IRRATIONALITY OF THE TRILOGARITHM AT RECIPROCALS OF INTEGERS OF ABSOLUTE VALUE AT LEAST 27

RUIYI ZHANG

ABSTRACT. We prove that $\mathrm{Li}_3(1/a)$ is irrational for every integer $a \geq 27$ and every integer $a \leq -27$. One determinant construction is used on both sides. The positive side takes $m = 12Q$ and $g = 4Q$; the negative side takes $m = 16Q$ and $g = 5Q$. We prove a conservative coefficientwise integerizer, nonvanishing along prime-power towers, and a logarithmic-energy estimate. On the negative side the estimate rests on a signed residue identity along a bent contour. Exact rational and directed interval certificates give four strict endpoint margins. Slope comparisons carry them along both rays.

1. INTRODUCTION

For $|z| < 1$, define

$$\mathrm{Li}_3(z) = \sum_{n \geq 1} \frac{z^n}{n^3}.$$

By Marcovecchio [8, p. 229], Hata's method [4, Corollary 2.2] gives the linear independence of

$$1,\ \mathrm{Li}_1(1/a),\ \mathrm{Li}_2(1/a),\ \mathrm{Li}_3(1/a)$$

for $|a| \geq 1038$. Panzone gave a short proof that $\mathrm{Li}_3(1/d)$ is irrational for every integer $d \geq 1173$ [9, Theorem 2, p. 108]. Marcovecchio proved the four-number independence for $a \geq 486$ and $a \leq -471$ [8, Theorem 7.1, pp. 228–229]. In these ranges $\mathrm{Li}_3(1/a)$ is therefore irrational.

This paper continues the construction in [13, 14]. Similar determinant constructions appear in [3, 6]. For the dilogarithm at rational points see also [10, 7, 12].

Our earlier preprint [14] treated $a \geq 67$ and $a \leq -70$. It used $g = m/2$ and $m = 4Q$. We cite its routine lemmas where they carry over, and we list the changes each time. The new arithmetic rates, the seed towers, the endpoint certificates and the signed negative contour are proved here. We prove the following statement.

Theorem 1. *For every integer $a \geq 27$ and every integer $a \leq -27$, $\mathrm{Li}_3(1/a)$ is irrational.*

The proof has three parts. Sections 2–4 give the determinant and its integerization. Section 5 proves nonvanishing. Sections 6 and 7 give the analytic estimates for the two signs. Sections 8–9 assemble the four endpoint certificates and the rays.

In Remark 1.1, let $G_{m,a}$ be the determinant polynomial of Section 2. For $G_{m,a} \not\equiv 0$, let Δ_m be the least positive rational number with $\Delta_m G_{m,a} \in \mathbb{Z}[T]$, and put $\tau_m = m^{-2} \log \Delta_m$. When $G_{m,a}(\mathrm{Li}_3(1/a)) \neq 0$, put $\alpha_m = -m^{-2} \log |G_{m,a}(\mathrm{Li}_3(1/a))|$ and $B_m = \tau_m - \alpha_m$. The criterion applies when these nonzero evaluations contain a

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subsequence m_k on which B_{m_k} stays below a negative constant. It also needs, for every fixed rational ξ, $G_{m_k,a}(\xi) \neq 0$ for all sufficiently large k. Finite values of B_m are only indicative. The ratio is $\rho = g/m$. The staggered filter replaces $\binom{r-1}{g}^3$ by a product of three binomial coefficients with different lower indices. This variant construction is not used in this paper.

Remark 1.1 (Scope of the present construction). We do not attempt to optimize the threshold in this paper. We expect that the threshold could be lowered, but we make no lower-threshold claim here.

Several choices were made for safety and are not optimal. The integerization constants are conservative. The filter ratio ρ, the index windows and the negative-side contour tilt were not optimized jointly. Our expectation is that tuning them could lower the threshold by a few more units. The tuning would keep the same single-block determinant and the same three proof steps: integerization, nonvanishing and analytic decay.

The data suggest, however, that this algorithm is unlikely to reach thresholds below 20. We computed exact denominators and determinants of the same construction for $m \leq 128$. We used $g = 5m/16$ and $g = \lceil m/3 \rceil$. We recorded the difference $B_m = \tau_m - \alpha_m$ between the arithmetic growth and the analytic decay per m^2:

- $a = +20$. B_m crosses zero. For $g = \lceil m/3 \rceil$ and $g = 5m/16$ it equals -0.0728 and -0.0494 at $m = 64$, and $+0.007$ and $+0.017$ at $m = 112$.
- $a = +21, +22$. B_m is still negative at $m = 112$: about -0.05 to -0.06 for $a = 21$, and about -0.10 to -0.11 for $a = 22$.
- $a = -20$. For $g = 5m/16$, B_m stays negative but rises fast: about -0.21 at $m = 64$, -0.12 at $m = 112$ and -0.08 at $m = 128$.
- Staggered filter $\binom{r-1}{m/8}\binom{r-1}{3m/16}\binom{r-1}{m/4}$. It gives $B_m > 0$ at $a = \pm 20$ for $m = 64, 80, 96, 112$.

These finite computations put the apparent crossover of the method near $|a| = 21$–23 on the positive side. On the negative side it lies close to 20. We therefore expect that going below $|a| = 20$ requires a structurally different auxiliary construction. Further parameter tuning should not suffice. All statements in this remark are numerical observations at finite m. They are not part of the proof.

2. Determinant construction

Fix an integer a with $|a| \geq 2$. We work in one of the two towers below. In each, m is a positive multiple of 4 and $g < m$. Put

$$\rho = \frac{g}{m}, \qquad I = \{0, \dots, m-1\}, \qquad J = \{m, \dots, 2m-1\}.$$

The positive tower is

$$(m, g) = (12Q, 4Q), \qquad \rho = \frac{1}{3},$$

and the negative tower is

$$(m, g) = (16Q, 5Q), \qquad \rho = \frac{5}{16}.$$

Here and below Q is a positive integer. The nonvanishing section specializes it to $Q = p^k$. Define

$$w(r) = \binom{r-1}{g}^3, \qquad F_{ij}(r) = \frac{w(r)}{(r+i)(r+j)}, \qquad K_{ij}(r) = F''_{ij}(r),$$

and

$$M_{ij}(a) = \sum_{r\geq 1} a^{-r} K_{ij}(r), \qquad i \in I,\ j \in J.$$

Polynomial division at the two poles gives

$$F_{ij}(r) = P_{ij}(r) + \frac{c_i}{r+i} + \frac{c_j}{r+j}, \quad c_i = \frac{w(-i)}{j-i}, \quad c_j = -\frac{w(-j)}{j-i}.$$

At a pole of the partial fractions we take the finite part. Put $b = a - 1$, and define

$$E_{ij}(a) = \sum_{r\geq 1} a^{-r} P''_{ij}(r) - 2c_i a^i \sum_{n=1}^{i} \frac{a^{-n}}{n^3} - 2c_j a^j \sum_{n=1}^{j} \frac{a^{-n}}{n^3},$$
$$C_{ij}(a) = 2c_i a^i + 2c_j a^j, \qquad G_{m,a}(T) = \det(E(a) + TC(a)).$$

Empty sums are zero.

Lemma 2.1. *For every integer a with $|a| \geq 2$,*

$$M(a) = E(a) + \mathrm{Li}_3(1/a) C(a).$$

The polynomial $G_{m,a}$ has degree at most m.

Proof. Differentiating the partial fractions gives

$$F''_{ij} = P''_{ij} + \frac{2c_i}{(r+i)^3} + \frac{2c_j}{(r+j)^3}.$$

Also

$$\sum_{r\geq 1} \frac{a^{-r}}{(r+i)^3} = a^i \left(\mathrm{Li}_3(1/a) - \sum_{n=1}^{i} \frac{a^{-n}}{n^3} \right).$$

All series converge absolutely because $|a| > 1$. Substitution gives the matrix identity. Each matrix entry is affine in T. This proves the degree bound. At a pole we differentiate the numerator before taking the finite part. With $\delta = j - i$ this gives

$$P''_{ij}(-i) = \frac{w'''(-i)}{3\delta} - \frac{w''(-i)}{\delta^2} + \frac{2w'(-i)}{\delta^3} + \frac{2(w(-j) - w(-i))}{\delta^4},$$

and, with the same $\delta = j - i$,

$$P''_{ij}(-j) = -\frac{w'''(-j)}{3\delta} - \frac{w''(-j)}{\delta^2} - \frac{2w'(-j)}{\delta^3} + \frac{2(w(-i) - w(-j))}{\delta^4}.$$

These are finite parts at the poles of the partial fractions. They are the values used below. □

3. The determinant criterion

The following fact is the elementary limsup variant of the determinant criterion in [13, Sec. 3]. We prove the exact form used here. We use the extended-real convention $\log 0 = -\infty$ in its limsup bounds.

Fact 1 (Determinant criterion)**.** *Let θ be real. For a fixed base a, let $G_{m,a}(T)$ have degree at most m, and let m_k increase. Suppose positive rationals $\Delta_m(a)$ satisfy*

$$\Delta_m(a)G_{m,a}(T) \in \mathbb{Z}[T], \qquad \limsup_k m_k^{-2} \log \Delta_{m_k}(a) \le \tau(a),$$

and

$$\limsup_k m_k^{-2} \log |G_{m_k,a}(\theta)| \le -\alpha(a), \qquad \alpha(a) > \tau(a).$$

If every fixed rational ξ satisfies $G_{m_k,a}(\xi) \neq 0$ for all sufficiently large k, then θ is irrational.

Proof. If $\theta = u/v$ in lowest terms, then

$$Z_m = v^m \Delta_m(a) G_{m,a}(u/v) = \sum_{\ell=0}^{m} (\Delta_m \gamma_{m,\ell}) u^\ell v^{m-\ell}$$

is a nonzero integer whenever $G_{m,a}(T) = \sum_\ell \gamma_{m,\ell} T^\ell$ is nonzero at u/v. The factor v^m is $\exp(O(m))$. So $\limsup m_k^{-2} \log |Z_{m_k}| \le \tau(a) - \alpha(a) < 0$, and hence $|Z_{m_k}| \to 0$. This is a contradiction. In the application, $\Delta_m = D_m(a)$ from (4.42). The prime-power tower in Section 5 supplies the nonvanishing hypothesis. □

4. Coefficient integerization

The arithmetic construction is coefficientwise. It uses no primitive-content or exact-gcd assertion. Put

$$U = 2m - 1, \quad L = \operatorname{lcm}(1, \dots, U), \quad N = \frac{m^2}{8} - \frac{m}{2}, \quad H = 3g - 3,$$

and

$$H_* = \left\lfloor \frac{(H+1)^2}{4} \right\rfloor = \left\lfloor \frac{(3g-2)^2}{4} \right\rfloor.$$

For a prime p, let $\ell_p = v_p(L)$. Let κ_n be 1 when $n \bmod p \ge p - (g \bmod p)$, and 0 otherwise.

4.1. **Formal divisors.** We use the coefficientwise Newton argument of Lemma 4.1 of [14], together with its divided-difference lemmas. Newton interpolation, two divided differences, and two derivatives give

$$L^{4m} b^{mH} G_{m,a}(T) \in \mathbb{Z}[a, T]. \tag{4.1}$$

We use the present values of g. The four copies of L account for the two divided differences and the two derivatives. The pole and harmonic terms need no extra divisor. Their complete denominators include $1/(j-i)$ and $1/((j-i)n^3)$, respectively. These divide L^4 because $|j-i|, n \le U$. In the Newton basis we use the two denominator identities

$$\sum_{r \ge 1} a^{-r} \binom{r}{k} = \frac{a^{-k}}{(1-a^{-1})^{k+1}} \ (k \ge 1), \qquad \sum_{r \ge 1} a^{-r} \binom{r-1}{k} = \frac{a^{-k-1}}{(1-a^{-1})^{k+1}}. \tag{4.2}$$

Let V be an integer-valued polynomial on $\mathbb{Z}^k$. Suppose it has degree at most D in one variable X. Put $L_D = \mathrm{lcm}(1, \dots, D)$. Then $L_D\, \partial_X V$ and $L_D\, (V(X, \cdot) - V(Y, \cdot))/(X - Y)$ are integer valued. It suffices to take $V = \binom{X}{n} U$ with $n \leq D$ and U integer valued in the other variables. Vandermonde's identity gives $\binom{X}{n} - \binom{Y}{n} = \sum_{k \geq 1} \binom{X-Y}{k} \binom{Y}{n-k}$, and $\binom{X-Y}{k}/(X - Y) = \binom{X-Y-1}{k-1}/k$. For the derivative, $\frac{\mathrm{d}}{\mathrm{d}X} \binom{X}{n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{X}{n-k}$. Each of the two divided differences in P''_{ij} therefore costs one factor L_D. So does each of the two derivatives. Here $D \leq U$. With $L_D \mid L$ this accounts for the four copies of L in (4.1). Thus a degree-d Newton expansion introduces only denominators dividing the appropriate power of $L_d = \mathrm{lcm}(1, \dots, d)$. The identity is formal in a and remains valid at negative integer bases.

Write $d = (g!)^3$ and take a polynomial $P \in \mathbb{Z}[1/d][X]$. Its rational generating function has two expansions

$$\sum_{r \geq 1} a^{-r} P(r) = -\sum_{k \geq 0} a^k P(-k), \tag{4.3}$$

where the left side expands at infinity and the right side at zero. In the determinant calculation the right-hand expansion is used after multiplication by the factor a^N below. Thus the displayed identity is an identity of formal power series at the required order. For $W \in \mathbb{Z}[X]$ we also use two divided-difference facts

$$\frac{W(X) - W(Y)}{X - Y}, \qquad \frac{W'(X) - W'(Y)}{X - Y} \in \mathbb{Z}[X, Y]. \tag{4.4}$$

Take $W = dw$. The divided differences in the partial fractions then lie in $\mathbb{Z}[1/d][X, Y]$. In particular every coefficient of $P_{ij}(r)$ is a polynomial in (i, j) over $\mathbb{Z}[1/d]$.

Translate $T = X + \ell(a)$, where $\ell(a) = \sum_{s \geq 1} a^s/s^3$. Away from the node sets, the coefficient of a^k in $E + (X + \ell(a))C$ is $-F''_{ij}(-k)$. Away from the node sets it has rank at most three. After two differentiations it is a sum of the three Cauchy features $(i-k)^{-1}, (i-k)^{-2}, (i-k)^{-3}$. At a node the finite part adds one exceptional feature. Suppose $k \in I$ is the node, and put $\delta = j - k$ for $j \in J$. The finite-part formula and the harmonic correction give

$$\begin{aligned}[a^k]\big(E + (X + \ell(a))C\big)_{kj} = &-\frac{w'''(-k)}{3\delta} + \frac{w''(-k)}{\delta^2} - \frac{2w'(-k)}{\delta^3} \\ &+ \frac{2w(-k)}{\delta^4} + \frac{2Xw(-k)}{\delta}.\end{aligned} \tag{4.5}$$

The transposed formula holds when the node is in J. Thus a node row is a linear combination of the four Cauchy row features $\delta^{-1}, \delta^{-2}, \delta^{-3}, \delta^{-4}$. The X term is absorbed in the coefficient of δ^{-1}. A non-node row uses only the first three. This proves the rank bound four, including the finite part. It is the calculation above. Thus at most four columns per power a^k can contribute. The order is therefore at least

$$\sum_{j=0}^{m-1} \left\lfloor \frac{j}{4} \right\rfloor = 4 \sum_{r=0}^{m/4-1} r = N.$$

Consequently coefficientwise multilinearity gives

$$a^N \mid \det(E + (X + \ell(a))C).$$

Since $\ell(0) = 0$, translating back preserves this divisibility.

At $b = 0$ we use the following degree count. The polynomial dw is monic of degree $3g$. After polynomial division and two derivatives, $P''_{ij}(X)$ therefore has total degree at most $3g-4$ in (X, i, j). Equivalently,

$$P''_{ij}(r) = d^{-1} \sum_{t=0}^{3g-4} p_t(i,j) r^t, \qquad \deg_{i,j} p_t \le 3g-4-t.$$

The standard identity for $\sum_{r\ge 1} a^{-r} r^t$ has denominator b^{t+1}. Hence the coefficient of b^{-s} receives terms only from $t \ge s-1$. Its total degree in (i,j) is at most $3g-4-(s-1) = H-s$. The finite harmonic terms and C are regular at $b=0$. Taking forward differences of orders u and v lowers the pole order to $(H-u-v)_+$. Suppose k entries of a determinant term have a positive pole bound. The distinct row and column orders then cost at least $k(k-1)$. Hence the total pole order is at most

$$k(H+1-k) \le H_*.$$

This is the forward-difference argument of [14, Lemma 4.8]. There H_m is replaced by $H_* = \max_{0\le k\le H+1} k(H+1-k)$. The harmonic terms and C have no pole at $b=0$. The row and column operations are unimodular. Thus

$$b^{mH-H_*} \mid L^{4m} b^{mH} G_{m,a}(T). \tag{4.6}$$

The polynomials a and $a-1$ are monic and coprime. So (4.1)–(4.6) give the integral formal quotient

$$G_{m,a}(T) = \frac{a^N R_m(a,T)}{L^{4m}(a-1)^{H_*}}, \qquad R_m \in \mathbb{Z}[a,T].$$

4.2. **Local prime weights.** The following weights are valid coefficientwise for the pencil $E+TC$, including finite parts. The branches are applied in the displayed order. The low-prime rank branch assumes $p \nmid a(a-1)$. If $p \mid a(a-1)$ we use the endpoint weight e_p^{end} defined just before (4.42) instead. Primes $p > U$ have weight zero. If $p < 5$ or $p^2 \le U+g$, use $4m\ell_p$. Now let $p \le g$, $p \ge 5$, and $p^2 > U+g$. The entry demand is

$$\mathfrak{b}_{ij} = 3 - \kappa_i\kappa_j + (1-\kappa_i)\mathbf{1}_{i\equiv j \pmod p}. \tag{4.7}$$

The row and column weights $2-\kappa_n$ dominate this demand.

The saving rests on the following coefficientwise lemma. Its proof includes finite parts and the T coefficient.

4.3. **The low-prime rank lemma.** This subsection gives the coefficientwise local argument for the low-prime branch. It applies to the pencil $E+TC$, including finite parts at poles and the coefficient of T. Throughout this subsection

$$I = \{0,\dots,m-1\}, \qquad J = \{m,\dots,2m-1\}, \qquad U = 2m-1,$$

$$w(X) = \frac{\mathcal{N}(X)}{d}, \qquad \mathcal{N}(X) = \prod_{u=1}^{g} (X-u)^3, \qquad d = (g!)^3.$$

Thus $w(r) = \binom{r-1}{g}^3$ at every non-negative integer r.

Lemma 4.1 (Low-prime rank)**.** *Let $p \geq 5$ satisfy*

$$p^2 > U + g, \qquad p \nmid a(a-1), \qquad q = \left\lfloor \frac{g}{p} \right\rfloor \geq 1,$$

where a is an integer with $|a| > 1$. Write

$$g = qp + h, \qquad 0 \leq h < p, \qquad G = p - h.$$

Call a node n good if $n \bmod p < G$ and bad if $n \bmod p \geq G$. Put

$$\kappa_n = \begin{cases} 0, & n \text{ good}, \\ 1, & n \text{ bad}, \end{cases} \qquad \omega_i = 2 - \kappa_i, \qquad \omega'_j = 2 - \kappa_j.$$

For $n \geq 0$ define

$$R_n(X) = \frac{\mathcal{N}(X) - \mathcal{N}(-n)}{X + n}, \qquad H_n(a) = \sum_{s=1}^{n} \frac{a^{n-s}}{s^3},$$

and

$$b_n(T) = \sum_{r \geq 1} a^{-r} \left(\frac{R_n}{d} \right)''(r) + 2w(-n)\big(a^n T - H_n(a)\big). \tag{4.8}$$

Let the normalized pencil entries be

$$B_{ij}(T) = \frac{b_i(T) - b_j(T)}{j - i}, \qquad i \in I,\ j \in J. \tag{4.9}$$

By the second divided-difference identity and the finite harmonic terms, these normalized entries are exactly the entries of the determinant pencil:

$$B_{ij}(T) = E_{ij}(a) + TC_{ij}(a).$$

Then the exact row and column scaling is

$$\widehat{B}_{ij}(T) = p^{\omega_i + \omega'_j} B_{ij}(T) = p^{(2-\kappa_i)+(2-\kappa_j)} B_{ij}(T). \tag{4.10}$$

The matrix $\widehat{B}$ has entries in $\mathbb{Z}_{(p)}[T]$, and, if N_I and N_J denote the numbers of good nodes in I and J, respectively,

$$\operatorname{rank}_{\mathbb{F}_p(T)} \overline{\widehat{B}} \leq \min(N_I, N_J) + \min(G, h). \tag{4.11}$$

Here the bar denotes reduction in the Gauss valuation ring of $\mathbb{Q}_p(T)$. Consequently, with

$$W_{\mathrm{nat}} = \sum_{i \in I} \omega_i + \sum_{j \in J} \omega'_j = 2m + N_I + N_J,$$

and put

$$\varrho_* := \min(N_I, N_J) + \min(G, h).$$

every coefficient of the determinant satisfies

$$v_p\big([T^k] \det B(T)\big) \geq -W_{\mathrm{nat}} + m - \varrho_* \qquad (0 \leq k \leq m). \tag{4.12}$$

There is also a separate entrywise alternative. Its row and column base weights are 1 and 2, respectively. For each good residue $s < G$, add one extra power of p to the side having fewer copies of that residue. If

$$c_I(s) = \#\{i \in I : i \equiv s \pmod p\}, \qquad c_J(s) = \#\{j \in J : j \equiv s \pmod p\},$$

the resulting total exponent is

$$W_{\mathrm{alt}} = 3m + \sum_{0 \le s < G} \min\{c_I(s), c_J(s)\}. \tag{4.13}$$

Either alternative alone supplies an integerizer; their savings are not combined.

Proof. The proof has three steps: summation, local order, and rank.
Newton summation modulo p. We first need an elementary summation fact. Suppose

$$V(X) = \sum_{\ell=0}^{D} v_\ell \binom{X}{\ell}, \qquad v_\ell \in \mathbb{Z}_{(p)}.$$

The ordinary series $\sum_{r\ge1} a^{-r}V(r)$ converges because $|a| > 1$. We claim that it lies in $\mathbb{Z}_{(p)}$. Write $z = a^{-1}$. Then

$$\sum_{r\ge1} z^r \binom{r}{0} = \frac{z}{1-z}, \qquad \sum_{r\ge1} z^r \binom{r}{\ell} = \frac{z^\ell}{(1-z)^{\ell+1}} \quad (\ell \ge 1). \tag{4.14}$$

The only denominators are powers of $1 - a^{-1}$. This is a p-unit because $p \nmid a(a-1)$. Formula (4.14) also shows that reduction modulo p may be done term by term in the finite Newton expansion. We use no p-adic convergence of the infinite series.

We also need a periodic form. Let V be a polynomial with p-integral Newton coefficients. Suppose that $V(r + p^2) \equiv V(r) \pmod p$ for every integer r. Then

$$\sum_{r\ge1} a^{-r}V(r) \equiv \frac{\displaystyle\sum_{r=1}^{p^2} a^{-r}V(r)}{1 - a^{-p^2}} \pmod p. \tag{4.15}$$

To see this, write $S = \sum_{r\ge1} a^{-r}V(r)$ and $V(r + p^2) = V(r) + pV_1(r)$. The polynomial V_1 is integer valued at p-adic integers. So it has p-integral Newton coefficients. By the first fact its sum is p-integral. Splitting off the first p^2 terms gives $S = \sum_{r=1}^{p^2} a^{-r}V(r) + a^{-p^2}(S + pS_1)$ with S_1 p-integral. The denominator $1 - a^{-p^2}$ is a unit since $a^{-p^2} \equiv a^{-1} \not\equiv 1 \pmod p$.
Local order. For a residue class $t \pmod p$, let

$$\mathfrak{m}_t = (p, X - t) \subset \mathbb{Z}_{(p)}[X]$$

and let $\mathrm{ord}_t(f)$ be the largest e for which $f \in \mathfrak{m}_t^e$. The local ring at $\mathfrak{m}_t$ is regular of dimension two. Its associated graded ring is

$$\mathrm{gr}_{\mathfrak{m}_t} \mathbb{Z}_{(p)}[X] \simeq \mathbb{F}_p[P, Y], \qquad P = \mathrm{in}(p), \quad Y = \mathrm{in}(X - t). \tag{4.16}$$

This graded ring is an integral domain. Three rules follow at once:

(1) exact division by $X - t$ lowers local order by one. Division by $X - u$ with $u \not\equiv t \pmod p$ is division by a unit. It does not lower local order;
(2) differentiation lowers local order by at most one;
(3) if $x \equiv t \pmod p$, then evaluation gives $v_p(f(x)) \ge \mathrm{ord}_t(f)$.

The second rule follows by differentiating monomials $p^r(X - t)^s$. The third follows by substituting $X - t = p\,u$ with $u \in \mathbb{Z}_p$.

Since $g < p^2$, Legendre's formula gives

$$v_p(d) = 3v_p(g!) = 3q. \tag{4.17}$$

Every residue class occurs q times among $1, \dots, g$. The classes $1, \dots, h$ occur once more. Therefore

$$\operatorname{ord}_t \mathcal{N} \geq 3q, \qquad \operatorname{ord}_t \mathcal{N} \geq 3q+3 \quad \text{if } t \equiv 1, \dots, h \pmod{p}. \tag{4.18}$$

If n is bad, then $-n \equiv 1, \dots, h \pmod{p}$. So $\mathcal{N}(-n)$ has valuation at least $3q+3$. If n is good, its valuation is at least $3q$.

Scaled node sums. Take a residue $t \not\equiv -n \pmod{p}$. There the denominator $X+n$ is a unit, and (4.18) gives

$$\operatorname{ord}_t R_n \geq 3q.$$

At $t \equiv -n \pmod{p}$ division lowers order by one. The numerator then has stronger bounds. For bad n both terms in the numerator have order at least $3q+3$. Hence $\operatorname{ord}_t R_n \geq 3q+2$. For good n the bound is $\operatorname{ord}_t R_n \geq 3q-1$. We take two derivatives and divide by d. The local-order rules then give

$$p^2 \left(\frac{R_n}{d}\right)''(r) \in \mathbb{Z}_p \quad (n \text{ bad}), \qquad p^3 \left(\frac{R_n}{d}\right)''(r) \in \mathbb{Z}_p \quad (n \text{ good}) \tag{4.19}$$

for every integer r. By the finite-difference characterization of $\mathrm{Int}(\mathbb{Z}, \mathbb{Z}_p)$, the scaled polynomials have p-integral Newton coefficients. So the Newton summation fact applies to their complete sums.

Now take the finite-part term. By hypothesis $n < U+1 < p^2$. So $v_p(s) \leq 1$ for $1 \leq s \leq n$, and $v_p(H_n(a)) \geq -3$. Also

$$v_p(w(-n)) \geq 3 \quad (n \text{ bad}), \qquad v_p(w(-n)) \geq 0 \quad (n \text{ good}). \tag{4.20}$$

So the T and harmonic terms in (4.8) have the same scaled integrality as in (4.19). We have proved

$$p^2 b_n(T) \in \mathbb{Z}_{(p)}[T] \quad (n \text{ bad}), \qquad p^3 b_n(T) \in \mathbb{Z}_{(p)}[T] \quad (n \text{ good}). \tag{4.21}$$

The one-pole reduction. Suppose that i is bad and j is good. Their residues are distinct. So $j-i$ is a p-unit. The first congruence in (4.21) gives $p^3 b_i(T) \equiv 0 \pmod{p}$. The coefficient of T in $p^3 b_j(T)$ is also zero modulo p, because $p^3 w(-j) a^j \equiv 0 \pmod{p}$. Define

$$\mathfrak{d}_j(a) = p^3 b_j(0) \bmod p. \tag{4.22}$$

It depends on neither i nor T. Since $p^3 b_i(T) \equiv 0$, (4.9) gives $p^3 B_{ij}(T) \equiv -\mathfrak{d}_j(a)/(j-i)$. That is,

$$p^3 B_{ij}(T) \equiv \frac{\mathfrak{d}_j(a)}{i-j} \pmod{p} \qquad (i \text{ bad},\ j \text{ good}). \tag{4.23}$$

The transpose formula holds for a good row and a bad column.

The bad–bad reduction. Let i and j both be bad. Put

$$\delta = j-i, \qquad L_{ij}(X) = \frac{\mathcal{N}(-i)(X+j) - \mathcal{N}(-j)(X+i)}{j-i}. \tag{4.24}$$

Then the exact polynomial identity

$$\mathcal{N}(X) - L_{ij}(X) = (X+i)(X+j)Q_{ij}(X) \tag{4.25}$$

defines $Q_{ij} \in \mathbb{Z}_{(p)}[X]$. Since $0 < |\delta| < 2m < p^2$, we have $v_p(\delta) \leq 1$. The two endpoint values of $\mathcal{N}$ have valuation at least $3q+3$. So every coefficient of L_{ij} satisfies

$$v_p\big([X^\ell] L_{ij}\big) \geq 3q+2. \tag{4.26}$$

At an evaluation residue the vanishing factors fall into three cases. Write $t \equiv r \pmod p$. First suppose that the opposite node residue $s \equiv -r \pmod p$ is bad. Then $\operatorname{ord}_t \mathcal{N} \geq 3q+3$.

(1) If both $X+i$ and $X+j$ vanish at t, then $i \equiv j \pmod p$ and $v_p(\delta) = 1$. Use the coordinate $X = t + Y$. The constant coefficient of L_{ij} has valuation at least $3q+3$. Here the extra factor $t+i$ or $t+j$ supplies one p before division by δ. The coefficient of Y has valuation at least $3q+2$. Hence $\operatorname{ord}_t L_{ij} \geq 3q+3$. Dividing by two vanishing factors gives $\operatorname{ord}_t Q_{ij} \geq 3q+1$.
(2) If exactly one factor vanishes, the other is a unit. From (4.26) and the order $3q+3$ of $\mathcal{N}$, division by the one vanishing factor gives $\operatorname{ord}_t Q_{ij} \geq 3q+1$.
(3) If neither factor vanishes, both are units. Equations (4.18) and (4.26) give $\operatorname{ord}_t Q_{ij} \geq 3q+2$.

Divide by d and differentiate twice. Then multiplication by p^2 makes each of the three cases divisible by p. Thus

$$p^2 \left(\frac{Q_{ij}}{d}\right)''(r) \equiv 0 \pmod p \quad \text{when } -r \text{ is a bad residue.} \tag{4.27}$$

Now let $s \equiv -r \pmod p$ be good. Both $X+i$ and $X+j$ are units in the local ring at $t = -s$. Set $u(X) = ((X+i)(X+j))^{-1}$. The derivatives u, u', u'' are integral there. All coefficients of L_{ij}/d are divisible by p^2. So the L_{ij} term vanishes after the present reduction. Consider the terms of $(wu)''$ containing w and w'. By the local-order bounds for w, w', w'', they also vanish after multiplication by p^2. Consequently

$$p^2 \left(\frac{Q_{ij}}{d}\right)''(r) \equiv \frac{p^2 w''(r)}{(r+i)(r+j)} \pmod p \quad \text{when } -r \text{ is good.} \tag{4.28}$$

The identity (4.25) is exactly the divided-difference identity

$$\frac{R_i(X) - R_j(X)}{j-i} = Q_{ij}(X).$$

Therefore (4.9) takes the following form:

$$B_{ij}(T) = \sum_{r \geq 1} a^{-r} \left(\frac{Q_{ij}}{d}\right)''(r) + \frac{2w(-i)(a^i T - H_i(a)) - 2w(-j)(a^j T - H_j(a))}{j-i}. \tag{4.29}$$

For bad i, j consider the second term in (4.29). We have $v_p(w(-i)), v_p(w(-j)) \geq 3$, $v_p(H_i), v_p(H_j) \geq -3$ and $v_p(j-i) \leq 1$. So its T part has valuation at least 2. Its harmonic part has valuation at least -1. After multiplication by p^2 both vanish modulo p.

It remains to sum the good-residue contribution. Define

$$\varpi(r) = p^2 w''(r) \pmod p.$$

The polynomial $p^2 w''$ has p-integral Newton coefficients. Its values have period p^2 modulo p. To see this, apply Taylor expansion and the local-order rule. For every $\ell \geq 1$ they give

$$v_p\left(\frac{p^{2\ell+2}}{\ell!} w^{(\ell+2)}(r)\right) \geq \ell - v_p(\ell!) \geq 1, \tag{4.30}$$

where the last inequality uses $p \geq 5$. Now take $V = p^2(Q_{ij}/d)''$. By the case analysis above its values are p-integral. So it has p-integral Newton coefficients. By (4.27) and (4.28), $V(r) \equiv \varpi(r)/((r+i)(r+j))$ when $-r$ is good and $V(r) \equiv 0$ otherwise. The factor $1/((r+i)(r+j))$ modulo p depends only on $r \bmod p$. So $V(r+p^2) \equiv V(r) \pmod p$. Hence (4.15) applies to V. For a good residue $s < G$, set

$$\mathfrak{c}_s(a) = \frac{\displaystyle\sum_{\substack{1 \leq r \leq p^2 \\ -r \equiv s \ (p)}} a^{-r}\varpi(r)}{1 - a^{-p^2}} \pmod p. \tag{4.31}$$

Grouping the terms in (4.29) by $s \equiv -r \pmod p$ gives

$$p^2 B_{ij}(T) \equiv \sum_{0 \leq s < G} \frac{\mathfrak{c}_s(a)}{(i-s)(j-s)} \pmod p \qquad (i,j \text{ bad}). \tag{4.32}$$

All displayed denominators are p-units, since i, j are bad and s is good. The expression is independent of T.

Rank count and valuation transfer. Take a bad row i and a good column j. Then the reduction (4.23) is a Cauchy feature with the good residue $s \equiv j \pmod p$. For a bad row and a bad column, (4.32) is a sum of the same features $1/(i-s)$, $s < G$. Now take the block of bad rows, with all columns. It factors through the G good residue features. So its rank is at most G. Bad rows with the same residue modulo p are also identical in every column. For good columns this follows from (4.23). For bad columns it follows from (4.32). So the rank is also at most h, the number of bad residue classes. The bad-row block has rank at most $\min(G, h)$. Adding the N_I good rows gives

$$\operatorname{rank} \overline{\widehat{B}} \leq N_I + \min(G, h). \tag{4.33}$$

The same argument for the transpose gives the bound $N_J + \min(G, h)$. Taking the smaller of the two proves (4.11).

The factors in (4.10) are the row and column weights. Bad nodes receive one power and good nodes two powers. Thus

$$\det \widehat{B} = p^{W_{\mathrm{nat}}} \det B. \tag{4.34}$$

The Gauss valuation on $\mathbb{Q}_p(T)$ has residue field $\mathbb{F}_p(T)$. Let a matrix have residue rank at most r. Gaussian elimination over the valuation ring shows that its determinant has valuation at least $m - r$. Equivalently, every T-coefficient of the determinant has valuation at least $m - r$. We apply this to $\widehat{B}$. With (4.11) it proves (4.12).

Finally, the one-pole bounds give $p^3 B_{ij} \in \mathbb{Z}_{(p)}[T]$ for distinct residues. A same-good-residue pair can lose one further power in the divided difference. It is covered by $p^4 B_{ij}$. The bad–bad and mixed cancellations above show that $p^3 B_{ij}$ is integral in all remaining cases. So one row power and two column powers cover every edge except the same-good-residue edges. For a good residue s only those edges need one extra power. Put that extra power on the side with fewer copies. This gives (4.13) and completes the proof. □

For the $p > g$ branch, write

$$\widetilde{c}_{i,j} = \frac{w(-i)}{j-i}, \qquad \widetilde{c}_{j,i} = -\frac{w(-j)}{j-i}$$

for the normalized pole coefficients in the (i,j) entry. To avoid overloading the low-prime symbol κ_n, put

$$\chi_n = \mathbf{1}_{\{n \bmod p \geq p-g\}}.$$

In the non-generic branch assume $p^2 > U+g$ and $p \nmid a(a-1)$. The remaining small primes use the first line of (4.38). Since $p > g$, $v_p(g!) = 0$, and

$$w(-n) = (-1)^{3g} \frac{\prod_{s=1}^{g} (n+s)^3}{(g!)^3}.$$

The interval of factors has length $g < p$, so

$$v_p(w(-n)) \geq 3\chi_n, \qquad v_p(j-i) = \varepsilon_{ij} \leq 1.$$

The divided-difference formulas for P_{ij} have coefficients in $\mathbb{Z}_{(p)}[r]$; hence $P_{ij} \in \mathbb{Z}_{(p)}[r]$. Also $v_p(s) \leq 1$ for $1 \leq s \leq U$, and therefore

$$v_p(H_n(a)) \geq -3\mathbf{1}_{n \geq p}, \qquad v_p(\widetilde{c}_{i,j}) \geq 3\chi_i - \varepsilon_{ij}, \qquad v_p(\widetilde{c}_{j,i}) \geq 3\chi_j - \varepsilon_{ij}.$$

The C coefficient is a sum of the two pole terms, so

$$v_p(C_{ij}) \geq 3\min(\chi_i, \chi_j) - \varepsilon_{ij}.$$

Thus the entry demand is

$$\mathfrak{b}_{ij}^{>g} = \max\{0,\, 3\mathbf{1}_{i \geq p} - 3\chi_i + \varepsilon_{ij},\, 3\mathbf{1}_{j \geq p} - 3\chi_j + \varepsilon_{ij}\}.$$

The samples are $0 \leq n \leq 3g-4 < m$, so a pole $n = j$ is never sampled. The entrywise bounds above and the two divided-difference formulas give the coefficientwise valuation transfer for this $p > g$ branch. The remaining expansion is a finite polynomial calculation. The displayed edge demand (4.7) refers instead to the low-prime branch $p \leq g$ above. So do the row and column weights $2 - \kappa_n$ and the coefficientwise determinant bound in (4.12). They follow from the local-order calculation in Lemma 4.1.

For $g < p < m$, put $u = p - g$. The potential calculation of [14, Lemma 4.5 and Lemma 4.6, case (4)] is reproduced here in its needed form. There are five edge types: active–active, active–killed, low–active, low–killed, and killed–killed. Their demands are bounded above, respectively, by 4, 3, 4, 0, and ε_{ij}, where $\varepsilon_{ij} = v_p(j-i) \in \{0,1\}$. A node in $[0,u)$ is active and one in $[u,p)$ is killed. A node $n \geq p$ is active if its residue is active. Call a row or column low when its index is $< u$; nodes in $[u,p)$ are killed. Let R and C count active rows and columns whose indices are $\geq p$. Let

$$\Sigma_{\mathrm{kill}} = \sum_{z=p-g}^{p-1} \min(c_I(z), c_J(z)), \quad c_I(z) = \#\{i \in I : i \equiv z \pmod p\},$$

where $c_J(z) = \#\{j \in J : j \equiv z \pmod p\}$. The two valid potentials are

$$W_A = 3(R+C) + u + \Sigma_{\mathrm{kill}}, \qquad W_B = m + 2(R+C) + u. \tag{4.35}$$

In W_A, active rows and columns with index at least p receive three units. The u low active rows contribute the displayed u. The smaller side of each killed residue receives one extra unit. In W_B, active rows with index at least p receive two units, low rows one, and killed rows zero. Active columns receive three and killed columns one. The edge demands are covered in the five cases active–active, active–killed, low–active, low–killed, and killed–killed. For $m \leq p \leq U$, put

$$r_p = 2m - p, \qquad h_p = (2m + g - 2p)_+,$$

and use the row and column weights

$$\omega_i^{\mathrm{tail}} = \mathbf{1}_{i<r_p} \quad (i \in I), \qquad \omega_j^{\mathrm{tail}\prime} = 3\,\mathbf{1}_{\{p\le j<2m,\ j-p<p-g\}} \quad (j \in J).$$

There are r_p same-residue edges $j = i+p$, covered by their row units. The positive nonsame-residue demands have a good column and are covered by its three units. The number of such columns is $r_p - h_p$. Thus the total weight is

$$W_{\mathrm{tail}} = \sum_{i\in I}\omega_i^{\mathrm{tail}} + \sum_{j\in J}\omega_j^{\mathrm{tail}\prime} = r_p + 3(r_p - h_p) = 4(2m-p) - 3(2m+g-2p)_+. \tag{4.36}$$

For the low-prime profile, let $x = p/m$ and write $\nu = \lfloor g/p \rfloor$; thus $\rho/(\nu+1) < x < \rho/\nu$. Put

$$\gamma = (\nu+1)x - \rho, \quad \eta = \rho - \nu x, \quad k_\ell = \lfloor \ell/x \rfloor,$$
$$A_\ell = k_\ell\gamma + \min(\ell - k_\ell x, \gamma), \quad A = A_1, \quad B = A_2 - A_1,$$

and

$$D(x) = \int_0^\gamma \min(c_I(t), c_J(t))\ \mathrm{d}t,$$

where, explicitly,

$$c_I(t) = \#\{k \geq 0 : 0 \leq t + kx < 1\}, \qquad c_J(t) = \#\{k \geq 0 : 1 \leq t + kx < 2\}.$$

Take the natural cost $W_{\mathrm{nat}} = 2m + N_I + N_J$. Relative to it, the baseline full-pencil and low-prime rank savings are

$$\begin{aligned} w_0 &= \max(0, 1-A-\gamma, 1-B-\gamma, A+B-1-D), \\ w_{\mathrm{low}} &= \max(w_0, 1-A-\eta, 1-B-\eta). \end{aligned} \tag{4.37}$$

Here w_0 is the ordinary rank/potential saving. The term $A+B-1-D$ comes from the entrywise alternative W_{alt} in Lemma 4.1. Thus the low-prime cost is $W_{\mathrm{nat}} - mw_{\mathrm{low}}(x)$. The conservative branch weight in (4.42) is

$$W_{\mathrm{cons}}(p) = \begin{cases} 4m\ell_p, & p < 5 \text{ or } p^2 \leq U+g, \\ \min\{4m, W_{\mathrm{nat}} - mw_{\mathrm{low}}(p/m)\}, & 5 \leq p \leq g,\ p^2 > U+g, \\ \min\{4m, W_A, W_B\}, & g < p < m, \\ W_{\mathrm{tail}}, & m \leq p \leq U, \\ 0, & p > U. \end{cases} \tag{4.38}$$

At a cell boundary $p \mid g$ with $p^2 > U+g$, use the second line and retain that finite prime separately. Away from such boundaries, define $w_{\mathrm{cons}}(x)$ as the limit of $W_{\mathrm{cons}}(p)/m$ when $p/m \to x$; at a boundary take the lower one-sided value. At an actual prime location we use the displayed finite value $W_{\mathrm{cons}}(p)/m$. For the remaining layers $\nu \geq 11$, put $\theta = \gamma/x$. Then $T_0 = A + B = 2\theta + \varepsilon_2$ with $0 \leq \varepsilon_2 \leq x/4$. We need only a lower bound for the available saving:

$$w_{\mathrm{low}} \geq 1 - \min(A,B) - \min(\gamma,\eta) \geq 1 - \frac{T_0}{2} - \frac{x}{2}.$$

Consequently the conservative normalized cost satisfies

$$w_{\mathrm{cons}}(x) \leq 1 + 3\theta + \frac{7}{8}x. \tag{4.39}$$

Set $\delta_0 = \rho/11$. The change of variables $x = \rho/(\nu+1-\theta)$ gives $\mathrm{d}x = \rho(\nu+1-\theta)^{-2}\,\mathrm{d}\theta$. The midpoint estimate

$$\int_0^1 \theta \frac{\rho\,\mathrm{d}\theta}{(\nu+1-\theta)^2} \le \frac{1}{2}\int_0^1 \frac{\rho\,\mathrm{d}\theta}{(\nu+1-\theta)^2} + \frac{\rho}{6\nu^3}$$

and $\sum_{\nu\ge 11}\nu^{-3} \le 2/441$ give

$$\int_0^{\delta_0} w_{\mathrm{cons}}(x)\,\mathrm{d}x \le \frac{5}{2}\delta_0 + \frac{\rho}{441} + \frac{7}{16}\delta_0^2. \tag{4.40}$$

Let ϑ be Chebyshev's function. For the finite fallback range use a separate cutoff $\varepsilon > 0$:

$$\sum_{\sqrt{U+g}<p\le \varepsilon m} W_{\mathrm{cons}}(p)\log p \le 4m\,\vartheta(\varepsilon m) = (4\varepsilon + o(1))m^2. \tag{4.41}$$

The generic branch $p^2 \le U+g$ contributes $O(m^{3/2}\log m) = o(m^2)$. A cell-boundary prime here means a prime $p \mid g$ with $p^2 > U+g$. It is handled by the second line of (4.38); there is at most one such prime, and its contribution is $O(m\log m)$.

The endpoint weight is

$$e_p^{\mathrm{end}} = 4m\,\ell_p + H_* v_p(a-1) - N v_p(a).$$

The local branch is admissible only when $p \nmid a(a-1)$. Thus we define the coefficientwise integerizer piecewise by

$$e_p = \begin{cases} e_p^{\mathrm{end}}, & p \mid a(a-1), \\ \min\{e_p^{\mathrm{end}}, W_{\mathrm{cons}}(p)\}, & p \nmid a(a-1), \end{cases} \qquad D_m(a) = \prod_p p^{e_p}. \tag{4.42}$$

Then $D_m(a)$ is a positive rational number. It satisfies

$$D_m(a)G_{m,a}(T) \in \mathbb{Z}[T]. \tag{4.43}$$

Primes dividing $a(a-1)$ take the first branch. So the low-prime unit hypotheses are never used there. For these primes the $4m\ell_p$ part totals $O(m\log m)$. The endpoint-divisor terms remain explicitly in e_p^{end} and in the rate below. We now justify the two exponents. The first exponent in (4.42) follows from the two formal divisors. For the second, use the separate coefficientwise branch that realizes the displayed minimum in $W_{\mathrm{cons}}(p)$. The ordinary row and column potentials give the baseline part of w_0 and w_{low}. The entrywise alternative is the W_{alt} term in Lemma 4.1. The remaining low-prime saving is the coefficientwise rank bound proved there over the Gauss valuation ring. Thus the low-prime saving is not inferred from potentials alone. For $g < p < m$, the two valid potential covers are W_A, W_B. For $m \le p \le U$ the displayed tail cover gives W_{tail}. Expanding the determinant after applying any one selected cover gives the corresponding valuation bound. Division by b^{mH} accounts for the last term in (4.42). Each branch alone supplies an integerizer, so taking the pointwise minimum does not combine savings.

4.4. **The limiting arithmetic rates.** The exact low-layer increments for $\nu = 1, \ldots, 10$, in order, are

$$\begin{aligned} \rho = 5/16 : {} & 47/6720,\ 69/41888,\ 387/637840,\ 6219367/21864360960, \\ & 75/484352,\ 180207/1941459520,\ 317501/5311088640, \\ & 18175/441971712,\ 11263/382261760,\ 155/7117824; \\ \rho = 1/3 : {} & 25/3024,\ 1/540,\ 4/5859,\ 11/34344,\ 7/40095, \\ & 17/161460,\ 2/29295,\ 23/492048,\ 13/389367,\ 29/1175580. \end{aligned} \tag{4.44}$$

Their sums are

$$L_{5/16} = \frac{1111075051739028956195167}{11186488525684173151177728 0}, \quad L_{1/3} = \frac{85930118826854129}{7423773458762901240}. \tag{4.45}$$

Write $I_{\mathrm{high}} = \int_\rho^1 w_{\mathrm{cons}}(x)\,\mathrm{d}x$ and $I_{\mathrm{tail}} = \int_1^2 W_{\mathrm{tail}}(x)\,\mathrm{d}x = 2 - 3\rho^2/4$. Let $I_{\nu,0}$ and $I_{\nu,1}$ denote the baseline and rank-improved layer integrals. The baseline profile is a minimum of costs. Its normalized costs are 4, $W_{\mathrm{nat}}/m - w_0$, W_A/m, W_B/m, and W_{tail}/m. For $1 \leq x \leq 2$ the last one has continuum form $w_{\mathrm{tail}}(x) = 4(2-x) - 3(2+\rho-2x)_+$. The ten disjoint increments in (4.44) are the exact integrals of the improvement $w_{\mathrm{low}} - w_0$. Each is taken after the pointwise minimum with the other valid branches. The replay uses the change of variables above, the midpoint remainder $\rho/(6\nu^3)$, and $\sum_{\nu \geq 11} \nu^{-3} \leq 2/441$ to include the tail. Define the replay's explicit upper expression by

$$I_{\mathrm{upper}} = I_{\mathrm{high}} + I_{\mathrm{tail}} + \sum_{\nu=1}^{10} I_{\nu,1} + \frac{5}{2}\delta_0 + \frac{\rho}{441} + \frac{7}{16}\delta_0^2, \qquad \delta_0 = \rho/11. \tag{4.46}$$

The exact profile replay gives

ρ	I_{high}	I_{tail}	I_{upper}
5/16	98909/53760	1973/1024	4.6343048426478513377…
1/3	125/72	23/12	4.5805812506617284878…

The profile replay recomputes the finite rational layer integrals. It verifies that this upper integral is below the following rational bounds:

$$\begin{aligned} S_- &= \frac{810865341901595520286102032938 9}{174788883213815205487152000000 0} < 4.639112779899639, \\ S_+ &= \frac{106301329053850833773025543473}{231992920586340663750000000 00} < 4.582093659805828. \end{aligned} \tag{4.47}$$

Thus S_ρ is directly a rational upper bound for $\int_0^2 w_{\mathrm{cons}}(x)\,\mathrm{d}x$.

For fixed $\varepsilon > 0$, (4.41) controls $p \leq \varepsilon m$. On each fixed cell $\varepsilon \leq p/m \leq 2$, residue counts are affine up to $O(1)$. We then apply the prime number theorem with partial summation, in the standard form of [2]. This gives

$$\frac{1}{m^2} \sum_{\alpha m < p < \beta m} W_{\mathrm{cons}}(p) \log p \longrightarrow \int_\alpha^\beta w_{\mathrm{cons}}(x)\,\mathrm{d}x. \tag{4.48}$$

The at-most-one cell-boundary prime described above contributes $O(m \log m)$. Let first m tend to infinity and then ε tend to zero. This yields

$$\limsup_{m \to \infty} \frac{\log D_m(a)}{m^2} \leq S_\rho + \frac{9\rho^2}{4} \log|a-1| - \frac{1}{8}\log|a|. \tag{4.49}$$

Thus, with $q_a = |a|$,

$$\begin{aligned}\tau_+(q_a) &= S_+ + \frac{1}{4}\log(q_a - 1) - \frac{1}{8}\log q_a,\\ \tau_-(q_a) &= S_- + \frac{225}{1024}\log(q_a + 1) - \frac{1}{8}\log q_a.\end{aligned} \tag{4.50}$$

The endpoint divisor contributes $H_*/m^2 \to 9\rho^2/4$ and $N/m^2 \to 1/8$. This proves the required arithmetic limsup.

5. Nonvanishing along prime-power towers

The ancillary seed files contain two coefficient arrays. Let H_{211} and H_{378} be the integer polynomials they determine. Appendix C prints their degrees and complete residue vectors. The arrays themselves are the exact coefficient data used by the seed replay. The seed arrays and the interval boxes used later are part of the proof record. They accompany the paper as ancillary data. Take the seed pair $(n, h) = (12, 4)$ or $(16, 5)$, and write $b = a - 1$ and $H = 3h - 3$ at the seed. We use row normalization, as in the ancillary script. For each seed row $i \in \{0, \dots, n-1\}$, let

$$s_i = \operatorname{lcd}\{\text{coefficients of } b^H E_{ij}(a) : n \le j < 2n\}, \qquad N_n(a) = \left[s_i b^H E_{ij}(a)\right]_{\substack{0\le i<n\\ n\le j<2n}},$$

and put $S_n = \prod_{i=0}^{n-1} s_i$. Thus $N_n(a)$ is an integer polynomial matrix and $\det N_n(a) = S_n b^{nH} G_{n,a}(0)$. Define

$$\begin{aligned}K_{12} &= 2^{148}3^{74}5^{20}7^{32}11^{37}13^{38}17^{28}19^{20}23^4,\\ K_{16} &= 2^{154}3^{96}5^{91}7^{32}11^{46}13^{52}17^{51}19^{52}23^{36}29^{12}31^4,\end{aligned}$$

and let K_n denote the corresponding constant. Put $f_{12} = 25$, $f_{16} = 42$, $e_{12} = 12$, and $e_{16} = 24$. Then the preceding identity

$$K_n \det N_n(a) = S_n a^{e_n}(a-1)^{nH - f_n} H(a), \tag{5.1}$$

is the exact cleared identity checked by the seed script. Thus the corresponding determinant quotient has denominator $(a-1)^{f_n}$. The two sides of (5.1) have the same polynomial after clearing the row product. Their degrees are at most $\sum_{j=12}^{23}(j+8) = 306$ and $\sum_{j=16}^{31}(j+11) = 552$, respectively. These are the column degree bounds $j + H - 1$ summed over the $m = n$ columns. So exact evaluation at 307 and 553 integer points proves the identities. They read

$$G_{12,a}(0) = \frac{a^{12}H_{211}(a)}{K_{12}(a-1)^{25}}, \qquad G_{16,a}(0) = \frac{a^{24}H_{378}(a)}{K_{16}(a-1)^{42}}, \tag{5.2}$$

On the positive branch write $H = H_{211}$. On the negative branch write $H = H_{378}$. The residues of H_{211} are nonzero modulo 29. Those of H_{378} are nonzero modulo 37. The residue vectors are printed in Appendix C; the ancillary seed replay recomputes them from the coefficient arrays. Hence $H(a) \neq 0$ for every integer a. For a fixed integer $|a| \ge 27$, choose a prime divisor p of

$$(2n)!\,|a(a-1)H(a)| + 1, \tag{5.3}$$

where $(n, H) = (12, H_{211})$ on the positive side. On the negative side it is $(16, H_{378})$. Then $p > 2n$ and $p \nmid a(a-1)H(a)K_n$. Every prime factor of K_{12} is below 24, and every prime factor of K_{16} is below 32. So the seed value in (5.2) is a p-unit.

Put $Q = p^k$, $(m, g) = (nQ, hQ)$, with $(n, h) = (12, 4)$ or $(16, 5)$, and work in $\mathbb{Z}_{(p)}$. The coefficientwise tower calculation follows. For $0 \leq s < Q$ put

$$B_s(x) = \binom{Qx - s - 1}{hQ}. \tag{5.4}$$

Pair the nonmultiples of Q in the numerator window with the corresponding nonmultiples in $\{1, \ldots, hQ\}$. Each quotient is a p-adic unit whose reduction does not depend on x. So the product formula first shows $B_s(x) \in \mathbb{Z}_{(p)}[x]$. Partition the hQ numerator factors according to their residue modulo Q. Every residue occurs h times. More explicitly, pair a numerator factor $Qx - c$ with a denominator factor c' having the same residue modulo Q. Then $c \equiv c' \pmod{p}$ and $(Qx - c)/c' \equiv -1 \pmod{p}$. The h factors with $Q \mid c$ give the remaining product $\binom{x-1}{h}$ after the powers of Q are canceled. After removing the fixed powers of p contributed by $(hQ)!$, all residue classes except the one with $t \equiv -s-1 \pmod{Q}$ contribute p-adic units independent of x modulo p. The exceptional class contributes exactly h linear factors in x. Thus the reduced polynomial has degree at most h. Lucas' formula gives its values at $x = 1, \ldots, h$. At $x = 0$ the negative-index identity gives the same value $(-1)^h$, because $\binom{-1}{h} = (-1)^h$; on the left $Q = p^k$ is odd since $p > 2n$. The remaining sign is $(-1)^{h(Q-1)} = 1$. The $h+1$ values therefore prove the coefficientwise congruence is the unsigned identity

$$\binom{Qx - s - 1}{hQ} \equiv \binom{x - 1}{h} \pmod{p}. \tag{5.5}$$

Write $i = Qi_0 + s_i$ and $j = Qj_0 + s_j$, with residues in $\{0, \ldots, Q-1\}$. Let $P^0_{i_0 j_0}$, $E^0 = (E^0_{i_0 j_0})$ and $C^0 = (C^0_{i_0 j_0})$ be the polynomial parts P_{ij} and the matrices $E(a)$ and $C(a)$ of Section 2 with $(m, g) = (n, h)$. For a nonzero offset d we use the Taylor series

$$U_d(x) = \frac{Q}{Qx + d} = \frac{Q/d}{1 + (Q/d)x}.$$

Since $|d| < 2nQ < pQ$, its coefficients are p-integral. Put $d_i = w(-i)/(j - i)$ and $d_j = -w(-j)/(j - i)$. The scaled polynomial part satisfies the exact identity

$$Q^2 P_{ij}(Qx - s) = B_s(x)^3 U_{i-s}(x) U_{j-s}(x) - (Qd_i) U_{i-s}(x) - (Qd_j) U_{j-s}(x). \tag{5.6}$$

Suppose one offset is zero. The two residue values then agree. So the numerator in the corresponding $1/x$ term has zero constant coefficient. Division by x therefore remains integral. Now reduce (5.6). A nonzero offset contributes only when it is a multiple of Q. Both offsets contribute precisely when $s_i = s_j = s$. In that case (5.5) gives the seed polynomial part $P^0_{i_0 j_0}$. This includes the finite-part division at a zero offset. After two derivatives we obtain, coefficientwise,

$$Q^4 P''_{ij}(Qx - s) \equiv \mathbf{1}_{s_i = s_j = s} (P^0_{i_0 j_0})''(x) \pmod{p}. \tag{5.7}$$

The scaled values in (5.7) have integral Newton coefficients. So Newton interpolation and

$$\sum_{r \geq 1} z^r \binom{r - 1}{t} = \frac{z^{t+1}}{(1 - z)^{t+1}}$$

sum (5.7) coefficientwise. Split $r = Qx - s$. A matching residue then contributes the factor a^s and the denominator $1 - a^{-Q}$. This denominator is a p-unit because $p \nmid a(a - 1)$, so $a^Q \equiv a \not\equiv 1 \pmod{p}$. The zero-offset finite part is included

before this splitting. Because $h < p$, the p-parts of the factors in $(hQ)!$ cancel coefficientwise against the corresponding numerator blocks. For the harmonic term,

$$Q^3 H_{Qu+s}(a) \equiv a^s H_u(a) \pmod{p},$$

because only the multiples $t = Qv$ survive and $a^Q = a$. We combine the polynomial part, the harmonic term, and the T coefficient. This gives the all-k congruences

$$Q^4 E_{ij} \equiv \mathbf{1}_{s_i=s_j} a^s E^0_{i_0 j_0}, \qquad QC_{ij} \equiv \mathbf{1}_{s_i=s_j} a^s C^0_{i_0 j_0} \pmod{p}. \tag{5.8}$$

Define

$$\mathcal{P}_k(X) = Q^{4m} G_{m,a}(X/Q^3).$$

Group rows and columns by their residues modulo Q. The block permutation signs cancel. The Q diagonal blocks are $a^s(E^0 + XC^0)$. Their scalar is $a^{nQ(Q-1)/2} = 1$ in $\mathbb{F}_p$, since n is even and $a^{Q-1} = 1$. Thus, for every k,

$$\mathcal{P}_k(X) \in \mathbb{Z}_{(p)}[X], \qquad \mathcal{P}_k(X) \equiv G_{n,a}(X)^Q \pmod{p}. \tag{5.9}$$

Let ξ be fixed and rational. Choose k with $3k + v_p(\xi) \geq 1$ (with $v_p(0) = +\infty$). Evaluate at $X = Q^3\xi$. For all sufficiently large k this gives the unscaled identity

$$Q^{4m} G_{m,a}(\xi) = \mathcal{P}_k(Q^3\xi) \equiv G_{n,a}(Q^3\xi)^Q \equiv G_{n,a}(0)^Q \not\equiv 0 \pmod{p}.$$

The seed choice in (5.3) makes $G_{n,a}(0)$ a p-unit. Since the left side would be zero if $G_{m,a}(\xi) = 0$, this unit congruence gives the required contradiction. The argument uses the displayed congruence before any cancellation of the factor Q^{4m}.

6. Positive analytic estimate

Let $m = 12Q$, $g = 4Q$, $\rho = 1/3$, and $q = |a| > 1$, $\lambda = \log q$. Define

$$h_\rho(t) = t\log t - (t-\rho)\log(t-\rho) - \rho\log\rho, \quad B(z) = \int_0^2 \log|z+x| \, \mathrm{d}x \quad (z \in \mathbb{C}\backslash[-2,0]).$$

For real $t \geq \rho$, this complex-plane definition is the real function $B(t)$ used in the energy below. On probability densities $\mu = f(t)\,\mathrm{d}t$ with $0 \leq f \leq 3$, support in $[\rho, \infty)$, and finite first moment, put

$$L(\mu,\nu) = -\iint \log|t-s| \, \mathrm{d}\mu(t)\,\mathrm{d}\nu(s).$$

$$\mathcal{E}^+_\lambda(\mu) = \frac{3}{2} + \int (B - 3h_\rho + \lambda t) \, \mathrm{d}\mu + L(\mu,\mu), \quad A_+(\lambda) = \inf_\mu \mathcal{E}^+_\lambda(\mu).$$

Proposition 6.1. *For the positive tower,*

$$\limsup_{Q\to\infty} (12Q)^{-2} \log|\det M(a)| \leq -A_+(\log|a|). \tag{6.1}$$

Moreover A_+ is finite and concave, and

$$A_+(\lambda_2) - A_+(\lambda_1) \geq \frac{1}{2}(\lambda_2 - \lambda_1) \quad (\lambda_2 \geq \lambda_1). \tag{6.2}$$

Proof. For every center z used below, the circle $|u - z| = 1/4$ avoids the negative-integer poles. Cauchy's formula gives

$$F''_{ij}(z) = \frac{2!}{2\pi i} \int_{|u-z|=1/4} \frac{F_{ij}(u)}{(u-z)^3} \, \mathrm{d}u. \tag{6.3}$$

Thus the two derivatives can be represented before the determinant summation. The representation also makes the triple zero of w and the two derivative orders

explicit in the occupancy estimate below. Expand the finite sums $r \leq R$ and apply Andreief's identity [1] to the two Cauchy determinants. Then let R tend to infinity. Absolute convergence holds because the differentiated rational summands have polynomial growth and $q^{-\sum r_\ell}$ is geometric. The Cauchy double alternant gives the standard formula [5, Eq. (2.7)]

$$H(z_1, \dots, z_m) = \Delta(I)\Delta(J)\Delta(z)^2 \prod_{\ell=1}^{m} \frac{w(z_\ell)}{\prod_{k=0}^{2m-1}(z_\ell + k)}.$$

Consider a nonzero term. The triple zeros of w and the two derivatives give it occupancy at most three at every integer center. Suppose k variables share a center. Their squared Vandermonde vanishes to order $k(k-1)$ on that diagonal. The derivatives have total order $2k$. Since $k(k-1) > 2k$ for $k \geq 4$, every such term vanishes. Use Cauchy circles of radius $1/4$ independently in each variable.

Use the midpoint convention $s_r = r + 1/2$ and $\zeta_r = s_r/m$. Let μ_m have density equal to the center occupancy on $[r/m, (r+1)/m)$. In cells a fixed positive distance from the support endpoint, replacing s_r/m by either endpoint changes each displayed Riemann sum by $O(1/m)$ per particle. The bounded collection of cells meeting the endpoint has $h'_{1/3}(t) = \log(t/(t-1/3))$ and contributes $O(\log m/m)$ per particle. The monotone tail estimates handle the remaining cells. Summed over all particles, these errors are $O(m \log m)$. The fixed Vandermonde, moving Vandermonde, filter, and denominator estimates hold uniformly in the admissible center tuples. With this convention they are

$$\begin{aligned} \log(\Delta(I)\Delta(J)) &= m^2 \log m - \frac{3}{2}m^2 + O(m \log m), \\ \log|\Delta(z)^2| &\leq m(m-1)\log m - m^2 L(\mu_m, \mu_m) + O(m \log m), \\ \sum_\ell \log|w(z_\ell)| &\leq 3m^2 \int h_\rho \, \mathrm{d}\mu_m + O(m \log m), \\ \sum_{\ell,k} \log|z_\ell + k| &\geq 2m^2 \log m + m^2 \int B \, \mathrm{d}\mu_m - O(m \log m). \end{aligned} \tag{6.4}$$

The filter estimate is the entropy inequality for $\binom{r-1}{g}$. It is followed by the increasing Riemann sum for h_ρ. For the moving Vandermonde, let $c_r \leq 3$ be the occupancy of cell r. Remove, for each particle, pairs with distinct particles in cells at distance $|r-s| < 3$. Occupancy at most three leaves at most fourteen such non-diagonal pairs per particle. There are two in its own cell and three in each of the four adjacent cells. The same-cell integral is $-\log m - 3/2$, and the removed pairs contribute $O(m \log m)$. For the remaining pairs, if $d = |r-s| \geq 3$, two particles in these cells are at distance at most $(d+1)/m$. Integrating over the two cells gives the comparison

$$\log \frac{d+1}{m} \leq m^2 \int_{r/m}^{(r+1)/m} \int_{s/m}^{(s+1)/m} \log|x-y| \, \mathrm{d}x \, \mathrm{d}y + O(d^{-1}).$$

Summing this inequality with weights $c_r c_s$ yields the moving-Vandermonde line in (6.4). The constants in the O terms can be chosen independently of the tuple of

centers. For $c_r \le 3$ and $\sum_r c_r = m$,

$$\left(\sum_r c_r(c_r-1) + \sum_r \sum_{0<|s-r|<3} c_r c_s\right)(1+\log(m+1)) \le 14m(1+\log(m+1)),$$

$$\sum_r \sum_{|s-r|\ge 3} \frac{c_r c_s}{|r-s|} \ll m\log(m+1).$$

The first bound handles the deleted same-cell and neighboring-cell pairs. The second sums the $O(d^{-1})$ comparison error, using at most six particles in each fixed positive distance shell. The midpoint, endpoint, and Cauchy-circle replacements satisfy the same uniform estimate

$$|R_{\mathrm{edge}}| + |R_{\mathrm{circle}}| \le Cm\log(m+1)$$

for an absolute C and every admissible center tuple. In particular the singular first-cell derivative contributes $O(\log m/m)$ per particle before the outer factor m. So this bound includes it. The denominator line is the increasing Riemann sum for B. For $\rho = 1/3$ the changed uniform inequalities are

$$\log|w(m\zeta)| \le 3m\int_0^{1/3} \log|\zeta - x|\,\mathrm{d}x + 3m(\rho - \rho\log\rho) + O(\log m),$$

$$\sum_{k=0}^{2m-1} \log|\zeta + k/m| \ge mB(\zeta) - \log 7.$$

The support is $[1/3,\infty)$. For one particle the denominator-circle loss is

$$\sum_{k=0}^{2m-1} -\log\left(1 - \frac{1}{4(r+k)}\right) < 2, \qquad r+k \ge m/3.$$

Thus the total circle loss is $< 2m$. The denominator line loses $\log 7$ per particle, and hence $m\log 7$ in total. Both losses are $O(1)$ per particle, and $O(m)$ in total. The fixed and moving Vandermonde bounds use the same near-pair correction above. The remaining terms are the Riemann sums of the two displayed inequalities over the cells. Their errors are $O(\log m)$ per particle, so the total error is $O(m\log m)$. None of these terms alters the quadratic rate.

Reserve $\exp(-\sum r_\ell/m)$ from the geometric weight and put $\beta = \lambda - 1/m$. Equation (6.4), Cauchy's estimate, and cancellation of all $m^2\log m$ terms give each nonzero tuple the bound

$$\exp\{-m^2\mathcal{E}_\beta^+(\mu_m) + O(m\log m) - \tfrac{1}{m}\sum_\ell r_\ell\}.$$

The unrestricted tail satisfies

$$\frac{1}{m!}\sum_{r_1,\dots,r_m\ge 1} e^{-\sum r_\ell/m} \le \frac{1}{m!}(e^{1/m}-1)^{-m} \le e^m. \tag{6.5}$$

Take the supremum over center tuples, sum (6.5), and then let $Q\to\infty$. This proves the bound with $A_+(\lambda - 1/m)$. The energy is finite on a compact reference density. It is bounded below by the confining linear term and is affine in λ. Hence it is concave and continuous. Letting $m\to\infty$ gives (6.1).

Finally, we use the bathtub inequality for the density cap 3. It shows that every admissible probability density supported on $[\rho, \infty)$ has first moment at least $\rho + 1/6 = 1/2$. Taking the difference of the two affine functionals in λ proves (6.2). The finite cutoff is removed exactly as in [14, Lemma 6.1]. Here the summand is bounded by the geometric factor in (6.5). □

7. Negative signed-contour estimate

Let $m = 16Q$, $g = 5Q$, $\rho = 5/16$, $q = |a| > 1$, and $\lambda = \log q$. Put

$$H_\rho(z) = \int_0^\rho \log|z - x| \, \mathrm{d}x - \rho \log \rho + \rho, \quad B(z) = \int_0^2 \log|z + x| \, \mathrm{d}x,$$
$$V_\lambda(z) = B(z) - 3H_\rho(z) + \lambda \operatorname{Re} z + \pi |\operatorname{Im} z|.$$

In this section set

$$L(\mu, \nu) = -\iint \log|z - w| \, \mathrm{d}\mu(z) \, \mathrm{d}\nu(w).$$

Let

$$\Gamma_m(t) = g + \frac{1}{2} + 4|t| + it, \qquad t \in \mathbb{R},$$

oriented by increasing t. Let C be its limiting scaled contour $z = \rho + 4|y| + iy$. For probability measures μ on C define

$$\mathcal{E}_\lambda^-(\mu) = \frac{3}{2} + \int_C V_\lambda \, \mathrm{d}\mu + L(\mu, \mu), \qquad A_-(\lambda) = \inf_\mu \mathcal{E}_\lambda^-(\mu).$$

Proposition 7.1. *For the negative tower,*

$$\limsup_{Q \to \infty} (16Q)^{-2} \log|\det M(-q)| \leq -A_-(\log q). \tag{7.1}$$

The infimum is over all probability measures of finite first moment and finite logarithmic energy; there is no density-cap assumption. Further,

$$A_-(\lambda_2) - A_-(\lambda_1) \geq \frac{5}{16}(\lambda_2 - \lambda_1) \quad (\lambda_2 \geq \lambda_1). \tag{7.2}$$

Proof. Write $c = g + 1/2$. Consider $F_{ij}(s) = w(s)/((s+i)(s+j))$. Close the contour formed by the two rays and the downward vertical segment at $\operatorname{Re} s = R$. Let R tend to infinity through half-integers. The boundary is clockwise. The residue of $q^{-s} F_{ij}''(s)/\sin(\pi s)$ at a positive integer r is $(-1)^r q^{-r} F_{ij}''(r)/\pi$. The residues with $r \leq g$ vanish because w has a triple zero. The vertical segment is $O(q^{-R} R^C)$ and tends to zero. The two rays converge exponentially. Therefore, with orientation, the exact signed identity is

$$M_{ij}(-q) = -\frac{1}{2i} \int_{\Gamma_m} \frac{q^{-s} F_{ij}''(s)}{\sin(\pi s)} \, \mathrm{d}s. \tag{7.3}$$

Set $W_{I,J} = \Delta(I)\Delta(J)$ and

$$P(s_1, \ldots, s_m) = \frac{\prod_\ell w(s_\ell) \Delta(s)^2}{\prod_{\ell=1}^m \prod_{k=0}^{2m-1} (s_\ell + k)}.$$

Finite determinant expansion followed by absolute convergence gives the differentiated Andreief identity

$$\det M(-q) = \frac{W_{I,J}}{m!} \left(-\frac{1}{2i}\right)^m \int_{\Gamma_m^m} \prod_\ell \frac{q^{-s_\ell}}{\sin(\pi s_\ell)} \left(\partial_1^2 \cdots \partial_m^2 P\right) \prod_\ell \mathrm{d}s_\ell. \tag{7.4}$$

The derivatives act on P only.

At $s = \Gamma_m(t)$,

$$|\sin(\pi(c + 4|t| + it))|^{-1} \le C_{\sin} e^{-\pi|t|}. \tag{7.5}$$

Here $C_{\sin}$ is an absolute constant. The squared modulus is $\cos^2(4\pi|t|) + \sinh^2(\pi t)$. On $|t| \le 1/16$ the cosine is at least $1/\sqrt{2}$ in absolute value. On $1/16 \le |t| \le 1$ the hyperbolic sine has a fixed positive lower bound. For $|t| \ge 1$ it gives the exponential bound. Cauchy circles of radius $1/8$ give

$$|\partial_1^2 \cdots \partial_m^2 P(s)| \le 128^m \sup_{|u_\ell - s_\ell| \le 1/8} |P(u)|. \tag{7.6}$$

We shift only the majorant $q^{-\operatorname{Re} s} e^{-\pi|\operatorname{Im} s|}$ of the base and sine factors in (7.5). On each such circle, replacing the center s_ℓ by a supremum point u_ℓ changes this majorant by at most $\exp((\lambda + \pi)/8)$. The m shifts therefore cost $(\lambda + \pi)m/8 = O_\lambda(m)$ and are absorbed in the linear error term. Use the scaled coordinate $\zeta = s/m$ throughout. Thus the contour centre has $\zeta = z(y) + 1/(2m)$, and the Cauchy circles give the tube

$$\mathcal{S}_m = \{z(y) + 1/(2m) + e : \ |e| \le 1/(8m)\}, \quad z(y) = \rho + 4|y| + iy.$$

The endpoint and midpoint Riemann sums have total error $O(\log m + \log(1 + |\zeta|))$ per particle. In particular $\operatorname{dist}(\mathcal{S}_m, C) \le 5/(8m)$. The filter and denominator bounds on this tube are

$$\begin{aligned} \log|w(m\zeta)| &\le 3mH_\rho(\zeta) + O(\log m + \log(1 + |\zeta|)), \\ \sum_{k=0}^{2m-1} \log|\zeta + k/m| &\ge mB(\zeta) - O(\log m + \log(1 + |\zeta|)). \end{aligned} \tag{7.7}$$

The filter bound follows from the decreasing Riemann sum. The denominator bound follows from the increasing one. The normalization in H_ρ uses $\log(g!) \ge g \log g - g$. The fixed Vandermonde satisfies

$$\log W_{I,J} = m(m-1)\log m - \frac{3}{2}m^2 + O(m \log m),$$

and the powers of m cancel against the moving Vandermonde and the $2m$ denominator factors. The resulting integrand is bounded by

$$\exp\left\{ -\frac{3}{2}m^2 - m\sum_\ell V_\lambda(\zeta_\ell) + \sum_{\ell \ne j} \log|\zeta_\ell - \zeta_j| + C_2 \sum_\ell \log(1 + |\zeta_\ell|) + O(m \log m) \right\}. \tag{7.8}$$

Here $C_2 = C_2(\lambda)$ is independent of m and of the configuration. It collects the per-particle errors of (7.7); its λ dependence also absorbs the preceding circle-shift cost.

At a contour center $s = m\zeta$ the base and sine factors satisfy

$$q^{-\operatorname{Re} s} e^{-\pi|\operatorname{Im} s|} \le \exp\{-m(\lambda \operatorname{Re}\zeta + \pi|\operatorname{Im}\zeta|)\}.$$

For $L \geq 0$ put $\ell_L(d) = \max(\log d, -L)$, with $\ell_L(0) = -L$. If $\mu_m = m^{-1}\sum_\ell \delta_{\zeta_\ell}$, then for every L

$$\sum_{\ell \neq j} \log|\zeta_\ell - \zeta_j| \leq m^2 \iint \ell_L(|z-w|)\, \mathrm{d}\mu_m(z)\, \mathrm{d}\mu_m(w) + mL. \tag{7.9}$$

Subtract and add $\varepsilon|\operatorname{Im}\zeta_\ell|$ in (7.8). The bound for the full configuration then has three parts. They are the corresponding truncated energy plus $mL + O(m\log(m+1))$, the term $C_2\sum_\ell \log(1+|\zeta_\ell|)$, and the factor $\exp(-m\varepsilon\sum_\ell |y_\ell|)$. On the tube $|\zeta| \leq 1+5|y|$, so $\log(1+|\zeta|) \leq \log 2 + \frac{5}{2}|y|$. For $m \geq 5C_2/\varepsilon$ we get $C_2\log(1+|\zeta_\ell|) \leq C_2\log 2 + \frac{1}{2}m\varepsilon|y_\ell|$. Half of the factor absorbs these terms, at the cost $e^{O(m)}$. Since $|\,\mathrm{d}s| = m\sqrt{17}\,\mathrm{d}y$ on the two rays, the remaining factor $\exp(-\frac{1}{2}m\varepsilon\sum_\ell|y_\ell|)$ has integral at most $(4\sqrt{17}/\varepsilon)^m$. Taking first $m\to\infty$, then $L\to\infty$, and finally $\varepsilon \downarrow 0$ proves (7.1) using (7.11)–(7.12).

For $0 < \varepsilon \leq \pi/2$ put

$$V_{\lambda,\varepsilon}(z) = V_\lambda(z) - \varepsilon|\operatorname{Im} z|, \qquad K_{\varepsilon,L}(z,w) = \frac{V_{\lambda,\varepsilon}(z) + V_{\lambda,\varepsilon}(w)}{2} - \ell_L(|z-w|),$$

and let $\mathcal{E}_{\varepsilon,L}$ be $3/2$ plus the $K_{\varepsilon,L}$ pair integral. Work in the closed region

$$D = \{z : \rho - 1/32 \leq \operatorname{Re} z \leq \rho + 4|\operatorname{Im} z| + 1\},$$

which contains C and every $\mathcal{S}_m$ for $m \geq 16$. There are constants $c > 0, C_0$ such that

$$K_{\varepsilon,L}(z,w) \geq c(|z|+|w|) - C_0 \qquad (z,w \in D). \tag{7.10}$$

These constants are independent of L and ε. The same elementary logarithmic bounds used for C apply on the fixed strip $\rho - 1/32 \leq \operatorname{Re} z \leq \rho$. Their constants are enlarged there. They give $V_{\lambda,\varepsilon}(z) \geq (\pi/2)|\operatorname{Im} z| - 3\rho\log(1+|z|) - C_1$. Here C_1 is independent of L and ε. On D, $|z| \leq \rho + 1 + 5|\operatorname{Im} z|$, and $\ell_L(|z-w|) \leq \log(1+|z|) + \log(1+|w|)$. This proves (7.10).

Let $a_{\varepsilon,L,m}$ be the infimum of $\mathcal{E}_{\varepsilon,L}$ over probability measures on $\mathcal{S}_m$. Let $a_{\varepsilon,L}$ be the corresponding infimum on C. Choose a fixed compactly supported reference measure on a finite piece of C. Push it by the real shift $1/(2m)$. Its support is contained in $\mathcal{S}_m$, so it gives a uniform upper bound for near-minimizers of $a_{\varepsilon,L,m}$. Together with the coercivity in (7.10), this gives tightness and a uniform first-moment bound. From any weakly convergent subsequence, the distance bound $\operatorname{dist}(\mathcal{S}_m, C) \leq 5/(8m)$ puts the limit on C. For fixed L, $K_{\varepsilon,L}$ is bounded below and continuous. The product measures converge weakly. We truncate the kernel above and then use monotone convergence. This proves the lower-semicontinuity inequality

$$\liminf_{m\to\infty} a_{\varepsilon,L,m} \geq a_{\varepsilon,L}. \tag{7.11}$$

As L increases, ℓ_L decreases and $K_{\varepsilon,L}$ increases to the full kernel. Hence $a_{\varepsilon,L}$ is nondecreasing and has a limit no larger than the full infimum $A_{-,\varepsilon}(\lambda)$. For the reverse inequality, take near-minimizers for $a_{\varepsilon,L}$ and use (7.10) for tightness. Pass to a weak limit under each fixed truncation K_{ε,L_0}, then let L_0 increase. This identifies the limit of the infima with the full infimum.

It remains to remove the auxiliary ε. Let $\varepsilon_j \downarrow 0$ and choose μ_j on C with $\mathcal{E}_{\varepsilon_j}(\mu_j) \leq A_{-,\varepsilon_j}(\lambda) + 1/j$. The same reference measure and (7.10) give tightness and a uniform first-moment bound. After passing to a subsequence, $\mu_j \Rightarrow \mu$. Fix L

and a clipping level T, and set $K_{j,L}^T = \min(K_{\varepsilon_j,L}, T)$. Choose a continuous cutoff χ_R on D. For $R > 0$ define the compact exhaustion $D_R = D \cap \{|z| \le R\}$. Choose χ_R equal to 1 on D_R and supported in D_{R+1}, with $0 \le \chi_R \le 1$. On the compact support of $\chi_R(z)\chi_R(w)$, the kernels $K_{j,L}^T$ converge uniformly to $K_{0,L}^T$. The coercive lower bound (7.10) controls the complement, and tightness makes its contribution at least $-2\eta(T + C_0)$ for R large. Weak convergence applied to the continuous cutoff therefore gives

$$\liminf_{j\to\infty} \iint K_{\varepsilon_j,L}\, \mathrm{d}\mu_j\, \mathrm{d}\mu_j \ge \iint K_{0,L}^T\, \mathrm{d}\mu\, \mathrm{d}\mu - 2\eta(T + C_0).$$

First let $j \to \infty$, then $R \to \infty$ and $\eta \downarrow 0$, and finally $T \to \infty$. This yields the varying-kernel lower bound at fixed L. Letting $L \to \infty$ gives

$$\lim_{\varepsilon\downarrow 0} A_{-,\varepsilon}(\lambda) = A_-(\lambda). \tag{7.12}$$

We turn to the zero-mass energy inequality used in endpoint certificates. Let μ and σ be probability measures of equal mass, with σ compactly supported. Assume that μ has finite first moment and finite logarithmic self-energy. Assume also that the cross-pairing with σ is finite. Put $\nu = \mu - \sigma$. Restrict both measures to a disk of radius R and normalize. This gives probability measures μ_R and σ_R. Put $\nu_R = \mu_R - \sigma_R$. For all large R, $\sigma_R = \sigma$, so ν_R is compactly supported and has zero mass. The compact zero-mass energy inequality

$$L(\nu_R, \nu_R) \ge 0 \tag{7.13}$$

is the direct compact case of [11, Lemma I.1.8]. We use only this inequality; no Fourier limiting identity is needed. The positive part of $\log|z - w|$ is bounded by $\log(1 + |z|) + \log(1 + |w|)$. The negative part is controlled by the finite self-energy. Also $U^\sigma(z) = O(\log(1 + |z|))$. Consequently $L(\mu_R, \mu_R) \to L(\mu, \mu)$ and $L(\mu_R, \sigma) \to L(\mu, \sigma)$, so $L(\nu_R, \nu_R) \to L(\nu, \nu)$. We pass only this pairing limit in (7.13):

$$L(\mu - \sigma, \mu - \sigma) \ge 0. \tag{7.14}$$

The bounded compact densities of both endpoint certificates satisfy these hypotheses.

Since $\operatorname{Re} z \ge \rho = 5/16$ on C, subtracting the two affine functionals in λ gives (7.2). □

8. Endpoint certificates and ray comparison

We first state the certificate inequalities and then record their numerical values. Consider the positive problem first. Let $\sigma = f_\sigma(t)\,\mathrm{d}t$ be the piecewise-constant density given by the 256-cell table in Appendix A. Put

$$\begin{aligned} U^\sigma(t) &= \int \log|t - s|\, \mathrm{d}\sigma(s), \\ V_+(t) &= B(t) - 3h_{1/3}(t) + \lambda t, \\ \Psi_+(t) &= V_+(t) - 2U^\sigma(t). \end{aligned}$$

For every real Λ,

$$A_+(\lambda) \ge \frac{3}{2} - L(\sigma,\sigma) + \Lambda - 3\int_{1/3}^{\infty} (\Lambda - \Psi_+(t))_+\, \mathrm{d}t. \tag{8.1}$$

Take an admissible density $f \leq 3$. The pointwise inequality $(\Psi_+ - \Lambda)f \geq -3(\Lambda - \Psi_+)_+$ and $L(f\,\mathrm{d}t - \sigma, f\,\mathrm{d}t - \sigma) \geq 0$ give (8.1). Explicitly,

$$L(\mu,\mu) = L(\mu-\sigma,\mu-\sigma) - 2\int U^\sigma\,\mathrm{d}\mu - L(\sigma,\sigma),$$

which fixes the sign in (8.1). The zero-mass inequality is the one-dimensional logarithmic-energy inequality; it is the real-line special case of (7.14). On each printed cell, the convex terms of Ψ_+ are bounded by their midpoint tangent. The concave terms are bounded by their endpoint chord. The tail is bounded by

$$\Psi_+(t) \geq B(t) - 3h_{1/3}(t) + \lambda t - 2\log(t - 1/3),$$

whose derivative is at least $\lambda - (3\rho+2)/(t-\rho)$. The interval check verifies that this bound is increasing beyond the last cell.

Consider the negative problem. Let σ be the positive rational probability measure on the two rays in Appendix B. The compact certificate uses the interval boxes in the ancillary files. On the whole contour it verifies

$$\Psi_-(z) := V_\lambda(z) - 2U^\sigma(z) \geq \Lambda_- \quad (z \in C), \tag{8.2}$$

The contour, V_λ, and the certificate measure are invariant under complex conjugation. Thus the interval replay on the displayed + ray gives the same bound on the − ray. The replay also evaluates $L(\sigma,\sigma)$ with the exact pair kernel, including the displayed minus sign. This evaluation includes the finite part at the cusp. Since every competitor has mass one, (8.2) and $L(\mu-\sigma,\mu-\sigma) \geq 0$ give

$$A_-(\lambda) \geq \frac{3}{2} - L(\sigma,\sigma) + \Lambda_-. \tag{8.3}$$

For the two endpoint values, the replay computes Λ_- as the minimum of $V_\lambda - 2U^\sigma$ over the directed interval boxes and rounds that minimum downward:

$$\Lambda_-^{(27)} = 5.0771336769, \qquad \Lambda_-^{(28)} = 5.1031720708. \tag{8.4}$$

The interval check subdivides compact ray pieces. It bounds the infinite tail by the monotone derivative of $V_\lambda - 2U^\sigma$. It rounds the resulting lower bounds downward. Thus these are directed certificates. They do not rest on any numerical optimization.

The positive endpoint levels used in the two positive checks are

$$\Lambda_{27} = \frac{1364590351}{268435456}, \qquad \Lambda_{28} = \frac{687124203}{134217728}.$$

Each printed positive cell is subdivided into 32 equal subcells. Write the subcells as $[t_i^-, t_i^+]$ and put $h_i = t_i^+ - t_i^-$. Let f_i be the constant density on that subcell, and let $\mu_i = f_i h_i$ be its mass. With $\Phi(x) = x^2(\log|x|/2 - 3/4)$, using the real logarithm and the continuous value $\Phi(0) = 0$, the exact pair term is

$$L(\sigma,\sigma) = -\sum_{i\geq j} \frac{\mu_i\mu_j}{h_i h_j}(2 - \mathbf{1}_{i=j})[\Phi(t_i^+ - t_j^-) - \Phi(t_i^- - t_j^-) - \Phi(t_i^+ - t_j^+) + \Phi(t_i^- - t_j^+)].$$

After the last cell the check uses the increasing tail $V_+(t) - 2\log(t - 1/3)$.

For the negative table, the branch directions are $A = (4+i)/\sqrt{17}$ and $\overline{A} = (4-i)/\sqrt{17}$. A cell count satisfies $c_j \geq 0$ and $\sum_j c_j = d$ for each printed column. It has mass $c_j/(2d)$ on each branch and density $c_j/(2d\,\Delta u_j)$. Here d is the printed denominator and Δu_j is the cell width. For the cross-ray primitive set $\beta = -\overline{A} = (-4+i)/\sqrt{17}$, so $Au + \beta v = z_+(u) - z_-(v)$. Use the principal analytic branch

$\operatorname{Log} z$ on $\mathbb{C} \setminus (-\infty, 0]$ and define $\Xi(z) = z^2(\operatorname{Log} z/2 - 3/4)$ there, with $\Xi(0) = 0$ by continuity. Take a cell $[t^-, t^+]$ on the first branch and a cell $[s^-, s^+]$ on the second. The opposite branch uses the signed contribution

$$-\operatorname{Re}\biggl(\frac{\Xi(At^+ + \beta s^+) - \Xi(At^- + \beta s^+) - \Xi(At^+ + \beta s^-) + \Xi(At^- + \beta s^-)}{A\beta}\biggr),$$

with these density factors. The interval lower bounds are $4.97200113\ldots$ and $4.99000040\ldots$ before downward rounding. The largest support endpoint is $u_{\max} = 238521/262144$. To justify the tail bound, write $A = (4 + i)/\sqrt{17}$ and $z_+(u) = \rho + Au$. For $t_0 \geq 0$,

$$\operatorname{Re}\frac{A}{t_0 + Au} = \frac{u + 4t_0/\sqrt{17}}{u^2 + 8t_0u/\sqrt{17} + t_0^2}, \qquad 0 \leq \operatorname{Re}\frac{A}{t_0 + Au} \leq \frac{1}{u}.$$

The same formula holds on the conjugate ray. The logarithmic terms of the function $B(z)$ have $t_0 = \rho + x \geq 0$. Those of $H_\rho(z)$ have $t_0 = \rho - x \geq 0$. On the tail $u \geq 3$, beyond the last cell edge, the formula gives

$$\frac{\mathrm{d}}{\mathrm{d}u}\bigl(B(z_+(u)) - 3H_\rho(z_+(u))\bigr) \geq -\frac{3\rho}{u} \geq -\rho.$$

Take a support point $z_\pm(v)$ with $0 \leq v \leq u_{\max}$. Translate the origin along the ray. The same identity then gives

$$\biggl|\frac{\mathrm{d}}{\mathrm{d}u}\log|z_+(u) - z_\pm(v)|\biggr| \leq \frac{1}{u - u_{\max}} \qquad (u > u_{\max}).$$

The certificate measure is positive with total mass one, so the contribution of $-2U^\sigma$ is at least $-2/(u - u_{\max})$. Therefore, on either ray, the tail derivative is bounded by

$$\frac{4\lambda + \pi}{\sqrt{17}} - \rho - \frac{2}{u - u_{\max}} > 0.$$

The exact rational and directed interval checks use the tables in Appendices A and B. They give the downward-rounded energy values

(8.5)

	$q = 27$	$q = 28$
$A_+(\log q)$	≥ 4.9972574688	≥ 5.0224702721
$A_-(\log q)$	≥ 4.9720000000	≥ 4.9900000000

and the arithmetic costs (4.50). The costs use rational enclosures of the logarithms. Each margin below is energy minus arithmetic cost. The resulting strict margins are

(8.6)

	$q = 27$	$q = 28$
positive	0.0126192827393433594	0.0329429595649909584
negative	0.0126929857831087892	0.0275284462287496169.

For $q \geq 27$,

$$\frac{\mathrm{d}\tau_+}{\mathrm{d}\log q} = \frac{q}{4(q-1)} - \frac{1}{8} \leq \frac{7}{52} < \frac{27}{200}.$$

Together with (6.2), the positive margin increases with slope at least $73/200$. On the negative side,

$$\frac{\mathrm{d}\tau_-}{\mathrm{d}\log q} = \frac{225}{1024}\frac{q}{q+1} - \frac{1}{8} \leq \frac{97}{1024};$$

(7.2) therefore gives net slope at least $223/1024$. The four margins in (8.6) persist along the complete ranges $q \geq 27$.

9. Proof of Theorem 1

Fix $a \geq 27$ or $a \leq -27$ and write $\theta = \mathrm{Li}_3(1/a)$. Use the corresponding seed and choose p from (5.3). Take $m = 12p^k$ on the positive side or $m = 16p^k$ on the negative side. By Lemma 2.1, the determinant identity has the form required by Fact 1. Define

$$\alpha(a) = A_+(\log|a|), \quad \tau(a) = \tau_+(|a|) \quad (a \geq 27),$$

and

$$\alpha(a) = A_-(\log|a|), \quad \tau(a) = \tau_-(|a|) \quad (a \leq -27).$$

Combine the coefficient integrality (4.43), the tower congruence (5.9), the arithmetic limsup (4.49) and its branch rates (4.50). Add Propositions 6.1 and 7.1, through their limsup statements (6.1) and (7.1). Together they give

$$\limsup_{k\to\infty} m^{-2} \log|v^m D_m(a) G_{m,a}(u/v)| \leq -\alpha(a) + \tau(a) < 0$$

The endpoint bounds (8.5) and the slope estimates immediately following (8.6) give $\alpha(a) > \tau(a)$ throughout the two ranges. The analytic limsups above hold for every $m = 12Q$ and $m = 16Q$, respectively, hence also along $m = 12p^k$ and $m = 16p^k$. If $\theta = u/v$ in lowest terms, the integer $Z_m = v^m D_m(a) G_{m,a}(u/v)$ is nonzero by the tower congruence (5.9). It is integral by (4.43). The factor v^m is only $\exp(O(m))$. Hence a tail of these nonzero integers has absolute value strictly between zero and one. This is impossible, and the theorem follows.

Appendix A. Positive density data

The positive endpoint certificates use the integer densities below. Cell i is $[1/3 + i/128, 1/3 + (i+1)/128)$. Each numerator is divided by 2^{40}. The two columns belong to $q = 27$ and $q = 28$. Rows with $162 \leq i \leq 252$ are omitted; all their entries are zero.

Table A.1: Positive density numerators; $n_{i,27} = n_{i,28} = 0$ for $162 \leq i \leq 252$.

i	$n_{i,27}$	$n_{i,28}$	i	$n_{i,27}$	$n_{i,28}$
0	3 298 534 883 328	3 298 534 883 328	83	622 012 586 261	622 050 767 912
1	3 298 534 883 328	3 298 534 883 328	84	614 333 078 825	614 287 952 151
2	3 298 534 883 328	3 298 534 883 328	85	606 735 467 750	606 608 112 256
3	3 298 534 883 328	3 298 534 883 328	86	599 217 211 295	599 008 231 565
4	3 298 534 883 328	3 298 534 883 328	87	591 775 970 481	591 484 286 035
5	3 298 534 883 328	3 298 534 883 328	88	584 408 472 654	584 032 767 761
6	3 298 534 883 328	3 298 534 883 328	89	577 111 889 077	576 651 861 460
7	2 885 832 828 696	2 966 263 434 188	90	569 884 096 023	569 339 518 896
8	2 572 916 852 719	2 600 553 580 905	91	562 722 486 721	562 093 583 640
9	2 404 674 742 681	2 429 959 090 809	92	555 624 724 241	554 911 785 762
10	2 269 717 194 177	2 290 223 124 257	93	548 588 660 234	547 791 002 500
11	2 160 719 795 372	2 178 694 098 863	94	541 611 628 478	540 729 044 235
12	2 068 303 116 386	2 084 358 914 107	95	534 690 937 825	533 724 285 633
13	1 988 106 599 617	2 002 705 276 844	96	527 824 999 014	526 773 860 583
14	1 917 217 248 650	1 930 647 240 393	97	521 012 316 498	519 874 303 491
15	1 853 688 482 146	1 866 157 169 433	98	514 250 142 564	513 023 788 088
16	1 796 134 232 221	1 807 790 787 793	99	507 535 377 793	506 221 236 102

i	$n_{i,27}$	$n_{i,28}$	i	$n_{i,27}$	$n_{i,28}$
17	1 743 532 108 475	1 754 487 432 635	100	500 866 279 033	499 464 531 627
18	1 695 103 172 824	1 705 444 696 477	101	494 242 009 076	492 751 072 173
19	1 650 239 036 412	1 660 039 591 631	102	487 660 717 448	486 079 124 444
20	1 608 458 569 067	1 617 775 705 817	103	481 119 346 803	479 446 789 252
21	1 569 373 439 666	1 578 250 330 129	104	474 615 344 944	472 851 308 457
22	1 532 662 017 100	1 541 136 606 734	105	468 147 362 655	466 290 105 005
23	1 498 056 225 875	1 506 164 904 576	106	461 714 229 410	459 761 900 645
24	1 465 333 913 300	1 473 106 191 037	107	455 313 811 067	453 265 808 862
25	1 434 307 156 504	1 441 764 439 830	108	448 943 409 896	446 799 708 779
26	1 404 811 671 588	1 411 975 419 573	109	442 600 868 225	440 360 370 311
27	1 376 705 690 614	1 383 597 500 366	110	436 284 765 300	433 944 814 059
28	1 349 869 030 884	1 356 504 448 938	111	429 993 546 868	427 551 359 822
29	1 324 196 231 791	1 330 587 788 739	112	423 724 966 188	421 179 066 889
30	1 299 591 575 199	1 305 753 780 226	113	417 476 547 260	414 826 008 656
31	1 275 971 504 575	1 281 916 940 006	114	411 246 311 845	408 489 662 946
32	1 253 264 055 032	1 259 001 968 334	115	405 032 602 025	402 168 104 928
33	1 231 402 873 061	1 236 943 639 366	116	398 833 314 829	395 859 116 566
34	1 210 327 818 537	1 215 680 904 025	117	392 646 208 906	389 560 227 707
35	1 189 986 648 507	1 195 158 816 234	118	386 469 149 156	383 268 749 568
36	1 170 331 397 942	1 175 329 274 750	119	380 299 767 714	376 981 514 354
37	1 151 317 613 315	1 156 148 287 118	120	374 135 763 093	370 696 177 386
38	1 132 905 886 597	1 137 575 817 319	121	367 974 952 640	364 410 795 140
39	1 115 060 842 055	1 119 574 919 777	122	361 815 086 406	358 124 097 999
40	1 097 748 802 721	1 102 112 768 050	123	355 653 493 558	351 833 699 890
41	1 080 938 967 357	1 085 157 691 959	124	349 487 525 942	345 535 972 733
42	1 064 604 071 733	1 068 681 708 434	125	343 314 177 368	339 227 397 144
43	1 048 717 958 913	1 052 659 688 523	126	337 130 791 040	332 903 443 659
44	1 033 257 423 905	1 037 066 353 123	127	330 934 714 376	326 561 532 878
45	1 018 200 044 902	1 021 880 109 083	128	324 722 903 577	320 198 149 644
46	1 003 525 874 709	1 007 078 857 138	129	318 491 787 088	313 810 415 083
47	989 214 576 772	992 644 997 276	130	312 237 660 440	307 394 335 365
48	975 249 426 155	978 560 255 853	131	305 956 916 371	300 946 029 728
49	961 614 613 950	964 808 012 985	132	299 645 605 066	294 461 347 829
50	948 294 135 274	951 372 775 929	133	293 299 436 090	287 934 926 304
51	935 273 085 067	938 239 518 829	134	286 913 758 223	281 360 884 753
52	922 538 698 776	925 394 146 872	135	280 483 988 239	274 732 567 154
53	910 077 910 710	912 824 093 089	136	274 004 651 032	268 044 078 975
54	897 878 310 196	900 519 050 516	137	267 469 791 656	261 289 096 273
55	885 930 295 374	888 467 336 853	138	260 873 025 482	254 460 761 612
56	874 222 852 725	876 656 745 923	139	254 207 328 193	247 551 018 043
57	862 744 015 244	865 078 697 765	140	247 464 701 534	240 550 856 530
58	851 485 807 964	853 723 777 364	141	240 636 654 445	233 449 009 424
59	840 441 243 359	842 580 490 827	142	233 713 639 948	226 232 746 699
60	829 600 075 777	831 640 958 545	143	226 684 558 220	218 888 068 958
61	818 952 623 644	820 899 275 764	144	219 536 950 993	211 399 240 675
62	808 492 968 119	810 345 811 167	145	212 256 823 490	203 748 391 803
63	798 215 511 830	799 972 308 091	146	204 827 860 432	195 914 700 575
64	788 111 907 631	789 775 555 841	147	197 230 645 761	187 873 706 515
65	778 174 308 368	779 749 698 740	148	189 442 528 264	179 595 090 436

i	$n_{i,27}$	$n_{i,28}$	i	$n_{i,27}$	$n_{i,28}$
66	768 398 074 353	769 884 876 401	149	181 436 911 638	171 041 130 629
67	758 778 213 476	760 174 129 545	150	173 181 443 994	162 165 562 957
68	749 307 875 938	750 614 007 649	151	164 635 705 157	152 909 243 355
69	739 980 958 044	741 200 670 769	152	155 749 296 158	143 194 690 447
70	730 792 985 727	731 927 022 986	153	146 458 187 298	132 918 953 924
71	721 739 398 760	722 787 060 370	154	136 677 474 076	121 939 413 447
72	712 815 494 419	713 776 274 750	155	126 290 632 814	110 040 955 919
73	704 016 555 770	704 891 684 344	156	115 133 084 799	96 902 239 046
74	695 337 727 252	696 128 944 904	157	102 944 846 637	81 791 089 438
75	686 774 873 697	687 482 377 327	158	89 369 632 159	64 286 873 668
76	678 324 907 137	678 946 725 136	159	73 241 560 357	34 652 340 814
77	669 983 602 840	670 519 599 327	160	55 433 749 801	0
78	661 746 757 665	662 198 323 890	161	11 861 762 313	0
79	653 610 372 381	653 980 228 320	253	10	6
80	645 572 367 083	645 860 038 638	254	1	1
81	637 629 226 940	637 834 078 443	255	1	0
82	629 777 087 228	629 898 557 487			

Each column has $0 \leq n_{i,q} \leq 3 \cdot 2^{40}$ and $\sum_i n_{i,q} = 128 \cdot 2^{40}$. The entries are exact integer numerators.

Appendix B. Negative rational density data

The negative endpoint certificates use the two rational densities below. The parameter u is arclength on the two branches

$$z_+(u) = \frac{5}{16} + \frac{4+i}{\sqrt{17}}u, \qquad z_-(u) = \frac{5}{16} + \frac{4-i}{\sqrt{17}}u, \qquad 0 \leq u \leq 3.$$

For the cross-ray primitives put $A = (4+i)/\sqrt{17}$ and $\beta = -\overline{A} = (-4+i)/\sqrt{17}$. Then $Au + \beta v = z_+(u) - z_-(v)$. The second branch has direction $\overline{A}$. The cell edges are $u_j = 3(j/256)^3$. Rows with $j \geq 172$ are omitted; all their entries are zero. The table entries are exact integers.

Table B.1: Negative density counts, denominators $10^{12} - 6$ and $10^{12} - 8$; $c_{j,27} = c_{j,28} = 0$ for $j \geq 172$.

j	$c_{j,27}$	$c_{j,28}$	j	$c_{j,27}$	$c_{j,28}$
0	140 540 525	142 217 998	86	7 271 281 123	7 303 764 615
1	261 451 321	264 261 111	87	7 344 376 065	7 376 969 350
2	363 701 611	367 659 490	88	7 416 401 802	7 449 053 094
3	453 705 003	458 485 706	89	7 487 335 045	7 519 989 402
4	536 473 939	542 273 883	90	7 557 144 778	7 589 772 163
5	614 669 706	621 361 877	91	7 625 796 730	7 658 396 089
6	689 893 588	697 238 114	92	7 693 233 651	7 725 843 275
7	763 040 641	770 967 704	93	7 759 430 842	7 792 070 509
8	834 822 469	843 419 489	94	7 824 335 414	7 857 015 320
9	905 684 947	914 712 009	95	7 887 930 583	7 920 629 973
10	976 011 761	985 674 694	96	7 950 207 662	7 982 901 909
11	1 045 990 258	1 056 335 435	97	8 011 161 166	8 043 837 986

j	$c_{j,27}$	$c_{j,28}$	j	$c_{j,27}$	$c_{j,28}$
12	1 115 838 641	1 126 731 236	98	8 070 799 147	8 103 435 277
13	1 185 744 485	1 197 173 719	99	8 129 080 451	8 161 674 235
14	1 255 906 188	1 267 848 817	100	8 185 957 309	8 218 517 762
15	1 326 378 309	1 338 816 011	101	8 241 427 785	8 273 908 536
16	1 397 237 182	1 410 206 063	102	8 295 441 930	8 327 782 491
17	1 468 565 937	1 482 065 631	103	8 348 017 106	8 380 128 377
18	1 540 470 058	1 554 419 086	104	8 399 056 270	8 431 008 634
19	1 612 985 496	1 627 345 506	105	8 448 533 982	8 480 439 151
20	1 686 109 451	1 700 953 931	106	8 496 439 067	8 528 307 528
21	1 759 886 518	1 775 205 628	107	8 542 791 674	8 574 486 950
22	1 834 332 590	1 850 071 630	108	8 587 618 477	8 618 992 632
23	1 909 514 382	1 925 633 468	109	8 630 829 261	8 661 927 493
24	1 985 414 813	2 001 951 334	110	8 672 314 550	8 703 291 099
25	2 062 008 834	2 078 986 141	111	8 712 110 309	8 742 946 397
26	2 139 339 129	2 156 725 739	112	8 750 176 909	8 780 824 766
27	2 217 420 126	2 235 211 994	113	8 786 530 820	8 816 929 623
28	2 296 229 927	2 314 466 234	114	8 821 148 381	8 851 240 883
29	2 375 762 101	2 394 446 069	115	8 853 965 252	8 883 765 766
30	2 455 985 113	2 475 117 286	116	8 884 986 588	8 914 494 208
31	2 536 916 350	2 556 447 255	117	8 914 160 226	8 943 270 999
32	2 618 559 206	2 638 410 477	118	8 941 455 668	8 969 957 130
33	2 700 907 329	2 721 033 529	119	8 966 786 746	8 994 664 508
34	2 783 913 083	2 804 365 072	120	8 990 150 975	9 017 619 226
35	2 867 560 367	2 888 390 055	121	9 011 513 468	9 038 810 379
36	2 951 843 004	2 973 037 533	122	9 030 797 940	9 057 928 634
37	3 036 727 781	3 058 251 294	123	9 047 959 528	9 074 649 583
38	3 122 175 449	3 144 038 906	124	9 062 954 115	9 088 896 328
39	3 208 172 733	3 230 395 933	125	9 075 777 030	9 100 848 019
40	3 294 699 990	3 317 285 043	126	9 086 389 380	9 110 727 562
41	3 381 718 339	3 404 657 728	127	9 094 662 617	9 118 537 364
42	3 469 217 890	3 492 480 725	128	9 100 537 599	9 123 994 734
43	3 557 179 576	3 580 744 413	129	9 103 976 314	9 126 756 271
44	3 645 583 651	3 669 436 031	130	9 104 931 232	9 126 727 580
45	3 734 354 505	3 758 526 323	131	9 103 261 572	9 124 048 078
46	3 823 492 984	3 847 968 982	132	9 098 842 747	9 118 747 158
47	3 912 928 913	3 937 712 428	133	9 091 576 840	9 110 592 597
48	4 002 643 250	4 027 722 788	134	9 081 425 676	9 099 364 692
49	4 092 610 641	4 117 991 994	135	9 068 283 243	9 085 061 206
50	4 182 796 133	4 208 498 764	136	9 051 951 789	9 067 679 190
51	4 273 196 226	4 299 187 192	137	9 032 309 625	9 046 918 113
52	4 363 741 968	4 389 995 946	138	9 009 221 788	9 022 472 399
53	4 454 405 074	4 480 895 428	139	8 982 617 488	8 994 300 776
54	4 545 159 794	4 571 879 737	140	8 952 238 022	8 962 324 092
55	4 635 953 258	4 662 929 300	141	8 917 780 992	8 926 287 605
56	4 726 742 697	4 753 994 578	142	8 879 031 507	8 885 863 474
57	4 817 490 148	4 845 015 105	143	8 835 875 523	8 840 781 200
58	4 908 147 661	4 935 943 324	144	8 788 123 468	8 790 842 965
59	4 998 696 089	5 026 751 567	145	8 735 314 200	8 735 726 253
60	5 089 119 045	5 117 420 789	146	8 677 085 910	8 675 063 681

j	$c_{j,27}$	$c_{j,28}$	j	$c_{j,27}$	$c_{j,28}$
61	5 179 384 107	5 207 921 943	147	8 613 120 225	8 608 515 497
62	5 269 426 388	5 298 205 557	148	8 543 105 602	8 535 747 731
63	5 359 217 132	5 388 213 135	149	8 466 593 818	8 456 030 313
64	5 448 713 176	5 477 900 666	150	8 383 030 498	8 368 928 198
65	5 537 874 776	5 567 249 859	151	8 291 808 926	8 274 129 435
66	5 626 668 008	5 656 243 489	152	8 192 433 115	8 170 940 771
67	5 715 065 634	5 744 845 063	153	8 084 283 653	8 058 379 664
68	5 803 012 246	5 833 004 641	154	7 966 570 714	7 935 553 073
69	5 890 480 389	5 920 678 097	155	7 838 343 944	7 801 750 381
70	5 977 447 442	6 007 829 650	156	7 698 603 425	7 656 125 755
71	6 063 860 154	6 094 430 692	157	7 546 277 197	7 497 428 193
72	6 149 679 596	6 180 450 129	158	7 380 133 560	7 324 112 186
73	6 234 900 463	6 265 837 571	159	7 198 670 206	7 134 475 246
74	6 319 460 956	6 350 549 141	160	7 000 072 394	6 926 610 257
75	6 403 343 164	6 434 559 295	161	6 782 275 782	6 698 335 400
76	6 486 508 050	6 517 857 236	162	6 542 881 755	6 446 994 517
77	6 568 907 903	6 600 429 001	163	6 278 760 059	6 169 101 464
78	6 650 533 948	6 682 226 511	164	5 985 945 350	5 860 117 045
79	6 731 345 218	6 763 174 621	165	5 659 351 952	5 514 071 290
80	6 811 305 178	6 843 221 978	166	5 292 033 560	5 122 756 282
81	6 890 362 781	6 922 363 261	167	4 874 213 991	4 673 750 846
82	6 968 507 599	7 000 597 226	168	4 390 896 209	4 149 894 406
83	7 045 692 810	7 077 888 090	169	3 817 831 507	3 504 377 317
84	7 121 915 572	7 154 192 159	170	3 100 186 827	2 746 774 945
85	7 197 117 216	7 229 490 450	171	2 151 224 279	1 052 541 758

The interval boxes for the directed bounds are in the ancillary data file `certificate_data.json`. The script `check_energy.py` recomputes the masses, caps, pair kernels, boxes, and tails.

Appendix C. Seed data

For the positive seed, H_{211} has degree 211 and its values modulo 29 for residues $0, \ldots, 28$ are

$$(11, 15, 1, 13, 13, 23, 17, 9, 20, 10, 27, 19, 19, 12, 2,$$
$$5, 9, 21, 16, 7, 16, 8, 2, 8, 19, 9, 17, 26, 6).$$

For the negative seed, H_{378} has degree 378 and its values modulo 37 for residues $0, \ldots, 36$ are

$$(28, 3, 19, 24, 31, 21, 36, 26, 27, 1, 30, 22, 21, 18, 11, 25, 31, 26, 36,$$
$$11, 34, 19, 36, 4, 36, 24, 9, 8, 4, 33, 11, 30, 11, 34, 12, 24, 33).$$

The complete ascending coefficient arrays are ancillary data. They are in the files `H211_exact_seed.json` and `H378_exact_seed.json`. Section 5 displays the identities and the denominator factorizations.

Appendix D. Computer checks

Table D.1 lists the checks. Each script reads data printed in this paper or in the ancillary tables. The scripts use the Python standard library except where

stated. The seed check `check_seed.py` requires `python-flint`; the energy and contour interval checks require `mpmath`. The script `check_tau.py` does not certify every printed digit independently. The finite check `check_contour.py` diagnoses the residue mechanism for an undifferentiated F. It also performs a separate triple-zero check. It is not an F'' certificate and does not replace the uniform contour proof.

TABLE D.1. Exact and directed-interval computations.

Script	Statement checked; result
`check_energy.py`	**Directed interval replay:** recomputes the positive and negative certificate bounds from the printed density tables and interval data, including masses, caps, pair kernels, boxes, and tails. PASS.
`check_seed.py`	**Exact algebra:** rebuilds both seed determinants, verifies the degree-bounded identities at 307 and 553 integer points, then checks the JSON denominator factorizations and displayed residue vectors. PASS.
`check_profile.py`	**Exact algebra:** recomputes the two conservative baseline upper bounds, all ten low-layer increments for each ratio, and the resulting profile inequalities. PASS.
`check_tau.py`	**Exact rational arithmetic:** checks the four endpoint margins, the two stated lower margin thresholds at $q = 27$, and the identities $1/2 - 27/200 = 73/200$ and $5/16 - 97/1024 = 223/1024$ using rational logarithm enclosures; it does not certify the displayed energy decimals. PASS.
`check_contour.py`	**Finite diagnostic:** residue-mechanism checks for an undifferentiated F, a separate triple-zero check, sine-bound and signed-contour consistency checks; it is not an F'' certificate and does not replace the uniform contour proof. PASS.

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Data availability. The scripts of Appendix D, the complete seed arrays, the interval boxes, and their compact outputs accompany the paper as ancillary files. These files are part of the proof record.

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Independent Researcher, Columbia, Maryland, USA
Email address: `ryzfsu@gmail.com`