in J} \tau(|mn - a|)^K \le 2^{24K}(1 + \log X)^{2^{25K}} \{3M_I\tau(n)^{25K+1} + X^{1/4}\}. \tag{3.19}
$$

Write $\widehat{I}_M$ for the integer support of ψ_I; it is contained in an interval of length at most $3M_I$. Sum (3.19) with $K = 5$ over $n \in I_T$. The n-sum of $\tau(n)^{126} \le \tau_{2^{126}}(n)$ is bounded by (3.17) and LEN. Since $M_I \ge X^{1/2}$, the term $N_I X^{1/4}$ is absorbed by $M_I N_I$. Thus

$$\sum_{n\in I_T}\sum_{m\in\widehat{I}_M} \tau(|mn-a|)^5 \ll M_I N_I L^{c_1}, \qquad c_1 = 2^{125}+2^{126}+2. \tag{3.20}$$

Here c_1 is an upper bound; the exponent $2^{125}+2^{126}-1$ already suffices.

For the prime source, Lemma B.3 takes T as the lower endpoint of the selected cell. Under threshold (B.24), it proves (B.27), namely $N_I \ge (2T)^{1-2^{-126}}$. In (3.17) take $Y' = \lfloor T \rfloor$ and $H' = N_I$. The integers in $I_T = (T, r_X T]$ of (B.22) are exactly $(Y', Y'+N_I]$. The stated bound implies $H' \ge (2Y')^{1-2^{-126}}$. The mesh uses the lower endpoint of the dyadic shell, which is a different scale. For the product source, LEN gives the lower length and the integer-interval convention. Thus (3.17) applies to $\tau_{2^{126}}$, after increasing $N_0(\sigma)$ for each fixed $\sigma > 0$. The same argument with $\varpi = \kappa/6$ treats intervals of length at least X^κ. It gives $\sum_m |\beta_m|^2 \ll M_I L^c$ by elementary means; Shiu's estimate [4] would also suffice.

Finite Heath–Brown identity [12]**.** For $U \ge 1$ and $n \le U^3$,

$$\Lambda(n) = \sum_{j=1}^{3}(-1)^{j-1}\binom{3}{j} \sum_{n=m_1\cdots m_j n_1\cdots n_j,\ m_i\le U} \mu(m_1)\cdots\mu(m_j)\log n_1. \tag{3.21}$$

For the proof put $M(s) = \sum_{m\le U}\mu(m)m^{-s}$. The series $1-\zeta M$ has no coefficients of index at most U. Hence $-(\zeta'/\zeta)(1-\zeta M)^3$ has none of index at most U^3. Expand the cube and use $-(\zeta'/\zeta)\zeta^j = (-\zeta')\zeta^{j-1}$. Let X be the dyadic shell of the tested coordinate, and $t \in (X, 2X]$. Take $U = 2X^{1/3}$. Then (3.21) applies, since $t \le 2X < U^3 = 8X$. For the product source use $U_i = 2X_i^{1/3}$ for the shell X_i of each coordinate. There are at most 24 variables. Roughness factors exactly over the product. The identity

$$1_{\mathbb{P}}(t) = \frac{\Lambda(t)}{\log t} - \sum_{k\ge 2,\ t=p^k}\frac{1}{k} \tag{3.22}$$

has its proper-prime-power error bounded at source level in §3.10.

Logarithms and boundaries. On one short box replace each $1/\log p$ by its value at the box reference point. The pointwise error is $O(L^{-C_{\mathrm{box}}-2})$ per coordinate. To sum the discrepancy absolutely, use

$$\sum_q |C_b(q)|\,|\mathcal{S}_t(q;a)| \ll_\sigma \tau(|t-a|) + L, \qquad t \ne a. \tag{3.23}$$

The first term counts divisors of $t-a$. The second uses $\sum_{q\le X^{0.6}} 1/\varphi(q) \ll L$. The unrestricted labelled coefficient is $O_\sigma(1)$. The number of prime labels is fixed, and the small-prime part is disjoint from them. For the prime source, the pointwise majorant in Appendix B gives (3.23) with $F_a(n) = d(a-n)+2(1+L)$. The source-wide logarithm-freezing estimate is (B.18): $|E_{\mathrm{LOG}}| \ll_\sigma NL^{18-C_{\mathrm{box}}}$. It is summed over tuples, each lying in one box, so its box-count factor is 1. Here the logarithm is $L_N = \log N$, and its divisor-moment input is (B.17). The box width is expressed

in $L_X = \log X$. Fixed powers of L_X and L_N are comparable. For the product source, the corresponding aggregate bound is a source condition stated in §3.10.

Write $L_N = \log N$. Consider the source cuts, the HB caps, and the monomial ordering and radial cuts of the product source. On each dyadic source shell, each boundary region has relative width $O(L_N^{-C_{\mathrm{box}}})$ in an active coordinate. Counting that coordinate first and summing the geometric shell scales gives a total of $O(NL_N^{5-C_{\mathrm{box}}} + N^{6/7}L_N^5)$ boundary indices, including the fixed divisor multiplicity. An active coordinate can be chosen among the positive-power factors, so the integer endpoint terms have a fixed power saving. For the prime source, Cauchy–Schwarz with this count and the source-wide second moment in (B.17) gives

$$E_{\mathrm{bdry}} \ll N(\log N)^{c_R - C_{\mathrm{box}}/2}. \tag{3.24}$$

For the prime source in Theorem 1.2, (3.24) holds with $c_R = 16$ by the global estimate (B.19) in Appendix B. This verifies (H6) for that source.

The prime-source estimate is invoked under Theorem 1.2, whose residue range supplies $a \geq N$. This lower bound is compatible with the local upper bound: $a \leq N\log N/3 \leq (2/3)X(\log N)^{B+1}$ when $X > N/(2(\log N)^B)$. The prime-source cut and $a \geq N$ give $|mn - a| \geq \varepsilon N/2$ on the doubled support. Thus resonant boxes are empty and (3.10) holds.

SW for the selected factor. For every $A_{\mathrm{SW}} > 0$, subinterval $I \subset I_T$, integer $d \geq 1$, and reduced class $l \bmod k$,

$$\sum_{n\in I,\ n\equiv l(k),\ (n,d)=1} \alpha_n - \frac{1}{\varphi(k)} \sum_{n\in I,\ (n,dk)=1} \alpha_n \ll_{A_{\mathrm{SW}}} T\tau(d)L^{-A_{\mathrm{SW}}}. \tag{3.25}$$

Here and below the interval indicator is used only once. Let $k > L^K$. After K is chosen large, the trivial bound is $O(T/k + 1 + T/\varphi(k)) \ll TL^{-A_{\mathrm{SW}}}$. The elementary bound $\varphi(k) \gg k/\log\log(k+2)$, or $\varphi(k) \gg k^{1/2}$, suffices. Let $k \leq L^K$ and put $y_1 = \exp(L^{2/3})$. For sufficiently large X, $z_0 > y_1$, so no selected integer has largest prime below y_1. Integers divisible by p^2, $p \geq y_1$, contribute $O(T/y_1)$. Every other integer has a unique form $n = Pm$, with $P > P^+(m)$, $P \geq y_1$, and $m \leq 2T/y_1$. For fixed m, its coefficient is $\mu(m)\mu(P) = -\mu(m)$, 1, or $\log(Pm)/L$. The last has bounded variation on the allowed prime interval. If $(m, dk) > 1$, both relevant counts vanish, or the coprimality with d removes the term. Otherwise $P \equiv l\bar{m}(k)$. Apply prime SW and partial summation, uniformly at endpoints $P \geq y_1$. The endpoint errors sum to $O_{A_{\mathrm{SW}}}(TL^{-A_{\mathrm{SW}}})$, since $\log y_1 = L^{2/3}$ and $\sum_{m\leq 2T/y_1} 1/m \ll L$. Excluding $P \mid dk$ costs at most $O((\omega(d)+\omega(k))T/y_1) \ll_{A_{\mathrm{SW}}} T\tau(d)L^{-A_{\mathrm{SW}}}$. Endpoint subtraction handles arbitrary I. This proves (3.25). All errors are absolute, of size $O(TL^{-A_{\mathrm{SW}}})$. Only the classical Siegel–Walfisz theorem is used for primes. The proof is uniform for every $\zeta \geq z_0$. A nonzero selected coefficient has all prime factors at least ζ. Raise the lower endpoint of the sum over the largest prime variable. Its SW bound is unchanged. If $\zeta > 2T$, the coefficient is zero.

3.5. **First dispersion with restoration before CRT.** Work first with the normalized convolution in (3.5). Split all three factors dyadically. Apply the triangle inequality before squaring. There are $O(L^3)$ triples, including the outer factor. For one triple put

$$\lambda_r = \sum_{r_3 r_2 = r} \gamma_3(r_3)\gamma_2(r_2), \quad r_3 \asymp R,\ r_2 \asymp S. \tag{3.26}$$

Since r is u_0-rough and at most $X^{0.6}$, $|\lambda_r| \le \tau(r) = O_\sigma(1)$. The arbitrary small-prime factor of C1 remains in the outer q, as the only smooth divisor. Let ψ_0 be a smooth nonnegative bump, equal to 1 on $[1,2]$, supported on $[1/2, 5/2]$. Cauchy–Schwarz in (q,m) gives

$$\begin{aligned} &|S|^2 \ll QM_I L^c S', \\ &S' = \sum_q \psi_0(q/Q) 1_{(q,a)=1} \sum_m \psi_I(m) \left| \sum_r \lambda_r \sum_n \alpha_n \mathcal{S}_{mn}(qr;a) \right|^2. \end{aligned} \tag{3.27}$$

Write $t = (q, r_1, r_2, n_1, n_2)$, $k_i = mn_i - a$, and

$$c(t) = \psi_0(q/Q) 1_{(q,a)=1} \lambda_{r_1} \bar{\lambda}_{r_2} \alpha_{n_1} \bar{\alpha}_{n_2}, \quad \omega(t) = 1_{(n_1, qr_1)=(n_2,qr_2)=1}. \tag{3.28}$$

Expanding the square gives $S' = S_1 - 2\Re S_2 + S_3$, where

$$\begin{aligned} S_3 &= \sum_t c(t) \sum_m \psi_I(m) 1_{qr_1|k_1} 1_{qr_2|k_2}, \\ S_2 &= \sum_t c(t) \sum_m \psi_I(m) 1_{qr_1|k_1} 1_{(mn_2,qr_2)=1}/\varphi(qr_2), \\ S_1 &= \sum_t c(t) \sum_m \psi_I(m) 1_{(mn_1,qr_1)=1} 1_{(mn_2,qr_2)=1}/(\varphi(qr_1)\varphi(qr_2)). \end{aligned} \tag{3.29}$$

Let $\mathcal{T}$ be the tuple set. Let $\mathfrak{B}$ be its union of cross-copy gcd failures

$$(n_1,n_2) > 1,\ (r_1,r_2) > 1,\ (n_1,r_2) > 1,\ (n_2,r_1) > 1,\ (q, r_1 r_2) > 1, \tag{3.30}$$

and let $\mathfrak{D} = \{|n_1 - n_2| < TL^{-C_M}\}$. Set $\mathfrak{G}' = \mathcal{T} \setminus (\mathfrak{B} \cup \mathfrak{D})$. On every tuple subset V define the formal sums

$$\begin{aligned} MT(V) &= \sum_{t \in V} c(t)\omega(t) \frac{M_I \widehat{\phi}_0(0)}{qr_1r_2\varphi(q)}, \\ \mathcal{Z}(V) &= \sum_{t \in V} c(t)\omega(t) 1_{n_1 \equiv n_2 (q)} \frac{M_I \widehat{\phi}_0(0)}{qr_1r_2}. \end{aligned} \tag{3.31}$$

Restoration lemma. Choose A_E, C_Σ, C_M in the order of §3.10. Then

$$S' = M_I E_2^* + \mathcal{Z}(\mathcal{T}) - MT(\mathcal{T}) + O(E_{\mathrm{GR}}), \tag{3.32}$$

$$\begin{aligned} E_{\mathrm{GR}} \ll{}& L^{3C_\Sigma + c} M_I N_I \left(\frac{N_I}{z_0 Q} + 1 \right) + C_\epsilon X^\epsilon M_I N_I \left(\frac{N_I}{u_0 Q} + 1 \right) \\ &+ \frac{M_I N_I^2}{QL^{A_E}}. \end{aligned} \tag{3.33}$$

We first prove the positive estimates needed for CRT. For S_i, let $\Upsilon_i(V)$ be the positive majorant. It replaces $c(t)$ by $|c(t)|$ and deletes coprimalities. Retain the two congruences for Υ_3, one for Υ_2. Let $|MT|(V), |\mathcal{Z}|(V)$ be the positive versions of (3.31). We write Υ_{exc} for the positive contribution of (S_3) from the complement of Σ^* defined in (3.37).

The total-variation progression estimate is

$$\left| \sum_{m \equiv b(k)} f(m) - k^{-1} \int f \right| \le V(f). \tag{3.34}$$

Compare $f(b + jk)$ to its cell average on $[b + jk, b + (j + 1)k]$. Bound the difference by the cell variation, and sum. Möbius expansion of the coprimality then gives $\sum_{(m,k)=1} \psi_I = (\varphi(k)/k) M_I \widehat{\phi}_0(0) + O(\tau(k))$.

From $n/\varphi(n) = \sum_{d|n} \mu^2(d)/\varphi(d)$ we get

$$\sum_{r \le Y} \frac{1}{\varphi(r)} \ll \log(2Y), \qquad \sum_{r \le Y, p|r} \frac{1}{\varphi(r)} \ll \frac{\log(2Y)}{p-1},$$

$$\sum_{q \asymp Q} \frac{1}{\varphi(q)^2} \ll L^2/Q, \qquad \sum_{q \asymp Q, p|q} \frac{1}{\varphi(q)^2} \ll L^2/(pQ). \tag{3.35}$$

For $\Upsilon_1, \Upsilon_2, |MT|$ the weight is bounded by $O_\sigma(M_I/(\varphi(q)^2 \varphi(r_1)\varphi(r_2)))$, since $qr_1 \ll QRS < M_I$. For $|\mathcal{Z}|$ it is $O_\sigma(M_I/(qr_1r_2)) 1_{n_1 \equiv n_2(q)} \omega(t)$. These two shapes prove

$$\Upsilon_1(\mathfrak{B}) + \Upsilon_2(\mathfrak{B}) + |MT|(\mathfrak{B}) + |\mathcal{Z}|(\mathfrak{B}) \ll L^c M_I N_I \left(\frac{N_I}{z_0 Q} + 1 \right). \tag{3.36}$$

For the first failure of (3.30), fix n_1. Choose one of its at most L primes $p \ge z_0$. Count n_2 in multiples of p. The first shape gives $N_I/z_0 + 1$. For the weight of $|\mathcal{Z}|$, $(n_1, q) = 1$. So the extra congruence gives a class modulo pq, with $N_I/(pq) + 1$ choices. For a common prime $p \ge u_0$ of r_1, r_2, fix r_1. Sum the p-divisible r_2 weight using (3.35). For $p \mid (n_1, r_2)$, fix n_1 and do the same weighted r_2 sum. For $p \mid (n_2, r_1)$, fix r_1. Then count n_2 in multiples of p, or modulo pq for the weight of $|\mathcal{Z}|$. For $p \mid (q, r_i)$, use the last estimate of (3.35). This covers all five failures, and $u_0 \ge z_0$ gives (3.36).

Split the unrestricted (t, m)-sum at

$$\Sigma^* = \{\tau(|k_1|)\tau(|k_2|) < L^{C_\Sigma}\}. \tag{3.37}$$

The complement is estimated on all tuples, coprime or not. Put $u = \tau(|k_1|)$ and $v = \tau(|k_2|)$. The exceptional indicator is at most uvL^{-C_Σ}. The two divisor choices $r_i \mid k_i$ cost at most uv. There are also at most u choices for $q \mid k_1$. The resulting weight is u^3v^2, of degree five. To keep the progression restriction while counting, split into $u \ge v$ and $v > u$. On the first part $u^3v^2 \le u^5$. For each fixed (m, n_1), sum over the at most u divisors q first. The condition $q \mid k_1$ and $(q, a) = 1$ implies $(m, q) = 1$. Thus $q \mid k_2$ forces $n_2 \equiv n_1 \pmod q$. There are at most $4N_I/Q$ such n_2, by (3.6). This bounds the first part by $L^{-C_\Sigma}(4N_I/Q)\sum u^5$. Exchanging the two copies $(n_1, r_1) \leftrightarrow (n_2, r_2)$ gives the same bound for the second part. Consequently

$$\Upsilon_{\mathrm{exc}} \ll L^{-C_\Sigma}\frac{N_I}{Q}\sum_{n\in I_T, m\in\widehat{I}_M}\tau(|mn-a|)^5 \ll \frac{M_I N_I^2}{Q}L^{c_1-C_\Sigma}. \tag{3.38}$$

The factor 4 from $4N_I/Q$ and the factor 2 from the two parts are absorbed in the implied constant.

For a modulus-related failure, fix (m, n_1). The number of (q, r_1) with $qr_1 \mid k_1 \neq 0$ is $\tau_3(|k_1|) \le \tau(|k_1|)^2 \ll_\epsilon X^{\epsilon/3}$. A common prime $p \ge u_0$ comes from r_1, n_1, or q. It forces n_2 into one class modulo pq. For $p \mid r_1, r_2$ use $p \nmid m$. For $p \mid n_1, r_2$ use $mn_2 \equiv a(p)$. For $p \mid n_2, r_1$ use $n_2 \equiv 0(p)$. If $p \mid q, r_2$, then $pq \mid qr_2 \mid k_2$ gives the class directly. If $p \mid q, r_1$ only, exchange the copies. There are $O(L)$ candidate primes and $O(N_I/(u_0Q) + 1)$ choices in each class. There are at most $\tau(|k_2|)$ remaining choices of r_2. Hence

$$\Upsilon_3(\mathfrak{B}_{\mathrm{modulus}}) \ll_\epsilon X^\epsilon M_I N_I(N_I/(u_0Q) + 1). \tag{3.39}$$

For $(n_1, n_2) > 1$ on Σ^*, there are at most L^{2C_Σ} pairs (q, r_1). There are at most L primes $p \mid n_1$, $p \ge z_0$. There are at most $2N_I/(z_0Q) + 1$ choices for n_2. There are at most L^{C_Σ} choices for r_2. Thus

$$\Upsilon_3(\mathfrak{B}_n \cap \Sigma^*) \ll L^{3C_\Sigma+1}M_I N_I(N_I/(z_0Q) + 1). \tag{3.40}$$

On the near-diagonal the same count replaces N_I/z_0 by TL^{-C_M}. The density shapes and (3.35) give the same or smaller bound. Therefore

$$\begin{aligned}\Upsilon_3(\mathfrak{D}\cap\Sigma^*) + \Upsilon_1(\mathfrak{D}) + \Upsilon_2(\mathfrak{D}) + |MT|(\mathfrak{D}) + |\mathcal{Z}|(\mathfrak{D})\\ \ll L^{3C_\Sigma+c}M_I N_I(TL^{-C_M}/Q + 1).\end{aligned} \tag{3.41}$$

Use the common constants from §3.10: $C_\Sigma = A_E + 2c_* + 3$ and $C_M = 5C_\Sigma + A_E + C_{\mathrm{box}} + c_* + 3$. They satisfy $C_\Sigma > A_E + c_1 + 2$ and $C_M > 3C_\Sigma + C_{\mathrm{box}} + A_E + c$. To check this, divide by the last term of (3.33). After this division, the right side of (3.41) is at most

$$L^{3C_\Sigma+c+A_E}\left(\frac{TL^{-C_M}}{N_I} + \frac{Q}{N_I}\right) \ll L^{3C_\Sigma+c+A_E+C_{\mathrm{box}}-C_M} + L^{3C_\Sigma+c+A_E}X^{-\sigma/3} = o(1).$$

Here LEN supplies $T/N_I \ll L^{C_{\mathrm{box}}}$, while (3.6) supplies $Q/N_I \le X^{-\sigma/3}$. The chosen C_M makes the first logarithmic exponent negative; the second term tends to zero for fixed σ. The same target absorbs (3.38), since $C_\Sigma > A_E + c_1 + 2$. This removes bad tuples and the near-diagonal from **all** of S_1, S_2, S_3, Z, MT before their main terms are compared.

Now use CRT on $\mathfrak{G}'$. Let $Q_1 = qr_1, Q_2 = qr_2, g = (r_1, r_2)$. The two congruences have one class modulo $[Q_1, Q_2] = qr_1r_2/g$ precisely when $(n_1, Q_1) = (n_2, Q_2) = 1$ and $n_1 \equiv n_2(qg)$. To see this, equate $a\bar{n}_1$ and $a\bar{n}_2$ modulo $(Q_1, Q_2) = qg$. On $\mathfrak{G}'$, $g = 1$ and q, r_1, r_2 are pairwise coprime.

Fixed cutoff. On the smoothed support put

$$k(t) = qr_1r_2 \le K_* = 40QR^2S^2, \quad H_2 = QR^2S^2X^\eta/M_I, \quad H_* = 40H_2. \tag{3.42}$$

These are fixed for the entire shell triple. Poisson gives exactly

$$
\begin{aligned}
E_2^* &= \sum_{t\in\mathfrak{G}'} \frac{c(t)\omega(t)1_{n_1\equiv n_2(q)}}{k(t)} \\
&\quad\times \sum_{1\le|h|\le H_*} \widehat{\phi}_0(hM_I/k(t))e(h(b(t)-m_0)/k(t)), \\
S_3(\mathfrak{G}') &= \mathcal{Z}(\mathfrak{G}') + M_I E_2^* + O(X^{-90}).
\end{aligned}
\tag{3.43}
$$

Put $\lambda = M_I/k(t)$. This is the Poisson prefactor of one progression sum. In (3.43) it appears as the factor M_I in front of E_2^* and the factor $1/k(t)$ inside E_2^*. Integration by parts gives $|\widehat{\phi}_0(u)| \le C_j|u|^{-j}$.

Here one may take $C_j = (2\pi)^{-j}\|\phi_0^{(j)}\|_1$. It is independent of the box and the tuple. The integer $j = \lceil 104/\eta\rceil + 1$ is fixed once σ is fixed. Choose X_0 after these seminorms and all fixed coefficient parameters. So the tail bound is uniform over the whole box family. If $H_* < 1$, then $\lambda > X^\eta$, since $\lambda H_* \ge X^\eta$.

The tail of one progression sum is at most $C_j'(\lambda H_*)^{1-j}$ if $H_* \ge 1$, and at most $C_j'\lambda^{1-j}$ otherwise. In either case it is $O(X^{-\eta(j-1)})$, since $\lambda H_* \ge X^\eta$. Take $j = \lceil 104/\eta\rceil + 1$. After increasing X_0, the weighted tail per tuple is at most X^{-103}. There are at most X^5 tuples. This proves (3.43). The cap (3.42) is the same for all tuples. So it survives the Mellin separation of the inner variable below.

On $\mathfrak{G}'$, (3.34) gives

$$
S_1(\mathfrak{G}') = MT(\mathfrak{G}') + O(N_I^2L^c/Q).
\tag{3.44}
$$

For S_2, retain $(m, r_2) = 1$. Möbius expansion over $e \mid r_2$ gives one class modulo qr_1e, so

$$
\sum_m \psi_I(m)1_{mn_1\equiv a(qr_1)}1_{(m,r_2)=1} = \frac{M_I\widehat{\phi}_0(0)\varphi(r_2)}{qr_1r_2} + O(\tau(r_2)).
\tag{3.45}
$$

Divide by $\varphi(qr_2) = \varphi(q)\varphi(r_2)$. The main term is again MT. The summed error is $O(RSN_I^2L^c)$. It is absorbed using $QRS < M_IX^{-89\sigma}$. Exchanging the two copies shows that $MT(\mathfrak{G}')$ is real. Complete $\mathcal{Z}$ and MT formally to $\mathcal{T}$, with errors bounded by (3.36), (3.41). This proves (3.32). CRT and the zero-mode identity are used on $\mathfrak{G}'$ only.

3.6. **Main-term cancellation and the full GR error.** Put $\alpha_n^{(r)} = \alpha_n 1_{(n,r)=1}$ and

$$
\Delta(\xi; q, b) = \sum_{n\equiv b(q)} \xi_n - \varphi(q)^{-1} \sum_{(n,q)=1} \xi_n.
\tag{3.46}
$$

Expanding the product of these two centered counts gives exactly

$$
\begin{aligned}
\mathcal{Z}(\mathcal{T}) - MT(\mathcal{T}) = &\sum_{q,r_1,r_2} \frac{\psi_0(q/Q)1_{(q,a)=1}\lambda_{r_1}\bar{\lambda}_{r_2}M_I\widehat{\phi}_0(0)}{qr_1r_2} \\
&\sum_{b \bmod q}^{*} \Delta(\alpha^{(r_1)}; q, b)\overline{\Delta(\alpha^{(r_2)}; q, b)}.
\end{aligned}
\tag{3.47}
$$

For the norm-normalized condition [6, (A2)], apply (3.25) to dr, with $\tau(dr) \le \tau(d)O_\sigma(1)$. If $\|\alpha^{(r)}\|_2^2 \le N_I L^{-A_3-1}$, the elementary estimate

$$\sum_{q\le 3Q}\sideset{}{^*}\sum_{b} |\Delta(\alpha^{(r)};q,b)|^2 \ll (T\log(3Q)+3Q)\|\alpha^{(r)}\|_2^2 \ll N_I T L^{-A_3} \tag{3.48}$$

suffices, since $\log(3Q) \le L$ is absorbed by the factor L^{-1}. If $\|\alpha^{(r)}\|_2^2 > N_I L^{-A_3-1}$, then $\|\alpha^{(r)}\|_2(2T)^{1/2} \gg TL^{-(C_{\mathrm{box}}+A_3+1)/2}$. Then (3.25) with exponent $A_{\mathrm{BFI}} + (C_{\mathrm{box}}+A_3+1)/2$ implies [6, (A2)]. The condition $Q < TX^{-29\sigma}$ implies the condition $3Q \le T(\log T)^{-B_1}$ of [6, Theorem 0(a)] eventually, which then gives (3.48) in this case too. Apply $|uv| \le (|u|^2+|v|^2)/2$ in (3.47), and sum the reciprocal r_i weights. This gives

$$|\mathcal{Z}(\mathcal{T}) - MT(\mathcal{T})| \ll M_I N_I^2 L^{C_{\mathrm{box}}+2-A_3}/Q. \tag{3.49}$$

Take $A_3 = A_E + C_{\mathrm{box}} + 2$. Increase it by a fixed constant to absorb any endpoint logarithm. Combine (3.27), (3.32), (3.33), (3.49), with $\epsilon = \ell_0\sigma^2/8$. For a requested local precision $A' > 0$, set $A_{\mathrm{work}} = A' + 4$ before choosing A_E, C_Σ, C_M as in §3.10. The resulting restoration contributions to $|S|$ are at most

$$XL^{-A_{\mathrm{work}}} + XL^C z_0^{-1/2} + XL^C(Q/N_I)^{1/2} + XX^{\epsilon/2}L^C\{u_0^{-1/2} + (Q/N_I)^{1/2}\}. \tag{3.50}$$

Both terms in braces are kept. Their power savings are at least $7\ell_0\sigma^2/16$ and $\sigma/6 - \ell_0\sigma^2/16$, respectively. Each exceeds c_σ, and $z_0^{-1/2}$ beats every logarithm. More explicitly, for fixed A_{work} and sufficiently large X, $XL^C z_0^{-1/2} \ll XL^{-A_{\mathrm{work}}}$, because $z_0^{-1/2} = X^{-1/(2(\log\log X)^3)}$ beats every fixed logarithm. Thus this term joins the first term in (3.51), rather than the fixed-power-saving term. The separate term $XL^C(Q/N_I)^{1/2}$ in (3.50) has saving $\sigma/6 > c_\sigma$, by $N_I \ge QX^{\sigma/3}$ in (3.6), and is therefore absorbed in the displayed power-saving term. Increase X_0 after the fixed exponents. This gives

$$|S| \ll XL^{-A_{\mathrm{work}}} + M_I Q^{1/2} L^c |E_2^*|^{1/2} + X^{1-c_\sigma}L^C. \tag{3.51}$$

This completes the first dispersion. A later saving for E_2^* changes only the middle term.

3.7. **Reciprocity, the complete mask, and the second square.** Split the combined products again as $r_i \leftarrow r_i s_i$, where $r_i \asymp R, s_i \asymp S$. The remaining support is

$$(q, r_1s_1r_2s_2) = (r_1s_1, r_2s_2) = (n_1,n_2) = 1,$$

$$(n_1, r_2s_2) = (n_2, r_1s_1) = 1, \quad (n_i, qr_is_i) = 1, \quad |n_1 - n_2| \ge TL^{-C_M}. \tag{3.52}$$

Solving the two CRT congruences and using $\bar{u}/v + \bar{v}/u \equiv 1/(uv) \pmod 1$ for $(u,v) = 1$ gives

$$\frac{ah\overline{n_1r_2s_2}}{qr_1s_1} + \frac{ah\overline{n_2qr_1s_1}}{r_2s_2} \equiv \frac{ah(n_1-n_2)\overline{n_2qr_1s_1}}{n_1r_2s_2} + \frac{ah}{qr_1s_1r_2s_2n_1} \pmod 1. \tag{3.53}$$

All inverses exist by (3.52). Put $f = (n_1 - n_2)/q \neq 0$. Then the outer tuple $\omega = (f, n_1, n_2, r_1, r_2, s_2)$ determines $q = (n_1 - n_2)/f$. The modular phase becomes

$$e\left(\frac{ahf\overline{n_2 r_1 s_1}}{n_1 r_2 s_2}\right). \tag{3.54}$$

Split $f \neq 0$ by sign and dyadic intervals $f \in \varsigma_f(F, 2F]$, starting at $F = 1/2$. Split $h \neq 0$ by sign and intervals $H_0 < |h| \leq \min(2H_0, H_*)$, starting at $H_0 = 1/2$. These disjoint intervals include $f, h = \pm 1$ exactly once. Write $\varsigma_f, \varsigma_h \in \{\pm 1\}$ for the signs of f and h. The cap on the last, shortened h-block depends on h alone. Always $F \ll T/Q$.

The nonmodular factor is exactly

$$\widehat{\phi}_0\left(\frac{hM_I}{qr_1s_1r_2s_2}\right) e\left[\frac{h}{qr_1s_1r_2s_2}\left(\frac{a}{n_1} - m_0\right)\right]. \tag{3.55}$$

Its dependence on (h, s_1) is only through $|h|/s_1$. For $v = |h|S/(H_0 s_1) \in [1/2, 2]$, choose a fixed smooth cutoff. It equals 1 there and is supported in $(1/4, 4)$. Rewrite (3.55):

$$\begin{gathered} W_{Y(\omega)}(v) = w(v)\widehat{\phi}_0(Y_2 v)e(Y_1 v), \\ Y_1 = \frac{\varsigma_h H_0(a/n_1 - m_0)}{Sqr_1r_2s_2}, \qquad Y_2 = \frac{\varsigma_h H_0 M_I}{Sqr_1r_2s_2}. \end{gathered} \tag{3.56}$$

Since $H_0 \leq H_* = 40H_2$,

$$|Y_1| + |Y_2| \ll (1 + a/X)L^{C_{\mathrm{box}}}X^\eta \ll L^{C_{\mathrm{box}}+K_{\mathrm{scale}}}X^\eta =: L_{\mathrm{osc}}. \tag{3.57}$$

Mellin inversion separates the ratio. Put $m(t) = \sup_\omega |\widetilde{W}_{Y(\omega)}(it)|$. The zeroth and second logarithmic derivative estimates give

$$m(t) \ll \min\{1, L_{\mathrm{osc}}^2/(1 + |t|)^2\}, \qquad \int_{\mathbb{R}} m(t)\, dt \ll L_{\mathrm{osc}}. \tag{3.58}$$

If $m(t) > 0$, divide the outer coefficient by $m(t)$. The norm bounds below are for this normalized coefficient. The factor $m(t)$ is restored before the t-integration. If $m(t) = 0$, the integrand is zero. So the integrable decay of m is kept.

The full inner coprimality mask is

$$(s_1, ac) = 1, \qquad (s_1, J_\omega) = 1, \quad c = n_1r_2s_2, \quad J_\omega = qn_2. \tag{3.59}$$

The second condition combines $(s_1, q) = 1$ and $(s_1, n_2) = 1$. To separate the second, split $\mathcal{P}_X = [u_0, X^3] \cap \mathbb{P}$ at random. Each prime goes independently and uniformly into one of two classes B_χ, R_χ. For an integer m, let $\mathcal{P}_X(m) = \{p \in \mathcal{P}_X : p \mid m\}$ and $\nu_X(m) = |\mathcal{P}_X(m)|$. If all prime factors of s lie in $\mathcal{P}_X$, then exactly

$$1_{(s,J)=1} = \mathbb{E}_\chi[2^{\nu_X(s)+\nu_X(J)} 1_{\mathcal{P}_X(s)\subseteq B_\chi} 1_{\mathcal{P}_X(J)\subseteq R_\chi}]. \tag{3.60}$$

A common prime makes the event impossible. Otherwise the event has probability $2^{-\nu_X(s)-\nu_X(J)}$. Every prime of a supported s_1 lies in $\mathcal{P}_X$. So $(s_1, J_\omega) = 1$

depends only on the primes of J_ω in $\mathcal{P}_X$. Since $s_1 \le X^{0.6}$ and $J_\omega \le X^{0.94}$, one may take

$$K_s = \lceil 0.6/(\ell_0\sigma^2)\rceil, \qquad K_J = \lceil 1/(\ell_0\sigma^2)\rceil. \tag{3.61}$$

Replace the inner coefficient by $\lambda_s^\chi = \lambda_s 2^{\nu_X(s)-K_s}1_{\mathcal{P}_X(s)\subseteq B_\chi}$ and each outer representation coefficient by $A_\omega^\chi = A_\omega 2^{\nu_X(J_\omega)-K_J}1_{\mathcal{P}_X(J_\omega)\subseteq R_\chi}$. Both multipliers have modulus at most 1. The original sum is $2^{K_s+K_J}\mathbb{E}_\chi W_\chi$. This factor is $O_\sigma(1)$, and $\mathbb{E}_\chi$ is an average. Fix χ before aggregating representations. The partition acts on λ_s and A_ω only, so (3.25) for α_n holds for each χ.

Set $b = n_2r_1, c = n_1r_2s_2$, and

$$I(b,c,f) = \sum_{s\asymp S,(s,ac)=1}\sum_h \lambda'_{s,t}\eta_{h,t}e(ahf\overline{bs}/c), \tag{3.62}$$

where $\lambda'_{s,t} = (S/s)\lambda_s^\chi(s/S)^{it}$, and $\eta_{h,t} = (|h|/H_0)^{-it}1_{|h|\le H_*}$ on the chosen signed block. Both are 1-bounded. Dyadically localize $b \asymp D, c \asymp C$. The outer support has $(b,c) = (b,p_0) = 1$. So the nonnegative majorant

$$\rho = \psi_0(b/D)\psi_0(c/C)1_{(b,c)=(b,p_0)=1}1_{f\in\varsigma_f(F,2F]} \tag{3.63}$$

equals 1 on it. Cauchy–Schwarz now gives

$$|W_{\chi,t}| \le \left(\sum_{b,c,f}|A^\chi_{b,c,f}|^2\right)^{1/2} W_4^{1/2}, \quad W_4 = \sum_{b,c,f}\rho|I|^2. \tag{3.64}$$

Fix (b,c,f). Factor $b = n_2r_1, c = n_1r_2s_2$; then q is determined. So there are at most $\tau(b)\tau_3(c)$ outer representations. For fixed n_1, n_2 the permitted $q \mid n_1 - n_2$ have divisor multiplicity. So there are $O(T^2R^2S)X^{o(1)}$ representations in total. A normalized representation coefficient has modulus $O_\sigma((QR^2S^2)^{-1})$. Summing over representations with Cauchy–Schwarz,

$$\left(\sum|A^\chi|^2\right)^{1/2} \ll X^{o(1)}\frac{T}{QRS^{3/2}}. \tag{3.65}$$

This bound holds for the coefficients A^χ of each class χ.

Expanding $|I|^2$, put

$$z = ss', \quad k = h_1s' - h_2s, \quad y = fk. \tag{3.66}$$

The terms with $k = 0$ have at most $X^{o(1)}H_*S$ choices of (s,s',h_1,h_2). To see this, write $(s,s') = g(u,v)$, $(u,v) = 1$, and $(h_1,h_2) = j(u,v)$. Sum (g,j) over their ranges. Thus

$$|W_{k=0}| \ll X^{o(1)}H_*SDFC \ll (T^2R^2S^3)X^{-8\sigma+\eta+o(1)}, \tag{3.67}$$

where the final inequality is $T^2R^2S < X^{1-8\sigma}$. For $k \ne 0$, split $y = \varepsilon_y m$, $m > 0$, and z into $Z \in \{S^2, 2S^2\}$. The coefficient independent of (b,c) is

$$\beta_{z,m}^{\chi,\varepsilon_y} = 1_{(z,ap_0)=1} \sum_{ss'=z} \lambda'_{s,t}\bar{\lambda}'_{s',t} \sum_{\substack{f\in\varsigma_f(F,2F],\ h_i \text{ in block} \\ f(h_1s'-h_2s)=\varepsilon_y m}} \eta_{h_1,t}\bar{\eta}_{h_2,t}. \tag{3.68}$$

Put $Y := H_*SF$. By (3.70), $|y| \le 8FH_*S$. So the coefficient is supported on $1 \le m \le 8Y$. The off-diagonal sum is exactly

$$\sum_{z,m} \beta_{z,m}^{\chi,\varepsilon_y} \sum_{b,c:(b,c)=(b,p_0)=(z,c)=1} \psi_0(b/D)\psi_0(c/C)e(\varepsilon_y am\overline{zb}/c). \tag{3.69}$$

3.8. **The W-prime norm, with endpoints.** For $S \ge 1$, $1 \le H \le S$, $F \ge 1/2$, and $a \ne 0$, let

$$\begin{aligned} B_{z,y} = \#\{&s_i \in (S,2S],\ (s_1s_2,a)=1, \\ &1 \le |h_i| \le H,\ f \in (F,2F] : \\ &s_1s_2 = z,\ f(h_1s_2-h_2s_1) = y \ne 0\}. \end{aligned} \tag{3.70}$$

For every nonzero integer v, define its a-supported part by $v_a = \prod_{p|a} p^{v_p(|v|)}$. In particular $y_a = \prod_{p|a} p^{v_p(|y|)}$. Set

$$\begin{aligned} \mathcal{T}_0 = \max \tau(z)\tau(|y|), \qquad P_0 = \prod_{p|a}(1-p^{\vartheta_{\mathrm{sp}}-1})^{-1}, \\ P_1 = P_0^2, \qquad P_2 = \prod_{p|a}(1-p^{\vartheta_{\mathrm{sp}}-1/2})^{-1}, \end{aligned} \tag{3.71}$$

where the maximum is over $z \le 4S^2, 0 < |y| \le 8FHS$. Then

$$\sum_{z,y\ne 0} B_{z,y}^2 (y_a)^{\vartheta_{\mathrm{sp}}} \le \mathcal{T}_0\, 2FP_0H^2S^2[16P_1 + 64(8+\log(2S))P_0 + 32P_2]. \tag{3.72}$$

Proof. For fixed $s = (s_1, s_2)$, let $N_s(k) = \#\{h : 1 \le |h_i| \le H, h_1s_2 - h_2s_1 = k\}$. Each (z, y) has at most $\tau(z)\tau(|y|)$ choices of (s, f). Therefore Cauchy–Schwarz and $(fk)_a = f_ak_a$ reduce the left side to

$$\mathcal{T}_0 \left(\sum_{f\le 2F} f_a^{\vartheta_{\mathrm{sp}}} \right) Q_0, \qquad Q_0 = \sum_s \sum_{k\ne 0} k_a^{\vartheta_{\mathrm{sp}}} N_s(k)^2, \quad \sum_{f\le 2F} f_a^{\vartheta_{\mathrm{sp}}} \le 2FP_0. \tag{3.73}$$

For the last inequality, majorize $f_a^{\vartheta_{\mathrm{sp}}}$ by $\sum_{d|a^\infty, d|f} d^{\vartheta_{\mathrm{sp}}}$. Then sum $\lfloor 2F/d \rfloor \le 2F/d$. It holds for every $F > 0$.

Put $g = (s_1, s_2)$, $s = g(u, v)$, $S' = S/g$. A fibre of $h_1s_2 - h_2s_1$ has at most $1 + 2H/S'$ points. It has at most one when $S' \ge 2H$. Expand

$$k_a^{\vartheta_{\mathrm{sp}}} = \sum_{d|a^\infty, d|k} \varphi_{\vartheta_{\mathrm{sp}}}(d), \quad \varphi_{\vartheta_{\mathrm{sp}}}(p^j) = p^{j\vartheta_{\mathrm{sp}}}(1-p^{-\vartheta_{\mathrm{sp}}}), \quad 0 \le \varphi_{\vartheta_{\mathrm{sp}}}(d) \le d^{\vartheta_{\mathrm{sp}}}. \tag{3.74}$$

Here $\varphi_{\vartheta_{\mathrm{sp}}}$ is multiplicative with $\varphi_{\vartheta_{\mathrm{sp}}}(1) = 1$; the prime-power formula is for $j \ge 1$.

If $A_d(s)$ counts $h \in [-H, H]^2$ with $k \ne 0, d \mid k$, then

$$Q_0 \le \sum_{d|a^\infty, d\le 4HS} d^{\vartheta_{\rm sp}} \left[\sum_s A_d(s) + \sum_{s: S'<2H} \frac{2H}{S'} A_d(s) \right]. \tag{3.75}$$

For the first sum write $h = ew$, w primitive, $\|w\|_\infty = j \le H$. There are at most $8j$ choices of w, and $e \le H/j$. The congruence is $d_e \mid w_1 s_2 - w_2 s_1$, $d_e = d/(d,e)$. Since $(s_1 s_2, d_e) = 1$, a contributing primitive vector has $(w_1, d_e) = 1$. The case $w_1 = 0$ forces $d_e = 1$. There are at most $2S$ choices for s_1, then $S/d_e + 1$ for s_2. By $\sum_{e\le E}(d,e) \le \tau(d)E$, this proves

$$\sum_s A_d(s) \le 16\tau(d) H^2 S^2/d + 16H^2 S. \tag{3.76}$$

For the second sum $(d,g) = 1$. At most $8HS'/d$ nonzero levels of $h_1 v - h_2 u$ are divisible by d. Each has at most $4H/S'$ points. Thus $A_d(s) \le 32H^2/d$. For each g, sum the at most $(S/g+1)^2$ pairs. Use the exact inequality

$$\sum_{g\le 2S} \frac{(S/g+1)^2}{S/g} = \sum_{g\le 2S} (S/g + 2 + g/S) \le S(8 + \log(2S)). \tag{3.77}$$

So before the d-sum, the second term of (3.75) is at most $64(8+\log(2S))H^3 S/d \le 64(8+\log(2S))H^2S^2/d$, using $H \le S$. Finally

$$\begin{aligned} \sum_{d|a^\infty} \tau(d) d^{\vartheta_{\rm sp}-1} &= P_1, \\ \sum_{d|a^\infty} d^{\vartheta_{\rm sp}-1} &= P_0, \\ \sum_{d\le 4HS, d|a^\infty} d^{\vartheta_{\rm sp}} &\le (4HS)^{1/2} P_2. \end{aligned} \tag{3.78}$$

The last contribution of (3.76) is at most $32H^2S^2P_2$, since $H \le S$. This proves (3.72), also for real nonintegral (S,H).

Uniformly for $|a| \le X^2$, elementary prime ordering gives

$$\begin{aligned} P_0, P_1, P_2 &\le \exp(C(\log X)^{23/32}) = X^{o(1)}, \\ \mathcal{T}_0 &\le \exp(C \log X/\log\log X) = X^{o(1)}. \end{aligned} \tag{3.79}$$

For $\alpha \in \{1-\vartheta_{\rm sp}, 1/2-\vartheta_{\rm sp}\}$ we have $\sum_{p|a} p^{-\alpha} \le \sum_{j\le\omega(a)} j^{-\alpha} \ll (\log X)^{1-\alpha}$. The bound for the products then follows from $-\log(1-v) \ll v$. The divisor bound is elementary.

To apply (3.72) to (3.68), use the cap $H = \min(2H_0, H_*)$. Empty blocks have zero coefficient. Increase X_0 in (3.6)–(3.7). Then

$$H_2/S \ll X^{-9\sigma+\eta} L^{C_{\rm box}}, \qquad H_* \le S. \tag{3.80}$$

The strict power margin in (3.6)–(3.7) absorbs the fixed logarithmic factor and the factor 40. For $S = 1/2$ this gives $H_* = 40H_2 \ll X^{-9\sigma+\eta} L^{C_{\rm box}} \to 0$. If the middle factor is supported at $s = 1 \in (1/2, 1]$, set $S = 1/2$. Then $H_* < 1$, every

nonzero-frequency block is empty, and $E_2^* = 0$. Thus Theorem 2.8 is needed only for $S \geq 1$; in that case a nonempty block has $H \geq 1$. Each summand of (3.68) has modulus at most 1, so $|\beta_{z,m}| \leq B_{z,\varepsilon_y \varsigma_f m}$. Since $p_0 \nmid a$, $(p_0 m)_a = m_a$ exactly. Thus

$$\sum_{n'' | a^\infty} (n'')^{\vartheta_{\mathrm{sp}}} \|\beta^\flat(n'')\|_2^2 \ll X^{o(1)} (H_* S)^2 F, \qquad \|\beta\|_2^2 \ll X^{o(1)} (H_* S)^2 F. \tag{3.81}$$

The second estimate is the same proof with $\vartheta_{\mathrm{sp}} = 0$. Here $S \geq 1$, as (3.72) requires; the case $S = 1/2$ was treated above. If γ_1 is supported at 1, take $q = 1, Q = 1$. If γ_3 is supported at 1, it is a fixed coordinate. If the full modulus is 1, the discrepancy is identically zero.

3.9. **Application of Theorem 2.8 and the spectral budget.** For nonzero (3.69), set

$$\begin{aligned} &r_T = z \asymp Z,\ s_T = p_0,\ R_T = Z,\ S_T = p_0/2, \\ &d_T = b,\ c_T = c,\ n_T = p_0 m,\ N_T = 8 p_0 Y, \quad a_T = \varepsilon_y a. \end{aligned} \tag{3.82}$$

Let the Theorem 2.8 coefficient equal $\beta_{z,m}$ at these indices and zero elsewhere. The support conditions give exactly

$$(zb, p_0 c) = 1 \iff (b,c) = (z,c) = (z,p_0) = (b,p_0) = 1, \qquad (a_T, z p_0) = 1. \tag{3.83}$$

Here $(z,c) = 1$ comes from $(s, ac) = (s', ac) = 1$. The p_0 conditions on (b,z) come from their roughness. The smoothed variable c may be divisible by p_0. Reducing the inverse modulo c gives the exact phase identity

$$e\left(\frac{a_T (p_0 m) \overline{z b}_{p_0 c}}{p_0 c}\right) = e\left(\frac{\varepsilon_y a m \overline{z b}_c}{c}\right). \tag{3.84}$$

We make the dyadic support of the smooth weight explicit. Choose a fixed smooth step χ. It is zero on $(-\infty, 0]$ and one on $[1, \infty)$. Split

$$\begin{aligned} \psi_{0,1}(v) &= \psi_0(v)[1 - \chi((v - 0.8)/0.1)], \\ \psi_{0,2}(v) &= \psi_0(v)\chi((v - 0.8)/0.1)[1 - \chi((v - 1.4)/0.2)], \\ \psi_{0,3}(v) &= \psi_0(v)\chi((v - 1.4)/0.2). \end{aligned} \tag{3.85}$$

They sum to ψ_0. They lie in the strict dyadic windows $(0.45, 0.90]$, $(0.80, 1.60]$, $(1.40, 2.80]$. Expanding the (b,c) bumps gives nine weights. Multiply by fixed smooth cutoffs equal to 1 on the supported n_T, z, p_0 indices. At the lower n_T endpoint use a cutoff zero below $1/2$ and one above 1. Its derivatives are $O_\nu(n_T^{-\nu})$. Each weight satisfies the Theorem 2.8 derivative conditions with $\epsilon_0 = \sigma/2000 > 0$. Take $\epsilon = \sigma/100$, $A_T = A_0$. Thus (3.14), exactly as stated, applies to (3.69).

We bound the four terms of its bracket. Put

$$\begin{aligned} V = T^4 R^4 S^6, \quad D_* = TR, \quad C_* = TRS, \\ Z \asymp S^2, \quad Y \ll X^{o(1)+\eta} T^2 R^2 S^3 / X. \end{aligned} \tag{3.86}$$

Since $\vartheta_{\mathrm{sp}} < 1/2$, all powers of (D, C) in (3.15) are nonnegative. First replace them monotonically by D_*, C_*. Then $C_* \le D_* Z$, and $Y \ll ZX^{-8\sigma+\eta+o(1)}$. Apply (3.81). Suppress only uniform $X^{o(1)}$ and p_0-powers. This gives

(3.87)

bracket term/V	upper bound
1	$K_A = A_0^{\vartheta_{\mathrm{sp}}} TQR^2S^5/X^2$
2	$(A_0/X)K_A$
3	$K_R = (TR/S)^{2\vartheta_{\mathrm{sp}}} TQR^2S^5/X^2$
4	$(TQR^2S^5/X^2)(T^2R^2/X)$.

We check the regular third term directly. Up to p_0-powers, its bracket before the coefficient norm is

$$\begin{aligned} C(CD\sqrt{Z})^{2\vartheta_{\mathrm{sp}}} Z^{1-\vartheta_{\mathrm{sp}}} (DZ)^{1-2\vartheta_{\mathrm{sp}}} &= CDZ^2 (C/Z)^{2\vartheta_{\mathrm{sp}}} \\ &\le T^2R^2S^5 (TR/S)^{2\vartheta_{\mathrm{sp}}}. \end{aligned} \tag{3.88}$$

The factor here is C/Z; the factor $C/(SZ)$ would lose $S^{2\vartheta_{\mathrm{sp}}}$. The bound $C \le DZ$ is applied after passing to the envelopes. The first term is at most $A_0^{\vartheta_{\mathrm{sp}}} CDZ^2$ times the weighted norm. The second uses $A_0 YZ$. The fourth uses D^2YZ. Substitute $(H_*S)^2 F \ll X^{o(1)+2\eta} QT^3R^4S^6/X^2$. This proves the other lines of (3.87).

The four packing inequalities (3.7) bound the exponents in (3.87) by $-15\sigma, -(15 - 1/20)\sigma, -15\sigma, -23\sigma$. The cutoff contributes at most 3η to J_*^2/V. The product in (3.14) is eventually at most $X^{10.01}$. This follows from $A_0 < X^{1.01}, T < X^{0.34}, R, S < X^{0.6}$ and the sharper Y bound. The bounds on R and S use only $RS \le X^{0.6}$. This comes from the full-modulus support $q \le X^{0.6}$ in (H3), since the full modulus has scale QRS; dyadic constants are absorbed after enlarging X_0. So the smoothness prefactor costs at most 0.19019σ in the exponent.

Before the cutoff loss, the exponents relative to $V^{1/2}$ (from the square-root row on, relative to $V^{1/4}$), in units of σ, are

(3.89)

J_*	$-299/40 = -7.475$
Theorem 2.8 prefactor	$+0.19019$
$W_{k\neq 0}$	-7.28481
$W_{k=0}$	-8
square root in (3.64)	-3.642405
Mellin integral	$+0.075$
combined fixed subpolynomial factors	$+0.11$
cutoff X^η after both roots	$+0.00075$.

The Mellin allowance follows from (3.57). Any fixed logarithmic power is eventually at most $X^{0.074\sigma}$. The remaining subpolynomial factors are the finitely many (h, f, D, C, Z) localizations, the divisor representation counts in (3.65), (3.71), the prime products (3.79), and powers of $p_0 \ll L$. Their product is uniformly $X^{o(1)}$. So it is at most $X^{0.11\sigma}$ after choosing X_0. The multiplier from the random partition is a fixed σ-dependent constant. By (3.58), (3.64), (3.65), (3.89),

$$|E_2^*| \ll L^C \frac{T^2}{Q} X^{-3.45\sigma}, \tag{3.90}$$

since $-3.642405 + 0.075 + 0.11 + 0.00075 = -3.456655 < -3.45$. For $S = 1/2$, $E_2^* = 0$ (see above). Inserting (3.90) in (3.51) gives a spectral contribution $M_I T X^{-1.725\sigma} L^C \ll X^{1-1.725\sigma} L^C$, since $M_I T \ll X$. As $c_\sigma < \sigma/2 < 1.725\sigma$, it is absorbed in $X^{1-c_\sigma} L^C$.

For a target A', use $A_{\mathrm{work}} = A' + 4$ on every shell triple, uniformly before summing. The triangle inequality over $O(L^3)$ triples costs the logarithm of the outer modulus and the two logarithms of the other factors. Finally restore the factor C_{slot} of (3.5). It is $O_\sigma(1)$. Equations (3.51), (3.90) prove Theorem 3.2.

3.10. **Global summation, prime powers, and normalization.** At fixed σ, assume (H7): the box family has the source and Heath–Brown coefficient bound

$$\sum_j |c_j| \le C_{\sigma, C_{\mathrm{box}}, \mathcal{C}_{\mathrm{src}}} (\log N)^{K_0}, \qquad K_0 \le 24 C_{\mathrm{box}} + 26. \tag{3.91}$$

Source-wide error bounds. Assume the source's aggregate logarithm-freezing bound and the boundary bound (3.24). With the exponents $c_{\log}$ and c_R of (H6), a sufficient pair of raw bounds is

$$E_{\log} \ll_\sigma N (\log N)^{c_{\log} - C_{\mathrm{box}}}, \qquad E_{\mathrm{bdry}} \ll_\sigma N (\log N)^{c_R - C_{\mathrm{box}}/2}.$$

They include all source labels and representation multiplicities. These bounds are separate from the interior-box hypotheses (H1)–(H5). The source-wide estimates enter through (B.29) for the prime source and through the auxiliary bound (3.95). They are not hypotheses of Theorem 3.2.

The source labels and prefix cells are $O_\sigma(1)$. There are at most 24 HB variables. Hence there are $O(L^{24})$ dyadic tuples, and each dyadic tuple contains $O(L^{24 C_{\mathrm{box}}})$ short boxes. The product shells number $O(\log\log N) \le O(L)$. The total count is $O_\sigma(L^{24 C_{\mathrm{box}} + 25})$. Since the coefficients are $O_\sigma(1)$, (3.91) holds with $K_0 = 24 C_{\mathrm{box}} + 26$; the count exponent is smaller. The three factor localizations are inside (3.13). The count includes every partition.

Choose parameters in the following order. Fix σ and the source combinatorics. For Theorem 1.2, through (3.92) and (B.29), use the requested exponent $A_{\mathrm{agg}} = A$ and choose $C_{\mathrm{box}} \ge \lceil 2A + 32 \rceil$; then set $A' = A + 24 C_{\mathrm{box}} + 31$ and $A_{\mathrm{work}} = A' + 4$. This is the parameter convention used in Theorem 1.2 and Theorem 1.1.

For the stronger source-wide bookkeeping in (3.95), which is outside Theorem 3.2, fix a separate final exponent $A > 0$ and put $A_{\mathrm{agg}} = A + 3$. Choose the box width $C_{\mathrm{box}} = C_{\mathrm{box}}(A_{\mathrm{agg}})$ with

$$C_{\mathrm{box}}(A_{\mathrm{agg}}) \ge \max\{2(A_{\mathrm{agg}} + c_R),\ A_{\mathrm{agg}} + c_{\log}\},$$

so that the boundary and logarithm-freezing errors are $O(N(\log N)^{-A_{\mathrm{agg}}})$. For the Iwaniec source, $c_R = 16$ by (B.19) and the logarithm-freezing exponent is $c_{\log} = 18$ by (B.18); since $A_{\mathrm{agg}} + 18 < 2A_{\mathrm{agg}} + 32$, the condition is the integer choice $C_{\mathrm{box}}(A_{\mathrm{agg}}) \ge \lceil 2A_{\mathrm{agg}} + 32 \rceil = \lceil 2A + 38 \rceil$. Set $A' = A_{\mathrm{agg}} + 24 C_{\mathrm{box}} + 31$ and $A_{\mathrm{work}} = A' + 4$. The value of A' is $K_0 + A_{\mathrm{agg}} + 5$ at the maximal count $K_0 = 24 C_{\mathrm{box}} + 26$ in (3.91). Let c_* dominate c_1 and every preliminary logarithmic loss denoted by c. This includes the fixed loss in (3.41) and the endpoint losses before BFI. It is chosen before A_E, C_Σ, C_M. Use the same c_* for every source and

every application below. Choose

$$\begin{aligned} A_E &\ge 2A_{\mathrm{work}} + 2C_{\mathrm{box}} + c_*, \\ C_\Sigma &= A_E + 2c_* + 3, \\ C_M &= 5C_\Sigma + A_E + C_{\mathrm{box}} + c_* + 3. \end{aligned}$$

Then choose A_3 and the required BFI/SW exponents. Later power-saving logarithmic losses may depend on C_Σ, C_M. The constant c_* is defined before them. Choose N_0 last. It is ineffective, as are the implied constants, because the Siegel–Walfisz theorem is ineffective.

The source may choose a larger width. Its fixed width is selected before the analytic precisions, with the same displayed formulas.

By (3.13), (H7) in the form (3.91), and $N/(\log N)^{K_{\mathrm{scale}}} \le X \le N$,

$$\begin{aligned} E_{\mathrm{II}} := \sum_j |c_j||S_j^{\mathrm{PSDT}}| &\ll N(\log N)^{K_0 - A'} + N^{1-c_\sigma}(\log N)^{K_0+C} \\ &\ll_{\sigma,A,C_{\mathrm{box}},K_{\mathrm{scale}},\mathcal{C}_{\mathrm{src}}} N(\log N)^{-A}. \end{aligned} \tag{3.92}$$

Every positive fixed power saving beats every fixed logarithm once N_0 is chosen. The spectral term $X^{1-1.725\sigma}L^C$ is already absorbed in (3.13), since $c_\sigma < \sigma/2$.

Source prime powers. For the direct source the correction in (3.22) has $O(N^{1/2}(\log N)^C)$ integers before discrepancy weights. For the product source assume at most four tested coordinates, each at least N^α, $\alpha = 100/1149$, and product $e < N$. A proper power coordinate has reciprocal mass

$$\sum_{N^\alpha \le p^k \le N,\ k\ge 2} p^{-k} \ll N^{-\alpha/2} + (\log N)N^{-2\alpha/3} \ll N^{-\alpha/2}. \tag{3.93}$$

For $k = 2$ compare with the tail of $\sum n^{-2}$. For $k \ge 3$ sum the integer k-power tail. There are $O(\log N)$ possible exponents. Fix one coordinate at t. For $1 \le r \le 4$, the remaining positive coordinates have at most $(N/t)(1+\log N)^{\max(r-2,0)}$ possibilities. For $r = 1$ no coordinate remains, and the single admissible completion is bounded by N/t. For $r = 2$ the remaining coordinate is counted directly. For $r \ge 3$ the reciprocal-sum argument gives the displayed logarithmic power. Take a union bound over the at most four possible power coordinates. For the product discrepancy write e for the tested product. Theorem 1.2 supplies $N \le a \le N\log N/3$. On the product support, $e < N \le a \ll N\log N$, so $1 \le |a-e| \ll N\log N$. Use (3.23) and $\tau(|a-e|) \ll_\epsilon N^\epsilon$ with $\epsilon = \alpha/4$. Then the whole source-level correction obeys

$$E_{R0} \ll N^{1-\alpha/4}(\log N)^C = N^{1-25/1149}(\log N)^C \ll_A N(\log N)^{-A}. \tag{3.94}$$

This bound precedes the Heath–Brown subdivision, so its box multiplicity is 1. For a product, the bound uses the coordinate floor N^α assumed immediately before (3.93). Thus (3.94) applies to every such product source.

The bounds (H6), the logarithm-freezing estimate, (3.92) (which uses (H7)), and (3.94), with the fixed choices above, give

$$\begin{aligned} &E_{\mathrm{II}} + E_{\mathrm{bdry}} + E_{\log} + E_{R0} \ll_{\sigma,A,C_{\mathrm{box}},K_{\mathrm{scale}},\mathcal{C}_{\mathrm{src}}} N(\log N)^{-A-3} = o(U_C), \\ &U_C = C(N)N/(\log N)^2, \quad C(N) \ge c > 0. \end{aligned} \tag{3.95}$$

The ratio is $O_{\sigma,A,C_{\mathrm{box}},K_{\mathrm{scale}},\mathcal{C}_{\mathrm{src}}}((\log N)^{-A-1})$.

For each source family, its exact source identities give the finite sum to which these bounds apply. The restoration, BFI, partition, Poisson and Theorem 2.8 errors enter once, inside E_{II}.

For any normalization $W_{\mathrm{norm}} \gg N/(\log N)^2$, (3.92) reads explicitly

$$\frac{\sum_j |c_j||S_j^{\mathrm{PSDT}}|}{W_{\mathrm{norm}}} \ll (\log N)^{K_0+2-A'} + N^{-c_\sigma}(\log N)^{K_0+C+2} = o(1). \tag{3.96}$$

Consider the original direct prime source, before HB. Its mass is not yet replaced by a logarithmic-integral (Li) mass. Its exact prime mass $X_A = \sum_p A_D(p)$ satisfies

$$\sum_{(p,q)=1} A_D(p) = X_A \tag{3.97}$$

because $p > N/(\log N)^B > q$ for $q < N^{e_Q}$ and sufficiently large N. Thus the formal and natural comparators coincide on that original source. This holds for the source itself, before the split into signed HB terms. If a Li mass X_{Li} is substituted, the exact difference is $(X_A - X_{\mathrm{Li}})/\varphi(q)$. Its weighted sum enters once, in the main-term mass contribution, outside the natural-comparator correction. For the composite product source, (3.97) may fail.

3.11. **The complete non-II cover.** These results are separate from Theorem 3.2. The source-to-box identity comes from the source itself. For a direct shell $[X, 2X)$, assume the following source conditions (SH1)–(SH5).

- (SH1) The coefficient $\gamma(q)$ is independent of tested variables, $(q, 2N) = 1$, $|\gamma(q)| \le C_\sigma$, $q \le N^{e_Q}$, $e_Q < 153/256$, and $(a, q) = 1$ on its support.
- (SH2) All tested restrictions are constant on each nonboundary box. Smooth factor weights have $\sup |w| + V(w) \ll 1$.
- (SH3) All HB factors are rough at the threshold $z^* = z_0(X)$ or $z_0(N)$.
- (SH4) The discrepancy is exactly (3.3).
- (SH5) The assignment priority is proper prime powers, boundary boxes, Type II, Type I-1, then Type I-2.

Put $\omega = \log(N/X)/\log N$ and $\kappa = 30 \log 2/\log N$. Increase N_0 until $\omega, \kappa \le \sigma/202$. Let V_i be the lower HB scales, and $e_i = \log V_i/\log N$. Then (3.21) implies

$$1 - \omega - \kappa < \sum_i e_i < 1 + \kappa, \qquad e_i \le 1/3 + \kappa \quad \text{on Möbius variables.} \tag{3.98}$$

There are at most six variables. Call a variable small below 60σ, and large above $1/3 + 14\sigma$. Suppose no variable is in the II window. Since $360\sigma < 1 - \omega - \kappa$, some variable is large. Since $1 + 42\sigma > 1 + \kappa$, at most two are large. Möbius variables are never large. Thus there are exactly one or two large smooth variables. For a window variable,

$$\log_X(2V) \le (1/3 + 14\sigma + \kappa)/(1 - \omega) \le 1/3 + 15\sigma, \tag{3.99}$$

which proves the shell-local window transfer. For one large variable,

$$e_* - e_Q - (1-\omega)/10 \ge 0.9 - 153/256 - 0.9\omega - \kappa - 300\sigma > 0.3023; \tag{3.100}$$

hence its scale $V \ge qX^{1/10}$. With two large variables, $N_1 \ge N_2 \ge X^{1/3}$. The at most four other factors are small. Their product m is split dyadically, $m \asymp M$, with

$$M \le X^{2.401\cdot 10^{-8}}, \qquad N_1N_2M \le 2^{30}X, \qquad q < X^{0.5977}. \tag{3.101}$$

These follow from $(240\sigma+\kappa)/(1-\omega) < 2.401\cdot 10^{-8}$, which uses $\sigma \le 10^{-10}$, and $e_Q/(1-\omega) < 0.5977$. Consider a product source with three or four coordinates at least $N^{100/1149}$. Its smallest coordinate has exponent between $100/1149$ and $(1+\kappa)/3$. It has at most six HB factors. One of them has exponent exceeding $(100/1149-\kappa)/6 > 60\sigma$. Each is below $1/3+14\sigma$. Thus, after proper prime powers and boundary boxes are removed, all its boxes are of type II.

Elementary sieve sandwich. For a finite prime set $\mathcal{P}$ below z, let $P = \prod_{p\in\mathcal{P}} p$, $E = \sum_{p\in\mathcal{P}} 1/p$, and

$$\lambda_d^{(J)} = \mu(d)1_{d\mid P,\ \omega(d)\le J}. \tag{3.102}$$

Let $r = \omega((n,P))$. For $r \ge 1$, Pascal's identity gives $\sum_{j\le J}(-1)^j\binom{r}{j} = (-1)^J\binom{r-1}{J}$. The sum is 1 for $r = 0$. Hence

$$\begin{gathered}\Psi^-(n) := \sum_{d\mid n}\lambda_d^{(2k-1)} \le 1_{(n,P)=1} \le \Psi^+(n) := \sum_{d\mid n}\lambda_d^{(2k)},\\ \Psi^+ \ge 0, \qquad \Psi^+ - \Psi^- = \binom{r}{2k} \ge 0.\end{gathered} \tag{3.103}$$

The weights are 1-bounded and supported on $d \le z^{2k}$. Also $\sum_{d\mid P,\omega(d)=j} 1/d \le E^j/j!$. Let $s = \log D/\log z \gg (\log\log X)^3$, and choose $2k = 2\lfloor s/2\rfloor$. Mertens' bound $E \le \log\log z + O(1)$, with $s \gg (\log\log X)^3$, gives $2k \ge s-2 \ge e^2E$ for large X. With Stirling,

$$E^{2k}/(2k)! \le (eE/(2k))^{2k} \le e^{2-s} \ll_A L^{-A}. \tag{3.104}$$

It is the sieve approximation used in I-1 and I-2.

Lemma 3.3 (Type I, one large variable (I-1))**.** *Assume (SH1)–(SH5). Consider a box with one large variable. Let the complementary coefficient satisfy $|\alpha_m| \ll \tau(m)^{B_0}$. Uniformly for every integer residue a coprime to each supported q,*

$$\sum_q |\gamma(q)| \left| \sum_m \alpha_m \sum_{v\in I,\ P^-(v)\ge z^*} w(v)\mathcal{S}_{mv}(q;a) \right| \ll_{\sigma,A,B_0} XL^{-A}. \tag{3.105}$$

Proof. If $(m,q) > 1$, the summand vanishes. Otherwise put $b = a\bar{m}(q)$. The progression and the reduced comparator contain only integers prime to q. Apply (3.102)–(3.104) to the q-dependent set $\mathcal{P}_q = \{p < z^* : p \nmid q\}$, with $D = X^{1/20}$. For every $d \mid P_q$, $(d,q) = 1$, and

$$\#\{v \le t : v \equiv b(q), d \mid v\} = t/(dq) + O(1),$$

$$\#\{v \le t : (v,q) = 1, d \mid v\} = (\varphi(q)/q)(t/d) + O(\tau(q)). \tag{3.106}$$

After comparator normalization, the upper and lower sandwiches have the same main term. Their gap gives discrepancy $\ll (t/q)e^{2-s} + 2D\tau(q)$, where $s \ge (\log\log X)^3/40$. Endpoint subtraction and partial summation cost $O(\sup|w| + V(w))$. By (3.100), $D\tau(q) \ll (V/q)X^{-1/20+o(1)}$. Thus the inner discrepancy is $O_A(V/(qL^A))$. Sum with $\sum_m |\alpha_m| \ll ML^C$, $MV \asymp X$, and $\sum_q 1/q \ll L$. Then request a larger A. This proves (3.105). The sieve weights use the prime set $\mathcal{P}_q$. □

Lemma 3.4 (Completed hyperbola)**.** *Let $c \bmod q$ be a unit. For integer intervals J_1, J_2, put*

$$\mathcal{D} = \#\{(u,v) \in J_1 \times J_2 : uv \equiv c(q)\} - \frac{\#J_1^* \#J_2^*}{\varphi(q)}. \tag{3.107}$$

The full-modulus Kloosterman-sum bound [17, Corollary 11.12]*,*

$$|S(h,k;q)| \le \tau(q) q^{1/2} (h,k,q)^{1/2}, \tag{3.108}$$

implies, uniformly in interval lengths and positions,

$$|\mathcal{D}| \ll \tau(q) q^{1/2} \log^2(2q) + \tau(q)^2 \log^2(2q)/\varphi(q). \tag{3.109}$$

Proof. Put

$$F_i(h) = \sum_{u \in J_i} e(hu/q).$$

Additive completion then gives

$$\mathcal{D} = q^{-2} \sum_{h,k} F_1(-h) F_2(-k) \left(S(h,kc;q) - \frac{c_q(h) c_q(k)}{\varphi(q)} \right).$$

The bracket vanishes if $h = 0$ or $k = 0$, so complete periods cancel. For $0 < h < q$, $|F_i(h)| \le q/(2\min(h, q-h))$, and

$$\sum_{0<h<q, d \mid h} |F_i(h)| \ll (q/d)\log(2q). \tag{3.110}$$

Use $(h,k,q)^{1/2} \le \sum_{d \mid (h,k,q)} d^{1/2}$ in (3.108) and sum $\sum d^{-3/2}$. For the principal part use $|c_q(h)| \le (h,q) = \sum_{d \mid (h,q)} \varphi(d)$, so its one-variable weighted sum is $O(q\tau(q)\log(2q))$. This proves (3.109). Modulo 1 the discrepancy is zero. □

Lemma 3.5 (Type I, two large variables (I-2))**.** *Assume (SH1)–(SH5) and (3.101). Consider two large smooth variables with nonnegative weights w_1, w_2. Let $|\alpha_m| \ll \tau(m)^{B_0}$. Uniformly for every integer residue a coprime to each supported q,*

$$\sum_{q \asymp Q} |\gamma(q)| \Big| \sum_m \alpha_m \sum_{n_i \in I_i,\ P^-(n_i) \ge z^*} w_1(n_1) w_2(n_2) \mathcal{S}_{mn_1n_2}(q;a) \Big| \ll_A XL^{-A}. \tag{3.111}$$

The smooth factor weights in the prime-source application of Appendix B are nonnegative. For a general real weight, write $w = (w + c) - c$, with $c = \sup |w|$, and apply the lemma to the nonnegative pieces.

Proof of Lemma 3.5. Again restrict to $(m, q) = 1$. Let $C(FG)$ and $M_q(FG)$ be the positive congruence and comparator functionals with the positive weights w_1, w_2. Put $\Delta = C - M_q$. Apply (3.103) to all primes below z^*, with $D_f = X^{\epsilon_I/2}$, $\epsilon_I = \delta/4 = 7.5\sigma$. Include the Brun–Bonferroni threshold for this smaller level in N_0. For every fixed logarithmic exponent A_{sv}, put

$$s_f = \frac{\log D_f}{\log z^*}, \qquad E = \sum_{p<z^*} \frac{1}{p}, \qquad J = 2\lfloor s_f/2 \rfloor,$$

and require $J \geq \max\{2, e^2E, A_{\mathrm{sv}} \log L\}$. At fixed $\sigma > 0$ this holds eventually, since $s_f \asymp \sigma(\log\log N)^3$ and $E = O(\log\log N)$. Hence $(z^*)^J \leq D_f$, $J - 1 \geq 1$, and

$$\frac{E^J}{J!} \leq \left(\frac{eE}{J}\right)^J \leq L^{-A_{\mathrm{sv}}}.$$

Choose A_{sv} after the fixed logarithmic losses in (3.115). So the same final choice of N_0 covers the use of (3.103)–(3.104) in I-2. Write $R_i = 1_{P^-(n_i) \geq z^*}$, $G_i = \Psi_i^+ - \Psi_i^- \geq 0$. Since $0 \leq R_i \leq \Psi_i^+$,

$$\Psi_1^+\Psi_2^+ - G_1\Psi_2^+ - \Psi_1^+G_2 \leq R_1R_2 \leq \Psi_1^+\Psi_2^+. \tag{3.112}$$

Since C, M_q are positive,

$$\begin{aligned} |\Delta(R_1R_2)| \leq{}& |\Delta(\Psi_1^+\Psi_2^+)| + |\Delta(G_1\Psi_2^+)| + |\Delta(\Psi_1^+G_2)| \\ &+ M_q(G_1\Psi_2^+) + M_q(\Psi_1^+G_2). \end{aligned} \tag{3.113}$$

The lower weights Ψ_i^- may be negative. The bound (3.113) contains only the nonnegative Ψ_i^+, G_i. Expand the first three terms over $d_i \leq D_f$. Their coefficients are bounded by 1. Terms with $(md_1d_2, q) > 1$ vanish. Otherwise put $c \equiv a\overline{md_1d_2} \pmod q$. This is a unit, and $u_i = n_i/d_i$ lies on the modular hyperbola $u_1u_2 \equiv c \pmod q$. Two-variable partial summation and (3.109) give a total $\ll_\epsilon D_f^2 q^{1/2} X^\epsilon$.

For the two main gaps, positivity lets us drop the restriction $(n_1n_2, q) = 1$. Full-integer progression counts and (3.104) give

$$\sum_{n\in I_i} G_i(n)w_i(n) \ll_{A_{\mathrm{sv}}} N_iL^{-A_{\mathrm{sv}}} + D_f, \quad \sum_{n\in I_i} \Psi_i^+(n)w_i(n) \ll N_i + D_f. \tag{3.114}$$

The progression counts range over all integers. Summing over (m, q) now gives

$$\ll MD_f^2Q^{3/2}X^\epsilon L^C + XL^{C-A_{\mathrm{sv}}} + MD_f(N_1 + N_2)L^C + MD_f^2L^C. \tag{3.115}$$

By (3.101), the first exponent is less than $2.401 \cdot 10^{-8} + 7.5 \cdot 10^{-10} + 1.5(0.5977) + \epsilon < 0.897$ for a sufficiently small fixed $\epsilon > 0$. Also $M(N_1+N_2) \leq 2^{31}X/N_2 \ll X^{2/3}$. So the exponents of the last gap terms are less than 0.668. Choose A_{sv} after the fixed log losses. This proves (3.111). Squareful moduli are covered by (3.108). □

Finally there are $O_\sigma(L^K)$ non-II direct boxes and shell pieces for a fixed K. Request (3.105), (3.111) with exponents larger than $K + A + 5$. Their sum is then $O(NL^{-A})$; in particular, divided by $N/(\log N)^2$ it tends to zero. For the prime source these are the exponents $A_i = K_I + A + 5$ of Appendix B. The I-1 and I-2 boxes are disjoint parts of the non-II contribution. Under its coordinate hypotheses, the product source has none. Their contributions lie outside E_{II}.

For the prime source, Section 4 supplies the packing construction and Proposition B.4 in Appendix B gives the prime-source estimate.

4. Three-slot packing for Iwaniec's remainder coefficients

This section proves a general packing theorem. It then verifies the hypotheses for the remainder coefficients of Iwaniec's linear sieve [10, Theorem 4]. The packing and coefficient identities provide the modulus factorization used in the prime-discrepancy estimates below. The functions below specialize Lichtman's optimization [1, Proposition 6.1, (6.2), (6.4)] to spectral exponent 7/32 and moving-residue exponent 1. We prove the labelled-atom statement here; its two-bin step is related to the corrected argument in Pascadi's Lemma 5.6 [8].

4.1. **A combinatorial packing theorem.** Put

$$\overline{H} = \frac{153}{256}, \qquad \mu = \frac{25}{128}, \qquad \vartheta = \frac{7}{32}.$$

Here ϑ is the fixed spectral exponent $2\theta_{\mathrm{RP}}$. Later in this section θ denotes the sieve level. For $x \geq 0$, define

$$v(x) = \min\left\{\frac{1+x}{2}, \frac{32-34x}{39}, \frac{57}{64} - \frac{3}{2}x\right\}.$$

For real x and $y \geq 0$, put $\psi(x,y) = x\mathbf{1}_{x\geq y}$. Also put

$$w(a,b,c) = \psi\left(\min\left\{\frac{132-68c}{214}, \frac{153-96c}{256}, 1-2b\right\}, \frac{1+a}{2}\right).$$

For an ordered triple $a \geq b \geq c \geq 0$, set

$$\begin{aligned} H(a,b,c) = \max\{&v(a), v(b), v(a+b), v(a+b+c), w(a,b,c), w(a,c,b),\\ &\psi(v(a+c), a+2b+c), \psi(v(b+c), 2a+b+c)\}, \end{aligned}$$

Define $\widetilde{H}(a,b,c) = H(a,b,c)$ if $a \leq \mu$. If $a > \mu$, put $\widetilde{H}(a,b,c) = v(a)$. For a finite non-increasing list $e_1 \geq \cdots \geq e_m \geq 0$, let $\bar{e}_i = e_i$ for $i \leq m$ and $\bar{e}_i = 0$ for $i > m$. Repeated values are distinct labelled atoms. The empty list is allowed.

A list $f_1 \geq \cdots \geq f_m \geq 0$ satisfies $\mathrm{WF}(y)$ if

$$f_1 + \cdots + f_{j-1} + 2f_j \leq y \qquad (1 \leq j \leq m). \tag{4.1}$$

Write

$$b_- = 2h - 1, \qquad b_A = \frac{32-39h}{34}, \qquad b_B = \frac{2-\vartheta-2h}{3}, \qquad b_+ = \min(b_A, b_B).$$

Theorem 4.1 (three-slot packing). Let $0 < \delta \leq 3 \cdot 10^{-9}$, $1/2 \leq h \leq \overline{H}$, $d_* = h - 2\delta$, and $\delta \leq g \leq 1/3 - \delta/2$. Suppose the labelled list $e_1 \geq \cdots \geq e_m \geq 0$ satisfies

$$\mathrm{WF}(d_*) \quad \text{and} \quad h \leq \widetilde{H}(\bar{e}_1, \bar{e}_2, \bar{e}_3). \tag{4.2}$$

Then the deterministic rule $\mathcal{R}$ below partitions the labels into $J_1 \sqcup J_2 \sqcup J_3$. If $\Sigma_s = \sum_{i \in J_s} e_i$, then

$$\begin{aligned} \Sigma_1 &\le g, \\ 2g + \Sigma_2 + 2\Sigma_3 &\le 1 - 4\delta, \\ \vartheta + g + \Sigma_1 + 5\Sigma_2 + 2\Sigma_3 &\le 2 - 4\delta, \\ (1+2\vartheta)g + \Sigma_1 + (5-2\vartheta)\Sigma_2 + (2+2\vartheta)\Sigma_3 &\le 2 - 4(1+\vartheta)\delta. \end{aligned} \tag{4.3}$$

The output depends only on the labelled list and on h, δ, g.

Lemma 4.2. For $x \ge 0$, $v(x) = \min\{(1+x)/2, 57/64 - 3x/2\}$. Thus v is concave, $v \le \overline{H}$, and $\widetilde{H} \le \overline{H}$.

Proof. Consider the middle affine function in the definition of v. Its difference with $(1+x)/2$ is $(25-107x)/78$, which is nonnegative for $x \le 25/107$. Its difference with $57/64 - 3x/2$ is $(1568x - 175)/2496$, nonnegative for $x \ge 175/1568$. As $175/1568 < 25/107$, it is never the strict minimum of all three. The minimum of the remaining two affine functions is concave and has maximum $(1+\mu)/2 = \overline{H}$. Each w is zero or at most $(153 - 96c)/256 \le \overline{H}$. Every ψ branch is zero or a value of v. This proves both upper bounds. □

Lemma 4.3. For $1/2 \le h \le \overline{H}$ and $0 < \delta \le 3 \cdot 10^{-9}$:

(1) $h \le v(b)$ for $b \ge 0$ exactly when $b_- \le b \le b_+$.
(2) $b_+ = b_B$, and $b_A > b_B$.
(3) $b_+ - b_- = (153 - 256h)/96 \ge 0$ and $b_+ - \mu = (153/128 - 2h)/3 \ge 0$.
(4) $d_* - b_+ - 2b_- \ge 3/256 - 2\delta > 0$ and $d_* - 3b_- \ge 3/256 - 2\delta > 0$.
(5) $\vartheta + 2(d_* + 2\delta) + 3b_B = 2$ and $(2+2\vartheta)(d_* + 2\delta) + (3 - 4\vartheta)b_A = 2$.

Proof. The three affine branches of v give $b \ge 2h - 1$, $b \le b_A$, and $b \le b_B$. These are closed inequalities. The difference $b_A - b_B = (567/16 - 49h)/102$ decreases with h and at $h = \overline{H}$ is $525/8704 > 0$, proving (2). The formulas in (3) and the first bound in (4) follow by expansion. Their minima occur at $h = \overline{H}$. For the second bound in (4), use $d_* - b_- = 1 - h - 2\delta$ and $3 - 5\overline{H} = 3/256$. For (5), use $2 + 2\vartheta = 39/16$, $3 - 4\vartheta = 17/8$, and $(3 - 4\vartheta)b_A = 2 - 39h/16$. Finally $3/256 > 6 \cdot 10^{-9} \ge 2\delta$. □

Lemma 4.4. If $h > 0$ and $w(x, y, z) \ge h$, then $z \le b_+ - b_-$.

Proof. Since $h > 0$, the activation in ψ is on. Hence $(132 - 68z)/214 \ge h$ and $(153 - 96z)/256 \ge h$. These give $z \le b_A - b_-$ and $z \le b_B - b_-$, respectively. □

Lemma 4.5 (two bins). Let $c_1, c_3 \ge 0$ and initial loads $u_0 \le c_1$, $w_0 \le c_3$. Suppose a finite list $f_1, \dots, f_k \ge 0$ satisfies

$$u_0 + w_0 + \sum_{i<j} f_i + 2f_j \le c_1 + c_3 \qquad (1 \le j \le k). \tag{4.4}$$

Process the atoms in order. Place each in bin 1 if it fits, and otherwise in bin 3 if it fits. Every atom is placed, and the final loads are at most c_1, c_3.

Proof. Suppose f_j is the first atom fitting neither bin. Then the current loads U, W satisfy $U + W = u_0 + w_0 + \sum_{i<j} f_i$, $U + f_j > c_1$, and $W + f_j > c_3$. Adding the last two inequalities contradicts (4.4). □

The rule permits zero atoms and repeated values. Its three cases are:

(1) If $g + b_+ \ge d_*$, apply Lemma 4.5 to the whole list. Use bin caps $g, d_* - g$ for slots 1 and 2, and leave slot 3 empty.

(2) If $g + b_+ < d_*$, test in order $K = \{1\}, \{1,2\}, \{1,2,3\}, \{1,3\}, \{2,3\}$. Put $K' = K \cap \{1, \dots, m\}$ and $b = \sum_{i \in K'} e_i$. Call K legal if $b_- \le b \le b_+$ and the remaining atoms, in their original order, satisfy $\mathrm{WF}(d_* - b)$. For the first legal K, place K' in slot 2. Apply Lemma 4.5 to the rest in slots 1 and 3, with caps $g, d_* - g - b$.

(3) If no K is legal, use caps $g, b_-, d_* - g - b_-$ for slots 1, 2, 3. Put atom 1 in slot 2 if it exists. Process subsequent atoms in order, placing each in the first slot in which it fits. Put the first atom fitting no slot in slot 2. Call the resulting slot-2 load b. Apply Lemma 4.5 to the remaining atoms in slots 1 and 3, with caps $g, d_* - g - b$.

Proof of Theorem 4.1. Write $\mathcal{R}$ for the rule just defined. Since $h \ge 1/2$, $b_- \ge 0$; also $d_* = h - 2\delta \ge 1/2 - 6 \cdot 10^{-9} > 1/3 \ge g$. It is enough to prove one of the following envelopes:

$$(E_{\mathrm{glob}}): \quad g \ge d_* - b_+, \quad \Sigma_1 \le g, \quad \Sigma_2 \le d_* - g, \quad \Sigma_3 = 0;$$
$$(E_{\mathrm{crit}}(b)): \quad b_- \le b \le b_+, \quad \Sigma_1 \le g, \quad \Sigma_2 \le b, \quad \Sigma_3 \le d_* - g - b.$$

The second, third, and fourth left sides in (4.3) have nonnegative load coefficients. Under $E_{\mathrm{crit}}(b)$, Lemma 4.3(5) gives

$$2g + \Sigma_2 + 2\Sigma_3 \le 2d_* - b \le 2d_* - b_- = 1 - 4\delta,$$
$$\vartheta + g + \Sigma_1 + 5\Sigma_2 + 2\Sigma_3 \le \vartheta + 2d_* + 3b \le \vartheta + 2d_* + 3b_B = 2 - 4\delta.$$

For the fourth load, similarly,

$$\begin{aligned}(1 + 2\vartheta)g + \Sigma_1 + (5 - 2\vartheta)\Sigma_2 + (2 + 2\vartheta)\Sigma_3 &\le (2 + 2\vartheta)d_* + (3 - 4\vartheta)b \\ &\le (2 + 2\vartheta)d_* + (3 - 4\vartheta)b_A \\ &= 2 - 4(1 + \vartheta)\delta.\end{aligned}$$

Here $3 - 4\vartheta = 17/8 > 0$. Under E_{glob}, use $g \ge d_* - b_+$ in the two expressions whose coefficient of g is negative. This gives

$$\begin{aligned}2g + \Sigma_2 \le g + d_* &\le \tfrac{1}{3} - \tfrac{\delta}{2} + \overline{H} - 2\delta = \tfrac{715}{768} - \tfrac{5}{2}\delta \le 1 - 4\delta, \\ \vartheta + 2g + 5\Sigma_2 \le \vartheta + 5d_* - 3g &\le \vartheta + 2d_* + 3b_+ = 2 - 4\delta, \\ (2 + 2\vartheta)g + (5 - 2\vartheta)\Sigma_2 \le (5 - 2\vartheta)d_* - (3 - 4\vartheta)g & \\ \le (2 + 2\vartheta)d_* + (3 - 4\vartheta)b_+ &\le 2 - 4(1 + \vartheta)\delta.\end{aligned}$$

The first line uses $1 - 715/768 = 53/768 > 3\delta/2$. In both envelopes $\Sigma_1 \le g$, proving the first inequality of (4.3). Equality at the case boundary belongs to case 1; all displayed endpoint inequalities are weak.

Consider case 1. The clause $\mathrm{WF}(d_*)$ of (4.2) is exactly the hypothesis (4.4) for the whole list. Here $u_0 = w_0 = 0$ and the total cap is d_*. Lemma 4.5 gives E_{glob}. Now suppose $g + b_+ < d_*$. If a tested set K is legal, the remaining list satisfies (4.4) with total cap $d_* - b$. Lemma 4.5 then gives $E_{\mathrm{crit}}(b)$.

We verify the rule when no tested set is legal. Two deletion facts will be used. First let K' be a leading set $\{1, \dots, k\}$. Subtract $b = \sum_{i \le k} e_i$ from every later inequality in $\mathrm{WF}(d_*)$. This proves $\mathrm{WF}(d_* - b)$ for the remaining list. Next let K' be $\{1,3\}$, $\{2,3\}$, or, when $m = 2$, the truncation $\{2\}$ of $K = \{2,3\}$. Let $b_- \le b \le b_+$. The only unselected atom among the first three has index $i_0 = 2$ or

1. If it is below b_-, it is first in the remaining list, and

$$2e_{i_0} + b < 2b_- + b_+ \le d_*$$

by Lemma 4.3(4). When $m = 2$ this is the whole remaining list. Now take a remaining rank at least four. Its original prefix sum contains both selected atoms. Subtracting b proves the reduced mask. Thus the nonconsecutive set is legal whenever these conditions hold.

Assume now that no tested set is legal. First, $\bar{e}_1 \le \mu$. Otherwise the definition of $\widetilde{H}$ and (4.2) give $h \le v(e_1)$. Then $b_- \le e_1 \le b_+$ by Lemma 4.3(1), and $\{1\}$ is legal. The same lemma and $b_+ \ge \mu$ show $\bar{e}_1 < b_-$; otherwise $\{1\}$ is legal. Hence every padded coordinate is below b_-. Since $\bar{e}_1 \le \mu$, the branch condition gives $\widetilde{H} = H$. A branch in its maximum is at least h. Neither $v(\bar{e}_1)$ nor $v(\bar{e}_2)$ can be that branch. Each would put the corresponding coordinate at least b_-. A branch $v(\bar{e}_1 + \bar{e}_2)$ or $v(\bar{e}_1 + \bar{e}_2 + \bar{e}_3)$ would make a leading set legal. If

$$\psi(v(\bar{e}_1 + \bar{e}_3), \bar{e}_1 + 2\bar{e}_2 + \bar{e}_3) \ge h,$$

then $v(\bar{e}_1 + \bar{e}_3) \ge h$, and $b = \bar{e}_1 + \bar{e}_3 \in [b_-, b_+]$. The only unselected atom among the first three is $e_2 < b_-$. So the deletion fact makes $\{1, 3\}$ legal, a contradiction. The other ψ branch similarly makes $\{2, 3\}$ legal. Therefore either $w(\bar{e}_1, \bar{e}_2, \bar{e}_3) \ge h$ or $w(\bar{e}_1, \bar{e}_3, \bar{e}_2) \ge h$. Lemma 4.4 gives, respectively,

$$\bar{e}_3 \le b_+ - b_-, \qquad \text{or} \qquad \bar{e}_2 \le b_+ - b_-. \tag{4.5}$$

In case 3, the slot-3 cap is

$$d_* - g - b_- = 1 - h - g - 2\delta \ge 1 - \overline{H} - \frac{1}{3} - \frac{3}{2}\delta = \frac{53}{768} - \frac{3}{2}\delta > 0.$$

Thus all three caps are nonnegative. Atom 1 fits slot 2 because $e_1 < b_-$. In the first w case, atom 2 cannot be a stopping atom. Slots 1 and 3 are empty. If e_2 fit neither, then $2e_2 > d_* - b_- \ge 2b_-$ by Lemma 4.3(4), contrary to $e_2 < b_-$. Thus any stopping atom has index $j \ge 3$ and $e_j \le e_3 \le b_+ - b_-$. In the second w case, a stopping atom has $j \ge 2$ and $e_j \le e_2 \le b_+ - b_-$. If there is no stop, the final loads are bounded by $(g, b_-, d_* - g - b_-)$. This is $E_{\mathrm{crit}}(b_-)$, also when $m = 0$.

If the stop is at j, let (u_0, v_0, w_0) be the loads before e_j. After placing e_j in slot 2, its load $b = v_0 + e_j$ satisfies

$$b_- < b \le b_- + (b_+ - b_-) = b_+.$$

The lower bound holds because e_j did not fit slot 2. The upper bound follows from (4.5). The j-th inequality in $\mathrm{WF}(d_*)$ gives $u_0 + w_0 + e_j \le d_* - b$. Since e_j did not fit slot 1, $u_0 + e_j > g$, so $w_0 < d_* - g - b$. For every later rank, the original mask is exactly the two-bin condition with total cap $d_* - b$. Lemma 4.5 completes the assignment and gives $E_{\mathrm{crit}}(b)$. The rule processes every atom once and has fixed tie priorities, so it is well defined. All cases imply (4.3). □

4.2. **A local ceiling and a small-atom criterion.** In this subsection write θ_{RP} for the exponent in the unramified Ramanujan bound and $\vartheta_{\mathrm{RP}} = 2\theta_{\mathrm{RP}}$. The Kim–Sarnak bound [13] gives $\theta_{\mathrm{RP}} = 7/64$ and $\vartheta_{\mathrm{RP}} = 7/32 = \vartheta$.

Proposition 4.6 (local ceiling and small-atom packing). Suppose $0 \le \vartheta_{\mathrm{RP}} \le 7/32$, that is $0 \le \theta_{\mathrm{RP}} \le 7/64$, and $0 \le t \le 1/3$. Let $x, y, z_3 \ge 0$ satisfy the

zero-loss local slot constraints

$$x - t \le 0, \qquad 2t + y + 2z_3 \le 1,$$

$$\vartheta_{\mathrm{RP}} + t + x + 5y + 2z_3 \le 2, \quad (1 + 2\vartheta_{\mathrm{RP}})t + x + (5 - 2\vartheta_{\mathrm{RP}})y + (2 + 2\vartheta_{\mathrm{RP}})z_3 \le 2,$$

and $x + y + z_3 \le 1 - t$. Then their maximum total load is

$$H_{\vartheta_{\mathrm{RP}}} = \frac{5 - \vartheta_{\mathrm{RP}}}{8} = \frac{5 - 2\theta_{\mathrm{RP}}}{8}.$$

For fixed t, equality holds precisely at $x = t$, $y = (1 - \vartheta_{\mathrm{RP}})/4$, $z_3 = (3 + \vartheta_{\mathrm{RP}})/8 - t$.

Suppose instead that $0 \le e_j \le u$ with $u \ge 0$. Let the non-increasing labelled atoms obey $\mathrm{WF}(h - 2\delta)$, with

$$\frac{1}{2} \le h \le H_{\mathrm{small}}(\vartheta_{\mathrm{RP}}, u) := \frac{5 - \vartheta_{\mathrm{RP}} - 3u}{8}, \qquad 0 < \delta \le \frac{1}{100}, \qquad 0 \le g \le \frac{1}{3}.$$

Then they admit a three-slot partition with the four load inequalities (4.3). Here ϑ_{RP} replaces ϑ. In particular, $H_{\mathrm{small}}(7/32, 1/500) = 19101/32000$ and $H_{7/32} = 153/256$.

Proof. Seven times $x - t \le 0$, three times $2t + y + 2z_3 \le 1$, and once $\vartheta_{\mathrm{RP}} + t + x + 5y + 2z_3 \le 2$ give $8(x + y + z_3) + \vartheta_{\mathrm{RP}} \le 5$. Equality forces equality in these three constraints, giving the stated point. Conversely, that point has $y \ge 25/128$, $z_3 \ge 1/24$, and total load plus t at most $23/24$. Substitution in the fourth constraint gives $2 - 3\vartheta_{\mathrm{RP}}(1 - \vartheta_{\mathrm{RP}})/4 \le 2$, proving the maximum and its equality cases.

For the small-atom assertion put

$$d = h - 2\delta, \qquad b_- = 2h - 1, \qquad b_B = \frac{2 - \vartheta_{\mathrm{RP}} - 2h}{3},$$

$$b_A = \frac{2 - (2 + 2\vartheta_{\mathrm{RP}})h}{3 - 4\vartheta_{\mathrm{RP}}}, \quad b_+ = \min(b_A, b_B).$$

The denominators are positive, and direct subtraction gives

$$b_B - b_- = \frac{5 - \vartheta_{\mathrm{RP}} - 8h}{3}, \qquad b_A - b_- = \frac{5 - 4\vartheta_{\mathrm{RP}} - (8 - 6\vartheta_{\mathrm{RP}})h}{3 - 4\vartheta_{\mathrm{RP}}}.$$

Hence $u \le b_+ - b_-$ is equivalent to

$$h \le \frac{5 - \vartheta_{\mathrm{RP}} - 3u}{8}, \qquad h \le \frac{5 - 4\vartheta_{\mathrm{RP}} - (3 - 4\vartheta_{\mathrm{RP}})u}{8 - 6\vartheta_{\mathrm{RP}}}.$$

The second upper bound exceeds the first by $[6\vartheta_{\mathrm{RP}}(1 - \vartheta_{\mathrm{RP}}) + 14\vartheta_{\mathrm{RP}}u]/[8(8 - 6\vartheta_{\mathrm{RP}})] \ge 0$. So the assumed bound on h gives $u \le b_+ - b_-$. Start with capacities $(g, b_-, d - g - b_-)$. They are nonnegative, since the last is at least $1 - 5/8 - 1/3 - 2\delta \ge 1/24 - 1/50 > 0$. Place atoms in order in the first slot in which they fit. If all fit, take $b = b_-$. Otherwise, put the first failed atom e_j in slot 2. Let b be the new slot-2 load. Its failure there and $e_j \le u$ give $b_- < b \le b_- + u \le b_+$. Let (U, V, W) be the previous loads. Then $\mathrm{WF}(d)$ gives $U + W + e_j \le d - b$. Failure in slot 1 gives $W < d - g - b$. For each later atom, subtract the fixed slot-2 load b from its original $\mathrm{WF}(d)$ inequality. Lemma 4.5 places the remaining atoms in slots 1 and 3 with capacities $g, d - g - b$. Both cases reach

$$\Sigma_1 \le g, \qquad \Sigma_2 \le b, \qquad \Sigma_3 \le d - g - b, \qquad b_- \le b \le b_+.$$

The last three left sides of (4.3), with ϑ_{RP} in place of ϑ, are at most

$$2d - b_- = 1 - 4\delta, \qquad \vartheta_{\mathrm{RP}} + 2d + 3b_B = 2 - 4\delta,$$

$$(2 + 2\vartheta_{\mathrm{RP}})d + (3 - 4\vartheta_{\mathrm{RP}})b_A = 2 - 4(1 + \vartheta_{\mathrm{RP}})\delta,$$

respectively. This proves the partition claim. □

These are a relaxed local spectral ceiling and a sufficient uniform small-atom packing criterion. The actual local inequalities have positive losses, so the relaxed ceiling is a supremum. For example, take the losses 29σ, 8σ, and 15σ in the first three slot constraints. They give

$$x + y + z_3 \le H_{\vartheta_{\mathrm{RP}}} - \frac{242\sigma}{8} - \frac{\vartheta_{\mathrm{RP}}\sigma}{160}$$

when the moving-frequency factor is $X^{\vartheta_{\mathrm{RP}}(1+\sigma/20)}$. The small-atom bound also needs strict room in an actual source. If $h = \rho\theta + 2\delta \le H_{\mathrm{small}}(\vartheta_{\mathrm{RP}}, u)$ and $\delta > 0$, then $\theta < H_{\mathrm{small}}(\vartheta_{\mathrm{RP}}, u)$. The next subsection applies this criterion to the structured coefficient family of Iwaniec's construction.

4.3. **Endpoint geometry for Iwaniec's coefficients.** Fix the parameters in Theorem 1.2 and put

$$\begin{gathered}\tau = \sigma^9, \qquad \rho = 1 + \tau, \qquad \Theta(u) = H_{\mathrm{small}}(\vartheta, u) = \frac{5 - \vartheta - 3u}{8}, \\ \eta_\Theta = \Theta(u) - \theta, \\ \delta = 30\sigma, \qquad d_* = \rho\theta, \qquad h = d_* + 2\delta.\end{gathered} \tag{4.6}$$

Proposition 4.6 gives the sufficient packing threshold $\Theta(u)$. The relaxed local ceiling is $H_\vartheta = 153/256$. The fixed support exponent for these coefficients is $\Theta(u)$, and $\Theta(u) < H_\vartheta$ because $u > 0$. The margin $\eta_\Theta = \Theta(u) - \theta$ leaves room for the smooth divisor and tail-prime ranges. Since the theorem also imposes $D^{\sigma^2} \le z$, every nonempty tail has the required roughness floor. Lemma 4.7 verifies $\rho\theta + 60\sigma < \Theta(u)$ and the hypotheses of Theorem 4.1. Let $D = N^\theta$, $z = N^u$, and $Y = D^{\sigma^2}$. Write $\mathcal{P}_N = \{p \text{ prime} : p \nmid 2N\}$ and $P_N(Y) = \prod_{p<Y,\ p\in\mathcal{P}_N} p$. Fix an Iwaniec sign. Write $\mathcal{D}^\nu(D)$ for its finite family of nonempty label vectors $\mathbf{D} = (D_1, \dots, D_k)$. They have $D_k \le \cdots \le D_1 < D^{1/2}$, grid coordinates in $\{D^{\sigma^2\rho^j} : j \ge 0\}$, and the cubic conditions

$$\begin{aligned}&\nu = +: \quad D_1 \cdots D_{2j} D_{2j+1}^3 < D \quad (2j+1 \le k), \\ &\nu = -: \quad D_1 \cdots D_{2j-1} D_{2j}^3 < D \quad (2j \le k).\end{aligned} \tag{4.7}$$

The prime labels in a vector range over $D_j \le p_j < \min(D_j^\rho, z)$, with $p_j \nmid 2N$. A vector with an empty coordinate interval contributes zero and may be omitted.

Lemma 4.7 (Iwaniec coefficient packing). For every contributing vector and every $\xi \in [60\sigma, 1/3 + 14\sigma]$, let $e_j = \log_N \min(D_j^\rho, z)$, sorted in non-increasing order. Then $\mathrm{WF}(d_*)$ holds and

$$h \le \widetilde{H}(\bar{e}_1, \bar{e}_2, \bar{e}_3). \tag{4.8}$$

Consequently Theorem 4.1 partitions the tail labels into three slots. Assign the Iwaniec smooth divisor $d_{AA} < D^\sigma$ to slot 1. Let Q, S, R be the resulting lower

dyadic slot scales. Then

$$\begin{aligned}\log_N Q &\le \xi - \left(1+\frac{7271}{256}\right)\sigma,\\ \log_N(T^2SR^2) &\le 1-60\sigma,\\ \log_N(A_{\max}^{\vartheta}TQS^5R^2) &\le 2-\left(15+\frac{95221}{1280}\right)\sigma,\\ \log_N\left((TR/S)^{2\vartheta}TQS^5R^2\right) &\le 2-\left(15+\frac{22407}{256}\right)\sigma,\end{aligned} \tag{4.9}$$

where $T=N^{\xi}$ and $A_{\max}=X^{1+\sigma/20}$ for $X\le N$.

Proof. Put $s_j=\log_N D_j$. The grid lower bound gives $s_j\ge\sigma^2\theta$. The cubic conditions in (4.7) imply $\sum_{i<j}s_i+2s_j<\theta$ for every rank j. At a cubic rank this follows by dropping one nonnegative copy of s_j. At an intermediate rank,

$$\sum_{i<j}s_i+2s_j=\sum_{i<j-1}s_i+s_{j-1}+2s_j\le\sum_{i<j-1}s_i+3s_{j-1}<\theta,$$

using $s_{j-1}\ge s_j$ and the cubic condition at rank $j-1$. For the first rank in the minus case use $2s_1<\theta$, which follows from $D_1<D^{1/2}$. Thus $\sum_{i<j}e_i+2e_j\le\rho(\sum_{i<j}s_i+2s_j)<\rho\theta=d_*$. Also $e_j\le u$, so $\bar e_1\le u<\mu$. For this range, the branch $w(\bar e_1,\bar e_2,\bar e_3)$ is active. Its activation threshold is at most $(1+u)/2\le 0.501$, while

$$\min\left\{\frac{132-68u}{214},\frac{153-96u}{256},1-2u\right\}=\frac{153-96u}{256}=H_{\mathrm{small}}(\vartheta,u)=\Theta(u).$$

Each branch in the minimum decreases with its argument, and $\bar e_2,\bar e_3\le u$. Take $0\le u\le 1/500$. Each other affine branch minus $(153-96u)/256$ is nonnegative at both endpoints. Hence it is nonnegative on the whole interval. In particular, the minimum defining $w(\bar e_1,\bar e_2,\bar e_3)$ is at least $\Theta(u)>0.501$, so its activation condition holds. Thus $\widetilde H\ge\Theta(u)$. Next, $\tau\theta+60\sigma<\eta_\Theta$, because $60\sigma\le 0.3\eta_\Theta$ and $\tau\theta<0.6\sigma^9\le 0.6\sigma<0.7\eta_\Theta$. Therefore

$$\frac12<h=\rho\theta+60\sigma=\Theta(u)-\eta_\Theta+\tau\theta+60\sigma<\Theta(u)<\overline{H}.$$

Also $0<\delta\le 3\cdot10^{-9}$. The range for ξ gives $g=\xi-\delta\in[\delta,1/3-\delta/2]$. Theorem 4.1 applies.

The smooth-only term is the empty-list case. Here $\widetilde H(0,0,0)=H(0,0,0)=\overline{H}$, since the w branch equals $\min\{132/214,153/256,1\}=\overline{H}$ and its activation threshold is $1/2$. Thus Theorem 4.1 applies to the empty list as well. The smooth divisor remains in slot 1, and slots 2 and 3 are units.

For the scale bounds, the slot-1 smooth divisor contributes at most $\sigma\theta<\sigma\overline{H}$ to the logarithmic load. The other slot loads are bounded by Σ_2,Σ_3. Thus

$$\begin{aligned}&\log_N Q\le\Sigma_1+\sigma\overline{H}, \qquad \log_N S\le\Sigma_2, \qquad \log_N R\le\Sigma_3,\\ &\log_N T=\xi=g+\delta, \quad \log_N A_{\max}\le 1+\sigma/20.\end{aligned}$$

The first line of (4.3) gives

$$\log_N Q-\xi\le-(30-\overline{H})\sigma=-\left(1+\frac{7271}{256}\right)\sigma.$$

The second gives

$$\log_N(T^2SR^2)\le 1-2\delta=1-60\sigma.$$

For the third scale, (4.3) gives

$$\log_N(A_{\max}^{\vartheta}TQS^5R^2) \le 2-3\delta+\left(\frac{\vartheta}{20}+\overline{H}\right)\sigma = 2-\left(15+\frac{95221}{1280}\right)\sigma.$$

For the fourth, it gives

$$\log_N((TR/S)^{2\vartheta}TQS^5R^2) \le 2-(3+2\vartheta)\delta+\overline{H}\sigma = 2-\left(15+\frac{22407}{256}\right)\sigma.$$

The exact constants in (4.9) hold for the lower scales. We round each of the three aggregate slot scales Q, R, S once, to a dyadic representative within a factor two. Each monomial below has total degree at most 8 in Q, R, S. So rounding multiplies it by at most 2^8. Individual prime labels are not rounded. This adds at most $8\log 2/\log N$ to its base-N exponent. Fix $\sigma > 0$. For sufficiently large N, each positive margin absorbs this cost. The tightest margin, for $Q < TX^{-29\sigma}$, is $103\sigma/256 - 8\log 2/\log N > 0$. □

For fixed $B > 0$ and $N/(2(\log N)^B) \le X \le N$, put $\omega = \log_N(N/X)$. Then $\omega \le (B\log\log N + \log 2)/\log N$, so eventually $\log_X N = (1-\omega)^{-1} < 1 + 0.01\sigma$. The first exponent in (4.9), after dividing by T, is negative. So conversion from base N to base X improves that relative inequality. Each other exponent is at most 2; conversion costs at most 0.02σ. The displayed margins absorb this cost.

For every contributing vector, the last-rank mask gives $\sum_j s_j < \theta$. Since each $s_j \ge \sigma^2\theta$, it also gives $k < \sigma^{-2}$. Thus $\prod_j D_j < D$, and $p_j < D_j^{\rho}$ gives

$$d\prod_j p_j < D^{\sigma}D^{\rho} = D^{1+\sigma+\tau} = N^{(1+\sigma+\tau)\theta} < N^{\Theta(u)},$$

where $(\sigma+\tau)\theta < \eta_\Theta$. All prime factors avoid $2N$. Thus the full coefficient is supported on $q < N^{\Theta(u)}$ with $(q, 2N) = 1$. Its coefficients are $O_\sigma(1)$. The Y-smooth part of a supported modulus determines d. The remaining labelled prime assignments number at most $k!$, and $k < \sigma^{-2}$. For each sign the Iwaniec family has fewer than $e^{8\sigma^{-3}}$ vectors. So the sum over vectors keeps a bound depending only on σ. The smooth-only term satisfies the same support and coefficient bounds.

The cubic-to-quadratic implication just proved is uniform in the vector length. We record the two bounds obtained above:

$$k < \sigma^{-2}, \qquad \sum_{j=1}^{k} s_j < \theta. \tag{4.10}$$

These bounds also control the coefficient regrouping below.

4.4. **Three-slot coefficient identity.** Fix a sign ν and a vector $\mathbf{D}$. The set of equal-label pairs in the Iwaniec grid is $I(\mathbf{D}) = \{(i,j) : 1 \le i < j \le k,\ D_i = D_j\}$. Distinct grid labels have disjoint half-open prime intervals $[D_i, \min(D_i^{\rho}, z))$. Adjacent grid values differ by the power ρ. Thus repeated primes can occur only in $I(\mathbf{D})$. Define

$$K_{\mathbf{D}}(\mathbf{p}) = \prod_{(i,j)\in I(\mathbf{D})}(1-\mathbf{1}_{p_i=p_j}).$$

For the Iwaniec coefficients $\lambda_d^{\nu}(\mathbf{D})$, define the unrestricted coefficient

$$C_{\mathbf{D}}^{\mathrm{full}}(q) = \sum_{\substack{d<D^{\sigma}\\ d\mid P_N(Y)}}\lambda_d^{\nu}(\mathbf{D})\sum_{\substack{D_j\le p_j<\min(D_j^{\rho},z)\\ p_j\nmid 2N}}\mathbf{1}_{q=d\prod_{j=1}^{k}p_j}. \tag{4.11}$$

The masked coefficient $C_{\mathbf{D}}^{\mathrm{mask}}$ is defined by inserting $K_{\mathbf{D}}(\mathbf{p})$ in the inner sum. The coefficient of the smooth-only term, the sole empty-tail term in the Iwaniec formula, is $C_\emptyset(q) = \sum_{d<D^\sigma, d\mid P_N(Y)} \phi_d^\nu \mathbf{1}_{q=d}$. Thus the full signed coefficient is

$$w_N^\nu(q) = C_\emptyset(q) + \sum_{\mathbf{D}\in\mathcal{D}^\nu(D)} C_{\mathbf{D}}^{\mathrm{mask}}(q). \tag{4.12}$$

Lemma 4.8 (slot factorization and collision support). Fix ξ in the range of Lemma 4.7 and let $J_1 \sqcup J_2 \sqcup J_3$ be its packing partition. All d and p_j sums below use the ranges in (4.11). Define

$$\gamma_1(m) = \frac{1}{|J_1|!} \sum_{d,(p_j)_{j\in J_1}} \lambda_d^\nu(\mathbf{D}) \mathbf{1}_{m=d\prod_{j\in J_1} p_j},$$
$$\gamma_s(m) = \frac{1}{|J_s|!} \sum_{(p_j)_{j\in J_s}} \mathbf{1}_{m=\prod_{j\in J_s} p_j}, \qquad s = 2, 3.$$

For $s = 2, 3$ set $\gamma_s = \mathbf{1}_{m=1}$ when J_s is empty; γ_1 always retains the smooth-divisor sum. Then

$$C_{\mathbf{D}}^{\mathrm{full}}(q) = C_{\mathrm{slot}}(\gamma_1 * \gamma_2 * \gamma_3)(q), \qquad C_{\mathrm{slot}} = |J_1|!|J_2|!|J_3|!, \tag{4.13}$$

and $|\gamma_s(m)| \le 1$. Consider the full Iwaniec family. The difference between the masked and unrestricted coefficients has pointwise size $O_\sigma(1)$. It is supported on moduli divisible by p^2 for some prime $p \ge Y$.

Proof. A labelled tuple $(d, (p_j)_{j\le k})$ splits uniquely into its three slot tuples. Multiplying the three normalized finite sum definitions by C_{slot} therefore gives (4.13). This regroups a finite sum and allows repeated primes. Fix a product m. Its prime factors below Y determine d uniquely: $d \mid P_N(Y)$, and every tail prime is at least $D_j \ge Y$. Once d is fixed, the labelled prime assignments within a slot number at most $|J_s|!$. The Iwaniec bounds $|\lambda_d^\nu(\mathbf{D})| \le 1$ give $|\gamma_s(m)| \le 1$; the same argument applies to γ_1. The smooth-only term has the corresponding bound from $|\phi_d^\nu| \le 1$. Since $k < \sigma^{-2}$, the scalar $C_{\mathrm{slot}} = |J_1|!|J_2|!|J_3|!$ is $O_\sigma(1)$.

Suppose the distinctness mask removes a tuple. Then some selected pair has $p_i = p_j \ge Y$, and its modulus is divisible by p_i^2. For fixed σ, (4.10) bounds the number of labels. The Iwaniec family has fewer than $e^{8\sigma^{-3}}$ vectors for each sign. The number of labelled assignments to a fixed modulus is at most $k!$. Hence the aggregate coefficient difference is $O_\sigma(1)$.

The prime-source cost of this collision support is estimated in (B.5)–(B.6). □

The smooth-only term $C_\emptyset$ is already a one-slot coefficient. Take $J_1 = J_2 = J_3 = \emptyset$, put $\gamma_1(m) = \sum_{d<D^\sigma, d\mid P_N(Y)} \phi_d^\nu \mathbf{1}_{m=d}$, and $\gamma_2 = \gamma_3 = \mathbf{1}_{m=1}$. Its coefficient identity and pointwise bound follow directly. The full coefficient w_N^ν in (4.12) is the sum of this term and the vector terms above. Every partition is fixed before the prime realizations are summed.

This section proves the packing inequalities and the slot identity for the unrestricted coefficient. It also gives the collision support needed in Appendix B. The prime-source estimate is proved there.

5. Application to Iwaniec's remainder weights

The source is the prime interval $I_N = (N/(\log N)^B, (1-\varepsilon)N)$ from Theorem 1.2. The residue satisfies $N \le a \le N \log N/3$ and is coprime to every supported modulus. The saving is $(\log N)^{-A}$ for each fixed $A > 0$.

5.1. **Iwaniec's remainder coefficients.** All parameters are those in Theorem 1.2; in particular $s = \log D/\log z = \theta/u$. Let $N > 2$ be even. Let $\mathcal{P}_N$ be the set of primes not dividing $2N$, and write

$$P_N(t) = \prod_{\substack{p<t \\ p\in\mathcal{P}_N}} p.$$

Use the grid $\mathcal{G}(D)$ in (2.28). The admissible vector sets $\mathcal{D}^\nu(D)$ are those in (2.29). The parameter substitution is (2.33a)–(2.33b), with $\epsilon_s = \sigma$ and $\delta_s = \tau$. The distinctness set is $I(\mathbf{D}) = \{(i,j) : 1 \le i < j \le k,\ D_i = D_j\}$, the equal-label tail-position pairs. For a vector $\mathbf{p} = (p_1, \dots, p_k)$ define

$$K_{\mathbf{D}}(\mathbf{p}) = \prod_{(i,j)\in I(\mathbf{D})} (1 - \mathbf{1}_{p_i=p_j}).$$

Let ϕ_d^ν and $\lambda_d^\nu(\mathbf{D})$ be the coefficients of Iwaniec [10, Theorem 4]. Take $R = 1$ and parameters $(\mathcal{P}_N, D, z, \sigma)$ at the cutoff $z = N^u$. The proof uses only their bound $|\cdot| \le 1$ and their support. It never compares coefficient values at different cutoffs. The labelled-tail cutoff of C1 in Section 3, used in (H1), is $u_L = Y = D^{\sigma^2}$. Since $\theta > 1/2 > \ell_0$, this satisfies $u_L = N^{\theta\sigma^2} \ge N^{\ell_0\sigma^2} = u_0$. The smooth divisor has $d < D^\sigma$ and $d \mid P_N(Y)$. Every tail interval starts at $D_i \ge Y$. The exact structured remainder coefficients are

$$\begin{aligned} w_N^\nu(q) = {} & \sum_{\substack{d<D^\sigma \\ d\mid P_N(Y)}} \phi_d^\nu \mathbf{1}_{q=d} \\ & + \sum_{\mathbf{D}\in\mathcal{D}^\nu(D)} \sum_{\substack{d<D^\sigma \\ d\mid P_N(Y)}} \lambda_d^\nu(\mathbf{D}) \sum_{\substack{p_i\in\mathcal{P}_N \\ D_i\le p_i<\min(D_i^{1+\tau},z)}} K_{\mathbf{D}}(\mathbf{p})\, \mathbf{1}_{q=dp_1\cdots p_k}. \end{aligned} \tag{W}$$

In particular, $\nu \in \{+,-\}$ selects one of the two signed remainders. Each is the full remainder, not an arbitrary well-factorable sequence. Formula (W) includes the smooth-only term. Every prime-tuple term keeps its smooth divisor. The smooth-only term retains its divisor when there is no tail. The estimate below applies separately to either signed remainder. When the directed lower-sieve inequality of Iwaniec [10, Theorem 4] is used, it takes $\nu = -$.

5.2. **Proof of the interior signed mean value.**

Proof of Theorem 1.2. The support and coefficient bounds in the theorem are proved in (B.2), using Iwaniec's cubic label conditions for every vector in (W). In particular, with $e_Q = (1+\sigma+\tau)\theta$,

$$q < D^{1+\sigma+\tau} = N^{e_Q} < N^{\Theta(u)}, \qquad (q, 2N) = 1,$$

including the smooth-only term. The choice $\sigma \le (\Theta(u) - \theta)/200$ gives the strict second inequality. Take the source $\mu_D = \mathbf{1}_{\mathbb{P}}\mathbf{1}_{I_N}$ of (B.1). By (B.3), $p > q$ for

every source prime p and every supported modulus q. The coprimality $(a,q)=1$ is assumed. So the natural comparator in (B.1) is exactly $\pi_{I_N}/\varphi(q)$, and

$$\Delta_D(q;a)=\pi_{I_N}(q,a)-\frac{\pi_{I_N}}{\varphi(q)}.$$

Before applying the dispersion theorem, (B.5)–(B.6) replace w_N^ν by $\widehat{w}_N^\nu(q)=\mathbf{1}_{(q,a)=1}\widetilde{w}_N^\nu(q)$, where $\widetilde{w}_N^\nu$ has the distinctness masks removed. By the hypothesis on the support of w_N^ν, the difference $w_N^\nu-\widehat{w}_N^\nu$ is a collision contribution. By (B.5), its complete signed discrepancy is $O_\sigma(N\log N/Y)$. The coprimality projection $\mathbf{1}_{(q,a)=1}$ factors across the three slots in (4.13), as shown after (B.6); this includes the smooth-only term. Thus the retained coefficient has the exact slot form (3.5). All its moduli are coprime to a. Appendix B extends the support envelope $q<N^{e_Q}$ of (B.2) to $\widetilde{w}_N^\nu$ and $\widehat{w}_N^\nu$. For each source scale $X>N/(2(\log N)^B)$, (B.3) then places this entire retained support below $X^{0.6}$. The residue bound gives $a/X<(2/3)(\log N)^{B+1}$. Hence C4 holds with $K_{\mathrm{scale}}=B+1$. This includes $a\le X^{1+\sigma/20}$ for large N. On the doubled complementary support, (B.28) gives $a-mn\ge\varepsilon N/2$. It uses $a\ge N$ and the upper source cut. Since $a\ll N\log N$ and $mn\ll N$ there, also $|mn-a|\le X^3$ for large N. These are both parts of the nonresonance condition (3.10).

After the collision projection, the prime-power contribution is (B.8). The logarithm-freezing error is (B.18). The source-boundary error is (B.19). Lemma B.3 proves the selected-interval lengths and integer endpoint convention in (B.22)–(B.27). Lemma 4.7 supplies the slot inequalities, while (B.20)–(B.21) classify every retained box. The internal Type-I and Type-II bounds, including the box and arithmetic costs, give (B.31). The integer choice $C_{\mathrm{box}}\ge\lceil 2A+32\rceil$ makes the logarithm-freezing and boundary errors in (B.18)–(B.19) $O(N(\log N)^{-A})$. The aggregate exponent $A_{\mathrm{agg}}=A$ here is the theorem convention; the separate $A_{\mathrm{agg}}=A+3$ bookkeeping used for (3.95) requires $C_{\mathrm{box}}\ge\lceil 2A+38\rceil$ and is outside Theorem 3.2. $A'=A+24C_{\mathrm{box}}+31$ and $A_{\mathrm{work}}=A'+4$ absorb the box count. Appendix B verifies the box-family bound (H7), that is (3.91) with $K_0=24C_{\mathrm{box}}+26$. The aggregate form of Theorem 3.2, namely (3.92), bounds the Type-II boxes. For each fixed admissible $\sigma>0$, the mask and proper-power terms have fixed power savings. Proposition B.4 combines these terms over the complete support of (W). This gives the assertion for each fixed $A,B,\varepsilon,u,\theta,\sigma$ in the stated ranges. □

5.3. **Proof of the logarithmic-integral comparator.**

Proof of Corollary 1.3. Put $L=\log N$. The coefficient bound in Theorem 1.2 and $\sum_{q\le x}1/\varphi(q)\ll\log(2x)$ give $\sum_q|w_N^\nu(q)|/\varphi(q)\ll_{\sigma,u,\theta}L$. The prime number theorem at the two endpoints of I_N gives

$$\pi_{I_N}=\mathrm{Li}((1-\varepsilon)N)-\mathrm{Li}(N/L^B)+O_A(NL^{-A-1})$$

for every fixed $A>0$. Strict endpoint conventions contribute at most two primes. Multiply this error by the reciprocal-totient sum and apply (1.2). □

Appendix A. Source checks

The source input used for the coefficients in formula (W) is [10, Theorem 4]. Its hypotheses are [10, (1)–(2)]. Its upper inequality permits $2\le z\le D$; its lower inequality permits $2\le z\le D^{1/2}$. Fix $0<u\le 1/500$ and $1/2<\theta<\Theta(u)$ as in

Theorem 1.2. Then $D = N^\theta$ and $z = N^u$ satisfy $2u < \theta$. Thus both ranges apply for all sufficiently large N. The choice $\sigma \le \sqrt{u/(2\theta)}$ also gives $Y = D^{\sigma^2} \le N^{u/2} < z$. The mesh is $\epsilon_s = \sigma$ and $\delta_s = \sigma^9 = \tau$. The smooth-prime cutoff is Y, as in (2.28)–(2.33b).

Theorem 4's printed argument list of λ_d^ν contains the mesh parameter and box tuple. It contains neither the sequence nor the cutoff. The construction in [10, (23)–(26)] collects universal Rosser and fundamental-lemma coefficients. The sequence enters through $r(\mathcal{A}, q)$. At each cutoff z, we use the coefficients and prime-tuple support supplied by that construction. The coefficient family is independent of the sequence. It may depend on z. The signed prime-source estimate is proved separately in Appendix B.

The O-constant in the main-term error of [10, Theorem 4] is not given numerically.

The proof of Theorem 1.2 uses no finite computation.

Appendix B. Prime-source estimate for Iwaniec's remainder weights

This section proves the source estimate used in Section 5. Equations (B.2)–(B.6) control the coefficients and distinctness masks. Equations (B.7)–(B.19) treat prime detection, logarithm freezing and boundary boxes. Equations (B.20)–(B.32) bound the retained boxes.

Fix the parameters in Theorem 1.2 and put

$$\tau = \sigma^9, \quad e_Q = (1+\sigma+\tau)\theta < \Theta(u) \le \frac{153}{256},$$
$$D = N^\theta, \quad Y = D^{\sigma^2}.$$

Put $s = \log D/\log z = \theta/u$, $z = N^u$, $L = \log N$, $H = N/L^B$, and use the width C_{box} in Theorem 1.2. Write $\mathcal{S}_n(q;a) = \mathbf{1}_{n\equiv a\ (q)} - \mathbf{1}_{(n,q)=1}/\varphi(q)$. All thresholds below may depend on the fixed parameters. For a product shell $[X, 2X)$, set $L_X = \log X$. The source measure is

$$\mu_D(n) = \mathbf{1}_{\mathbb{P}}(n)\mathbf{1}_{H<n<(1-\varepsilon)N},$$
$$\Delta_D(q;a) = \sum_n \mu_D(n)\left(\mathbf{1}_{n\equiv a\ (q)} - \frac{\mathbf{1}_{(n,q)=1}}{\varphi(q)}\right). \tag{B.1}$$

Lemma B.1 (Modulus support and coefficient size)**.** *For either sign in* (W),

$$\operatorname{supp} w_N^\nu \subset \{q < N^{e_Q} : (q, 2N) = 1\}, \qquad \sup_q |w_N^\nu(q)| \le C_{\sigma,u,\theta}. \tag{B.2}$$

If $[X, 2X)$ *meets* $(H, (1-\varepsilon)N)$*, then, for all sufficiently large* N*,*

$$q < N^{e_Q} \le X^{3/5} \quad (q \in \operatorname{supp} w_N^\nu), \qquad H > N^{e_Q}. \tag{B.3}$$

Proof. By the definition of $\mathcal{D}^\nu(D)$ in (2.29), every label vector in (W) satisfies the cubic label conditions of Iwaniec's sieve remainder [10, Theorem 4]. Write $s_i = \log_N D_i$ for a label vector $\mathbf{D} = (D_1, \dots, D_k)$. The cubic conditions in Iwaniec imply $\sum_i s_i < \theta$. For the plus sign and odd k, use the last odd-rank condition. For even k, use the condition at rank $k-1$ and $s_k \le s_{k-1}$. For the minus sign, if $k = 1$ then $s_1 < \theta/2$. If k is even, use the last even-rank condition. If $k \ge 3$ is odd, use rank $k-1$ and $s_k \le s_{k-1}$. Thus every term of (W) satisfies

$$q < N^{\sigma\theta+(1+\tau)\sum_i s_i} < N^{\theta(1+\sigma+\tau)} = N^{e_Q}.$$

All prime factors in d and in the tail avoid $2N$.

Each grid coordinate has the form $D^{\sigma^2(1+\tau)^h}$ and is less than $D^{1/2}$. Thus there are at most $1 + 2\tau^{-1}\log(\sigma^{-2})$ choices per position, since $\log(1+\tau) \geq \tau/2$. Also $k < \sigma^{-2}$ by the last-rank cubic condition and $s_i \geq \sigma^2\theta$. With $\tau = \sigma^9$ and $0 < \sigma \leq 10^{-10}$, the total number of vectors of all lengths is at most

$$\sum_{0\leq k<\sigma^{-2}} \bigl(1 + 2\sigma^{-9}\log(\sigma^{-2})\bigr)^k < \exp(8\sigma^{-3}).$$

The logarithm of the left side is at most $\log(1+\sigma^{-2}) + \sigma^{-2}\log(1+4\sigma^{-9}\log(1/\sigma))$. With $t = 1/\sigma \geq 10^{10}$, this is at most $(3+11t^2)\log t < 8t^3$, using $\log t \leq t/2$. For fixed q, the primes below Y determine the smooth divisor d uniquely. The remaining prime factors can be assigned to the k labelled positions in at most $k!$ ways. The Iwaniec coefficients have absolute value at most 1. Summing over the displayed finite vector family proves the multiplicity bound, including the empty-tail term.

If the shell meets the source interval, then $X > H/2$ and $\log X > L - B\log L - \log 2$. Impose

$$(3/5 - e_Q)L \geq (3/5)(B\log L + \log 2). \tag{B.4}$$

It gives $e_Q L \leq (3/5)\log X$, proving the first claim of (B.3). The second follows from $N^{1-e_Q}/L^B \to \infty$. □

Here $p > H > N^{e_Q}$, and the theorem assumes $(a,q) = 1$ on the coefficient support. So the natural comparator in (B.1) is $\pi_{I_N}/\varphi(q)$. Thus (B.1) is the exact prime-source discrepancy for this source.

Removing the distinctness masks. Let $\widetilde{w}_N^\nu$ be obtained from (W) by replacing each $K_{\mathbf{D}}$ by 1, and put $\widehat{w}_N^\nu(q) = \mathbf{1}_{(q,a)=1}\widetilde{w}_N^\nu(q)$. The counting argument in Lemma B.1 also gives $\sup_q |\widetilde{w}_N^\nu(q)| + \sup_q |\widehat{w}_N^\nu(q)| \ll_{\sigma,u,\theta} 1$. Its termwise product bound applies to every unmasked term. So $\widetilde{w}_N^\nu$ and $\widehat{w}_N^\nu$ are also supported on $\{q < N^{e_Q} : (q,2N) = 1\}$, and (B.3) applies to them. Their support need not lie in $\operatorname{supp} w_N^\nu$. The theorem assumes $w_N^\nu(q) = 0$ when $(q,a) > 1$. Hence $w_N^\nu - \widehat{w}_N^\nu$ is supported on the terms removed by a distinctness mask. This holds even when an unrestricted collision modulus is not coprime to a. Every such modulus is divisible by p^2 for some tail prime $p \geq Y$. The coefficient difference is $O_\sigma(1)$. Put $Q_{\mathrm{supp}} = N^{e_Q}$.

$$\sum_{\substack{q\leq Q_{\mathrm{supp}}\\ p^2\mid q \text{ for some prime } p\geq Y}} \left(\frac{N}{q} + 1 + \frac{N}{\varphi(q)}\right) \ll \frac{N\log(2Q_{\mathrm{supp}})}{Y-1}. \tag{B.5}$$

For the proof, write $q = kp^2$, use $\varphi(kp^2) \geq \varphi(k)\varphi(p^2)$, and sum $1/k$ and $1/\varphi(k)$ up to Q_{supp}. Then use $\sum_{p\geq Y} 1/(p(p-1)) \ll 1/(Y-1)$. The left side bounds the discrepancy of each removed term. Its progression count is at most $N/q + 1$. Consequently

$$\sum_q w_N^\nu(q)\Delta_D(q;a) = \sum_q \widehat{w}_N^\nu(q)\Delta_D(q;a) + O_\sigma(NL/Y). \tag{B.6}$$

As $Y = N^{\theta\sigma^2}$, this error has a fixed power saving for each fixed $\sigma > 0$. For a fixed Iwaniec vector, the exact factorization (4.13) survives the projection. Replace each slot factor $\gamma_i(m)$ by $\gamma_i(m)\mathbf{1}_{(m,a)=1}$. Here $(m_1m_2m_3, a) = 1$ if and only if each $(m_i, a) = 1$. The bounds $|\gamma_i| \leq 1$ are unchanged. Equivalently, the smooth divisor

and each tail prime are restricted to avoid a within their existing summation ranges. Thus the projected coefficient still has the labelled form (3.4). Every modulus to which the dispersion theorem is applied is coprime to $2a$.

Prime detection and the rough Heath–Brown identity. The shells meeting the source satisfy $H/2 < X < N$. Set

$$z_0 = z_0(X) = X^{1/(\log\log X)^3}, \qquad U = 2X^{1/3}.$$

For sufficiently large N, every $n \in [H, (1-\varepsilon)N]$ has $n \geq z_0$. Write $\Lambda_{z_0}(n) = \Lambda(n)\mathbf{1}_{P^-(n)\geq z_0}$. For every integer $n \geq z_0$, and so on the whole source range,

$$\mathbf{1}_{\mathbb{P}}(n) = \frac{\Lambda_{z_0}(n)}{\log n} - \sum_{k\geq 2} \frac{1}{k}\mathbf{1}_{n=p^k,\ p\geq z_0}. \tag{B.7}$$

The identity follows by checking primes, prime powers, and all other integers $n \geq z_0$. It fails for primes below z_0, which lie outside the source range. The proper powers contribute to the weighted discrepancy

$$O_\sigma(N^{1/2+o(1)}), \tag{B.8}$$

in total. There are $O(N^{1/2}\log N)$ such integers in the source range $H < n < (1-\varepsilon)N$. The theorem's range $N \leq a \leq NL/3$ gives $a \neq n$ there. Each contributes at most $O_\sigma(d(a-n)+L)$, where $d(m)$ is the divisor function.

For $n \leq 8X$, the order-three rough Heath–Brown identity is

$$\Lambda_{z_0}(n) = \sum_{j=1}^{3}(-1)^{j-1}\binom{3}{j} \sum_{\substack{m_1\cdots m_j n_1\cdots n_j = n \\ m_i\leq U,\ m_i, n_i \in \mathcal{R}_{z_0}}} \mu_{z_0}(m_1)\cdots\mu_{z_0}(m_j)\log n_1. \tag{B.9}$$

Here $\mathcal{R}_{z_0}$ consists of 1 and the z_0-rough integers, and $\mu_z(m) = \mu(m)\mathbf{1}_{P^-(m)\geq z}$. To verify (B.9), put

$$\zeta_z(s) = \sum_{r\in\mathcal{R}_z} r^{-s}, \qquad M_U(s) = \sum_{m\leq U}\mu_z(m)m^{-s}.$$

The coefficients of $\zeta_z M_U$ vanish except at 1 through degree U. Thus all coefficients of $(1-\zeta_z M_U)^3$ at $n \leq U^3$ vanish. Multiply the expansion of $1-(1-\zeta_z M_U)^3$ by $-\zeta_z'/\zeta_z$ to obtain (B.9). Every variable is rough; this includes the logarithmic variable n_1.

The tested indices for shell X and pattern j are

$$T_{X,j} = \left\{ \begin{array}{l} \mathbf{v} = (m_1,\dots,m_j,n_1,\dots,n_j) \in \mathcal{R}_{z_0}^{2j} : \\ \quad X \leq \displaystyle\prod_i v_i < 2X, \quad H < \prod_i v_i < (1-\varepsilon)N, \\ \quad m_i \leq U \end{array} \right\}. \tag{B.10}$$

There are at most six variables. Their coefficient is

$$c_j(\mathbf{v}) = (-1)^{j-1}\binom{3}{j}\mu_{z_0}(m_1)\cdots\mu_{z_0}(m_j)\log n_1.$$

The shell, pattern, and tuple are part of the index. The identity's range $n \leq 8X$ is automatic on $T_{X,j}$.

For every nonnegative G and each $j \in \{1,2,3\}$,

$$\sum_X \sum_{\mathbf{v}\in T_{X,j}} G\left(\prod_i v_i\right) \le \sum_{n<N} \tau_{2j}(n)G(n) \le \sum_{n<N} \tau_6(n)G(n). \tag{B.11}$$

The product determines its shell. Roughness restrictions can be dropped in this upper bound. Consequently the sum over all three patterns is at most $3\sum_{n<N}\tau_6(n)G(n)$. The factor 3 is absorbed in later implied constants.

For $n < N$, (B.2) and its extension to $\widehat{w}_N^\nu$ after (B.5) give the pointwise majorant

$$\left|\sum_q \widehat{w}_N^\nu(q)\mathcal{S}_n(q;a)\right| \le C_\sigma F_a(n), \qquad F_a(n) = d(a-n) + 2(1+L).$$

The congruence term counts $q \mid a-n$. The natural-comparator term uses $\sum_{q\le N^{\epsilon_Q}} 1/\varphi(q) \ll L$. On the prime source, (B.1) equals $\sum_n \mu_D(n)\mathcal{S}_n(q;a)$. Apply (B.7) to this identity while keeping $\mathcal{S}_n(q;a)$ attached to the tested integer. Each HB tuple then contributes the natural PSDT discrepancy $\mathcal{S}_{\prod_i v_i}(q;a)$. This holds also when its product shares factors with q. The proper-power terms use the same discrepancy and obey (B.8).

Equal-ratio boxes and faces. For each product shell put

$$\delta_X = L_X^{-C_{\mathrm{box}}}, \qquad J_X = \left\lceil \frac{\log 2}{\log(1+\delta_X)} \right\rceil, \qquad \Delta_X = 2^{1/J_X} - 1.$$

For sufficiently large N, $\delta_X \le 1/3$. The equal-ratio partition divides every dyadic coordinate interval $[2^a, 2^{a+1})$ into the J_X cells

$$[2^a(1+\Delta_X)^h,\ 2^a(1+\Delta_X)^{h+1}), \qquad 0 \le h < J_X.$$

It covers the full interval, including its last cell. Also

$$\frac{\delta_X}{3} \le \delta_X(1-2\delta_X) \le \Delta_X \le \delta_X. \tag{B.12}$$

For the lower bound, set $\ell = \log(1+\delta_X)$. Since $J_X < \log 2/\ell + 1$,

$$\log(1+\Delta_X) > \frac{\ell}{1+\ell/\log 2} \ge \ell(1-\ell/\log 2).$$

Use $\delta_X - \delta_X^2/2 \le \ell \le \delta_X$ and $1/2 + 1/\log 2 < 2$. This proves (B.12).

The mesh is imposed on the ambient integer axes before roughness is tested. Keep $\{1\}$ as an atom. A product box is a Cartesian product of at most six cells. Let V_i be its lower endpoints and put $e_i = \log_N V_i$; the atom has $V_i = 1$. Define $\lambda_{\mathcal{B}} = \sum_i \log V_i$. Its tested-side faces are

$$n = H, \quad n = (1-\varepsilon)N, \quad n = X, \quad n = 2X, \quad m_i = 2X^{1/3}, \tag{B.13}$$

where $n = \prod_i v_i$ and m_i ranges over the Möbius variables. The first two are the source cuts. The next two are product-shell edges, and the last are the HB caps. Roughness is part of the coefficient, not a face. A box is called boundary if one predicate in (B.13) is not constant on it. Every other box is either empty or a full Cartesian sum.

Lemma B.2 (Lattice thin-shell count)**.** *Let $s \geq 2$, $a \in \mathbb{Z}^s \setminus \{0\}$ with $|a_i| \leq 2$, and $J = \{i : a_i \neq 0\}$. Let $1 \leq Y_0 \leq N$, $0 < \eta_{\mathrm{dist}} \leq 1/4$, and let $\mathcal{L}$ be γ-separated. Put $K_\gamma = (1 - e^{-\gamma/2})^{-1}$. Then*

$$\begin{aligned} &\#\{\mathbf{v} \in \mathbb{Z}^s_{\geq 1} : \prod_i v_i \leq N, \mathrm{dist}(\sum_i a_i \log v_i, \mathcal{L}) \leq \eta_{\mathrm{dist}}, \max_{i \in J} v_i \geq Y_0\} \\ &\qquad \leq s \left[4\eta_{\mathrm{dist}} K_\gamma N (1+L)^{s-1} + \left(\frac{4L}{\gamma} + 2 \right) \frac{N}{Y_0} (1+L)^{s-2} \right]. \end{aligned} \tag{B.14}$$

For a single face, take $K_\gamma = 1$ and replace $4L/\gamma + 2$ by 1.

Proof. Assign a tuple to the least active coordinate i with $v_i \geq Y_0$. Fix the other coordinates, with product P, and put $Z = N/P$. The nonempty fibre has $P \leq N/Y_0$. Let $q = |a_i| \in \{1, 2\}$. The admissible values of v_i lie in intervals of relative length at most $e^{2\eta_{\mathrm{dist}}/q} - 1 \leq 4\eta_{\mathrm{dist}}$. Their logarithmic left endpoints have spacing at least γ/q. After clipping to $[Y_0, Z]$ and merging overlaps, there are at most $2L/\gamma + 2$ components. The separated left endpoints give total length at most $4\eta_{\mathrm{dist}} K_\gamma Z$. Counting integer endpoints adds at most $2L/\gamma + 2$. For a singleton face there is one component. Since $2L/\gamma + 2 \leq 4L/\gamma + 2$, this gives the coefficient $4L/\gamma + 2$ in (B.14). Sum over the fixed coordinates using

$$\sum_{P \leq N} \frac{\tau_{s-1}(P)}{P} \leq (1+L)^{s-1}, \qquad \sum_{P \leq N/Y_0} \tau_{s-1}(P) \leq \frac{N}{Y_0} (1+L)^{s-2}.$$

This proves (B.14). □

Boundary and logarithm-freezing estimates. For every shell meeting the source,

$$L - B \log L - \log 2 < \log X < L. \tag{B.15}$$

Thus, since C_{box} is fixed, for every $\epsilon > 0$, $(\log X)^{-C_{\mathrm{box}}} \leq (1+\epsilon) L^{-C_{\mathrm{box}}}$ once N is large. Also $\log n$ and each nonempty box lower-corner logarithm are $(1 + o(1))L$ on the source.

A boundary box lies within $6\delta_X$ in logarithmic distance of a face in (B.13). The source and product-shell faces give two singleton families and the lattice $(\log 2)\mathbb{Z}$. The cap faces give $\log 2 + (\log 2/3)\mathbb{Z}$. In each boundary tuple, an active coordinate is at least $N^{1/7}$ for large N. For a product face, the largest of at most six factors is at least $(H/2)^{1/6}$. On a cap face the capped factor is at least $(H/4)^{1/3}$. Apply Lemma B.2 with $s \leq 6$, $Y_0 = N^{1/7}$, and $\eta_{\mathrm{dist}} \ll L^{-C_{\mathrm{box}}}$. The number of face families is finite and independent of the number of boxes. Let $\mathcal{U}$ be the set of boundary indices $(X, j, \mathbf{v})$ across shells and patterns for which $\mathbf{v} \in T_{X,j}$ lies in a boundary box. It excludes ambient Cartesian points outside the source-restricted set (B.10). The number of boundary indices is therefore

$$\#\mathcal{U} \ll N L^{5 - C_{\mathrm{box}}} + N^{6/7} L^5 \ll N L^{5 - C_{\mathrm{box}}}. \tag{B.16}$$

The elementary divisor estimates

$$\begin{aligned}\sum_{n<N}\tau_6(n)&\le N(1+L)^5,\\ \sum_{n<N}\tau_6(n)F_a(n)^2&\ll N(1+L)^{26},\\ \sum_{n<N}\tau_6(n)F_a(n)&\ll N(1+L)^{19}\end{aligned}\tag{B.17}$$

follow from $\tau_6^2\le\tau_{36}$, $d^2\le\tau_4$, $d^4\le\tau_{16}$, the divisor-sum bound $\sum_{n\le x}\tau_k(n)\le x(1+\log x)^{k-1}$, and Cauchy–Schwarz. For the last bound, apply Shiu's short-interval estimate [4] to $d(m)^2\le\tau_4(m)$: $\sum_{a-N<m\le a}d(m)^2\ll N(1+L)^3$ uniformly for $N\le a\le NL/3$. For $a\le 2N$ the global divisor-sum bound suffices. For $a>2N$ the interval length N exceeds $a^{2/3}$ for all sufficiently large N. Combining this with $\sum_{n<N}\tau_6(n)^2\ll N(1+L)^{35}$ gives the exponent 19. The bound for F_a^2 follows by summing $d(m)^4$ up to $a\le NL/3$. In particular, the LOG part from replacing $1/\log n$ by the lower-corner value $1/\lambda_{\mathcal B}$ on a box satisfies

$$|E_{\mathrm{LOG}}|\ll_\sigma NL^{18-C_{\mathrm{box}}}.\tag{B.18}$$

Here $0\le\log n-\lambda_{\mathcal B}\le 6\delta_X$, $|c_j(\mathbf v)|\le 3L$, and $|1/\log n-1/\lambda_{\mathcal B}|\ll L^{-C_{\mathrm{box}}-2}$. Sum the resulting $O(L^{-C_{\mathrm{box}}-1})F_a(n)$ over the tuples using (B.17). A tuple belongs to exactly one box, so there is no box-count factor.

For the boundary part, $|c_j(\mathbf v)|/\lambda_{\mathcal B}\le 3+o(1)$. Cauchy–Schwarz, (B.16), and the second estimate in (B.17) give

$$|E_{\mathrm{boundary}}|\ll_\sigma NL^{16-C_{\mathrm{box}}/2}.\tag{B.19}$$

Classification and interval lengths. The thresholds in (B.4), together with

$$\sigma L\ge 202(B\log L+\log 2),\qquad \sigma L\ge 202(\log 2+6),\tag{B.20}$$

ensure $\omega_X:=\log_N(N/X)\le\sigma/202$ and $\kappa_X:=(\log 2+6\log(1+\Delta_X))/L\le\sigma/202$. In this appendix κ_X plays the role of the κ in (3.98). For a box meeting $T_{X,j}$,

$$\begin{aligned}1-\omega_X-\kappa_X\le\sum_i e_i&\le 1-\omega_X+\kappa_X,\\ e_i&\le\frac{1-\omega_X}{3}+\kappa_X\quad\text{on each Möbius variable.}\end{aligned}\tag{B.21}$$

Here $\prod_i V_i\ge(\prod_i v_i)/(1+\Delta_X)^6$ and $\prod_i V_i\le\prod_i v_i<2X$. The Möbius bound follows from $V_i\le m_i\le 2X^{1/3}$. The corner error is independent of the lower-cut exponent B.

Use the window

$$\mathcal W=[60\sigma,\,1/3+14\sigma].$$

A box is Type II if some $e_i\in\mathcal W$. Select the largest such e_i, with ties broken by the fixed variable order. Suppose no exponent lies in $\mathcal W$. A box is Type I-1 when it has exactly one large variable and that variable is not a Möbius variable. It is Type I-2 when it has exactly two large variables and neither is a Möbius variable. There are no other nonboundary boxes. In fact, a large Möbius variable would imply $14\sigma\le\kappa_X-\omega_X/3\le\sigma/202$. If all variables were small, (B.21) would give $1-\sigma/101\le 360\sigma$. If three were large, it would give $1+42\sigma\le 1+\sigma/202$. Each is impossible.

Lemma B.3 (Lengths on retained Type-II boxes)**.** *Let $\mathcal{B}$ be a nonboundary Type-II box. Its selected cell is*

$$[T, r_X T), \qquad r_X = 1 + \Delta_X,$$

Let M be the product of the other lower endpoints. At least one complementary cell is nonunit, since the source product exceeds $H > N^{1/3+15\sigma}$ eventually. There are $1 \le r \le 5$ such cells. Set

$$I_T = (T, r_X T], \quad N_I = \lfloor r_X T \rfloor - \lfloor T \rfloor, \quad M_I = M(r_X^r - 1). \tag{B.22}$$

Then, for sufficiently large N,

$$N_I \asymp T L_X^{-C_{\mathrm{box}}}, \qquad M_I \asymp (X/T) L_X^{-C_{\mathrm{box}}}, \tag{B.23}$$

The PSDT interval I_T is contained in $(T, 2T]$. It has exactly the same selected integer coefficients as the half-open cell $[T, r_X T)$.

Proof. The real length of the selected cell is $h_N = T\Delta_X$. Both

$$\lceil r_X T \rceil - \lceil T \rceil \quad \text{and} \quad N_I$$

differ from h_N by less than 1. The only integer endpoints at which the half-open cell and I_T can differ are powers of two. Every internal mesh endpoint is $2^a 2^{h/J_X}$ with $0 < h < J_X$, and is irrational. Write $h/J_X = u/v$ in lowest terms with $v > 1$. If $2^{u/v}$ were rational, its 2-adic valuation would give $u = v\, v_2(2^{u/v})$, contradicting $(u, v) = 1$. At powers of two the selected rough coefficient vanishes, since $z_0 > 2$. Thus the two integer conventions agree after multiplication by the coefficient.

Put

$$\sigma L \ge 2^{130}(C_{\mathrm{box}} + 1)(\log L + 1). \tag{B.24}$$

Then $h_N \ge T/(3L_X^{C_{\mathrm{box}}}) > 2$, so $N_I \ge T/(6L_X^{C_{\mathrm{box}}})$ and $T/N_I \le 6L_X^{C_{\mathrm{box}}} \le 6L^{C_{\mathrm{box}}}$. Also $\prod_i V_i \asymp X$, hence $M \asymp X/T$ and $M_I \asymp L_X^{-C_{\mathrm{box}}} M$, proving (B.23). Write $x = \log_N X = 1 - \omega_X$, $\xi = \log_N T$, and $\Sigma_2 = \log_N S$, $\Sigma_3 = \log_N R$ for the slot scales in Lemma 4.7. The scale Q_{slot} is the first-slot scale in that lemma; Q_{supp} is the modulus cutoff in (B.5). The inequalities in (4.9) and $M_I \asymp XT^{-1}L_X^{-C_{\mathrm{box}}}$ give

$$\begin{aligned}
&\log_N(Q_{\mathrm{slot}}/T) \le -\left(1 + \frac{7271}{256}\right)\sigma < -29\sigma x,\\
&2\xi + \Sigma_2 + 2\Sigma_3 \le 1 - 60\sigma,\\
&\log_N \frac{M_I X^{-89\sigma}}{Q_{\mathrm{slot}} RS} \ge x - 2\xi - \Sigma_2 - \Sigma_3 + \left(1 + \frac{7271}{256}\right)\sigma - 89\sigma x - o(1)\\
&\qquad\qquad \ge \left(\frac{7271}{256} - 28\right)\sigma - \omega_X + \Sigma_3 + 89\sigma\omega_X - o(1) > 0.
\end{aligned} \tag{B.25}$$

Here $x = 1 - \omega_X$, $\omega_X \le \sigma/202$, and $\log L_X/L = o(1)$ as $N \to \infty$ for fixed σ, C_{box}. The terms $o(1)$ in (B.25) are taken in this sense. For fixed σ they are eventually smaller than $\sigma/1000$. Thus $Q_{\mathrm{slot}} < TX^{-29\sigma}$ and $Q_{\mathrm{slot}} RS < M_I X^{-89\sigma}$. It follows that

$$N_I \ge Q_{\mathrm{slot}} X^{\sigma/3}, \qquad M_I \ge X^{1/2} \tag{B.26}$$

for all sufficiently large N. For the first inequality, the ratio is at least

$$(6L_X^{C_{\mathrm{box}}})^{-1} X^{(29-1/3)\sigma};$$

for the second, use the bound $T \le X^{1/3+15\sigma}$ of (B.30) in (B.23).

Let $m_0 = M(1 + r_X^r)/2$, and let ϕ_0 be the fixed smooth cutoff in (3.9). Then

$$\psi_I(m) = \phi_0\left(\frac{m - m_0}{M_I}\right)$$

equals 1 on $[M, Mr_X^r]$ and is supported on $[M - M_I/2, Mr_X^r + M_I/2]$. It majorizes the full complementary interval without changing its length parameter.

Since $T \geq N^{60\sigma}$ and $L_X \leq L$, (B.24) gives

$$T^{2^{-126}} \geq 12L_X^{C_{\mathrm{box}}}.$$

Together with $N_I \geq T/(6L_X^{C_{\mathrm{box}}})$ this proves the integer-length condition

$$N_I \geq (2T)^{1-2^{-126}}. \tag{B.27}$$

This is the selected interval length hypothesis (LEN) in Section 3.2. □

The source cuts are constant on each retained box by the definition of a non-boundary box. Let n_0 be a tested product in a retained Type-II box, and let T, M, r be as in Lemma B.3. Every selected integer is at most $r_X T$. The complementary cutoff ψ_I is supported below $M[1 + \frac{3}{2}(r_X^r - 1)]$. As $TM \leq n_0$ and $1 \leq r \leq 5$, their product is at most

$$n_0 r_X[1 + \tfrac{3}{2}(r_X^r - 1)] \leq n_0(1 + 11\delta_X) \leq (1 - \varepsilon/2)N$$

for sufficiently large N, using $\Delta_X \leq \delta_X \to 0$. Thus on this enlarged support the upper cut leaves

$$a - mn \geq \varepsilon N/2 \geq X(\log X)^{-C_{\mathrm{box}}}. \tag{B.28}$$

The final inequality follows from $X < N$ and $\delta_X = (\log X)^{-C_{\mathrm{box}}} < \varepsilon/2$. Thus the resonant class is absent. Also $a \leq X^{1+\sigma/20}$ for all sufficiently large N.

For each fixed Iwaniec vector, Lemma 4.7 and the exact slot identity (4.13) give the three-factor coefficient required by (3.5). The smooth-only term has all three prime slots empty. Every tail factor is at least $Y = N^{\theta\sigma^2} > N^{\ell_0\sigma^2} = u_0$. This is the roughness threshold u_0 of (3.2), which bounds the tail cutoff u_L in C1. The smooth divisor is attached to the first slot. The Section 4 packing lemma verifies the spectral inequality (3.7) directly, as displayed in (4.9). Separately, (B.25)–(B.26) verify the modulus and interval inequalities in (3.6); (B.27) gives the integer-length condition (LEN), and (B.30) gives the selected-window bounds used below. Also

$$\log_X N = \frac{1}{1 - \omega_X} \leq 1 + \frac{\sigma}{20},$$

and $\log_X T \leq 1/3 + 15\sigma$ by the first inequality in (B.30) below. The selected window also gives $X^{60\sigma} \leq T$.

The interior estimate. Let E_{int} be the sum of the HB contributions on retained Type-I-1, Type-I-2, and Type-II boxes. It is taken after the finite Iwaniec vector sum. The arguments above prove

$$\begin{aligned}\left|\sum_q w_N^\nu(q)\Delta_D(q;a)\right| &\leq C_\sigma\big(NL/Y + N^{1/2+o(1)} + NL^{18-C_{\mathrm{box}}} \\ &\qquad + NL^{16-C_{\mathrm{box}}/2}\big) + |E_{\mathrm{int}}|.\end{aligned} \tag{B.29}$$

The first two errors in (B.29) have fixed power savings. The choice $C_{\mathrm{box}} \geq \lceil 2A+32 \rceil$ in Theorem 1.2 makes the last two $O(NL^{-A})$. Both estimates are over the union of tuples.

The tested measure μ_D is the prime indicator on the interval of (B.1). It does not depend on the Iwaniec vector. The sieve factors belong to the modulus coefficient in (W). The prime-detection and rough Heath–Brown identities are (B.7)–(B.10). Every variable, including the logarithmic one, is z_0-rough. The prime-power terms are estimated before the box partition.

The exponent constraints (B.20)–(B.21) imply that the residual cells are empty by the three contradictions above. The Type-II selected exponent obeys

$$\begin{aligned} \log_X T &\leq \frac{1/3+14\sigma}{1-\omega_X} \leq 1/3+15\sigma, \\ \frac{1-\omega_X}{2} - \log_N T &\geq \frac{1}{6} - (15+1/202)\sigma > 0. \end{aligned} \tag{B.30}$$

The second line records $T < X^{1/2}$, the Type-II upper-length condition; the first line supplies the sharper bound used in (B.23). By (B.2), its extension to $\widetilde{w}_N^{\nu}$ and $\widehat{w}_N^{\nu}$ after (B.5), and the parameter choice in Theorem 1.2, $q < N^{e_Q}$ with $e_Q < \Theta(u) \leq \Theta(0) = 153/256 < 0.6$. For a Type-I-1 box, (3.100) with $e_Q < 153/256$ gives $e_* - e_Q - (1-\omega_X)/10 > 0.3023$. Take a Type-I-2 box. At most four of its at most six variables are small, each with $e_i < 60\sigma$. Their product m has upper endpoint at most $N^{240\sigma}(1+\Delta_X)^4 \leq N^{240\sigma+\kappa_X}$, so $\log_N M \leq 240\sigma + \kappa_X$. Since $\log_N X = 1-\omega_X$, $q < N^{e_Q}$, $\kappa_X, \omega_X \leq \sigma/202$ by (B.20), and $0 < \sigma \leq 10^{-10}$,

$$\begin{aligned} \log_X M &\leq \frac{240\sigma+\kappa_X}{1-\omega_X} \leq \frac{(240+1/202)\sigma}{1-\sigma/202} < 2.401 \cdot 10^{-8}, \\ \log_X q &< \frac{e_Q}{1-\omega_X} < \frac{153/256}{1-\sigma/202} < 0.5977. \end{aligned}$$

Also $N_1, N_2 \geq X^{1/3}$ by the large-variable definition, and $N_1 N_2 M \leq 2^{30} X$ by the Heath–Brown product support and its dyadic scale convention. Hence (3.101) holds, and with (B.30) and (3.100) the hypotheses (3.99)–(3.101) of the Type-I lemmas hold.

The packing lemmas give, for each Iwaniec vector, the exact three-slot coefficient and the packing inequalities (3.6)–(3.7). They include the empty-tail term and retain repeated-prime multiplicity before the mask error (B.6). The selected rough unit, Möbius and normalized-log coefficients are the three classes in (3.8). The proof of (3.25) in Section 3 applies to each class, with every auxiliary coprimality parameter d and every subinterval. Its bound is $T\tau(d)L^{-A_{\mathrm{SW}}}$ for every fixed $A_{\mathrm{SW}} > 0$. Its application to $\alpha^{(r)}$ and the small-norm case in (3.47)–(3.48) give the norm-normalized BFI passage. The complementary short-interval second moment and the remaining dispersion steps are those of Theorem 3.2. On these boxes, (B.22)–(B.28) verify their interval, smoothing and nonresonance data. In the local theorem take $K_{\mathrm{scale}} = B+1$. In particular $X > N/(2L^B)$ and $a \leq NL/3$ imply $a \leq (2/3)XL^{B+1} \leq X^{1+\sigma/20}$ for large N. The comparator is the natural PSDT comparator on every HB product. No coprimality of a composite product is asserted.

For the chosen width, the global box count in (3.91) has $K_0 = 24C_{\mathrm{box}} + 26$. The source labels, Iwaniec vectors and prefix cells number $O_\sigma(1)$, and the HB coefficients are $O(1)$. There are at most six variables. So there are $O(L^6)$ dyadic tuples, within

the $O(L^{24})$ of §3.10, and each dyadic tuple contains $O(L^{6C_{\mathrm{box}}})$ short boxes, within the $O(L^{24C_{\mathrm{box}}})$ there. This verifies (H7) for the prime source. Choose the local Type-II precision $A' = K_0 + A + 5 = A + 24C_{\mathrm{box}} + 31$ and $A_{\mathrm{work}} = A' + 4$, as in (3.92). Then Theorem 3.2 with (H7), that is (3.92), bounds the sum over all Type-II boxes. The bound is $O(NL^{-A-5})$ plus a fixed power-saving term. Lemmas 3.3 and 3.5 give $O(NL^{-A})$ for Type-I-1 and Type-I-2 when requested with $A_i = K_I + A + 5$ and $K_I = 6(C_{\mathrm{box}} + 1) + 2$. Thus

$$|E_{\mathrm{int}}| \ll NL^{-A} + N^{1-c_\sigma} L^{K_0+C}, \qquad c_\sigma = \ell_0\sigma^2/4 > 0. \tag{B.31}$$

Here $C = C(\sigma, C_{\mathrm{box}}, A', K_{\mathrm{scale}}, \mathcal{C}_{\mathrm{src}})$ is the logarithmic exponent from (3.13). Here the local spectral, restoration, endpoint and slot-localization costs are included once in Theorem 3.2.

Proposition B.4 (Iwaniec prime-source estimate)**.** *Take the parameters of Theorem 1.2, either Iwaniec sign ν, and a sufficiently large even N. Let w_N^ν be the exact coefficient (W). Let a satisfy the residue conditions of Theorem 1.2, and let Δ_D be (B.1). First split off the LOG correction algebraically. Then every remaining source summand is assigned exactly once to one class. The classes are mask, proper-power, boundary, Type-II, Type-I-1 and Type-I-2. Every retained box has the scale and coefficient properties stated above, and*

$$\left|\sum_q w_N^\nu(q)\Delta_D(q;a)\right| \ll_{\sigma,A,u,\theta,B,\varepsilon,C_{\mathrm{box}}} NL^{-A}. \tag{B.32}$$

Proof. The Iwaniec mask partition and support are (B.2)–(B.6). The explicit identities (B.7)–(B.11) partition the remaining tested measure into proper powers and HB tuples. The equal-ratio mesh puts each tuple in exactly one box. A face-straddling box belongs to the boundary class. On every other box both source predicates are constant. The exponent-polytope argument above puts every nonempty such box in exactly one of II, I-1 and I-2. The displayed tie rule selects one II factor. LOG is split off algebraically before that assignment. The errors are (B.6), (B.8), (B.18), (B.19) and (B.31). Their sum is $O_{\sigma,A,u,\theta,B,\varepsilon,C_{\mathrm{box}}}(NL^{-A})$. □

Use of AI tools

The author devised the method of proof. Codex (OpenAI) and Claude Code (Anthropic) are the AI tools used. They assisted with derivation checks, code, typesetting, and review. The author checked every proof step, table, and reference. The author takes full responsibility for the content of the paper.

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