IRRATIONALITY OF THE TRILOGARITHM AT RECIPROCALS OF LARGE INTEGERS

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ABSTRACT. We prove that $\mathrm{Li}_3(1/a)$ is irrational for every integer $a \geq 67$ and every integer $a \leq -70$. Earlier ranges are $|a| \geq 1038$ (Hata), $a \geq 1173$ (Panzone), and $a \geq 486$, $a \leq -471$ (Marcovecchio). For $4 \mid m$ we form an $m \times m$ determinant. Its entries are linear forms in 1 and $\mathrm{Li}_3(1/a)$. The kernel has only triple poles, so no other polylogarithm enters. We bound a common denominator of the coefficients. The determinant does not vanish along a prime-power subsequence that depends on a. A logarithmic energy bounds its size. Exact density tables with directed rounding give the energy bounds at $a = 67$ and $a = -70$. A slope comparison extends them to all $a \geq 67$ and all $a \leq -70$.

1. INTRODUCTION

For $|z| < 1$, define

$$\mathrm{Li}_3(z) = \sum_{n \geq 1} \frac{z^n}{n^3}.$$

Hata [3] constructed explicit rational approximations to polylogarithms at $1/q$. By Marcovecchio [5, p. 229], Hata's method [3, Corollary 2.2] gives the linear independence over $\mathbb{Q}$ of

$$1,\ \mathrm{Li}_1(1/a),\ \mathrm{Li}_2(1/a),\ \mathrm{Li}_3(1/a)$$

for every integer a with $|a| \geq 1038$. Panzone [6, Theorem 2, p. 108] gave a short proof that $\mathrm{Li}_3(1/d)$ is irrational for every integer $d \geq 1173$. He attributes this result to Hata. Marcovecchio [5, Theorem 7.1, p. 228; p. 229] extended the linear independence to $a \geq 486$ and $a \leq -471$. In particular $\mathrm{Li}_3(1/a)$ is irrational in these ranges.

Theorem 1. *For every integer $a \geq 67$ and every integer $a \leq -70$, $\mathrm{Li}_3(1/a)$ is irrational.*

Theorem 1 concerns $\mathrm{Li}_3(1/a)$ alone. In their ranges, the results of Hata and Marcovecchio give the stronger linear independence of four numbers.

The method is the determinant method of [8] for the dilogarithm, carried to weight three. We use only the determinant criterion of [8, Section 3]; Fact 1 restates it with proof. The analytic estimate uses Andreief's identity [1], Cauchy's double alternant [4], logarithmic energy [7], and the prime number theorem [2].

Let m be a positive multiple of 4. Let $G_{m,a}(T)$ be the determinant polynomial of Section 2. Section 4 gives a common denominator with limiting logarithmic rate

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$\tau(a)$, where

$$S = \frac{643}{96} + 4\log 2 - \frac{25}{8}\log 5, \qquad \tau(a) = S + \frac{9}{16}\log|a-1| - \frac{1}{8}\log|a|.$$

Section 5 proves nonvanishing along $m = 4p^k$, with one prime p for each a. Section 6 bounds the determinant by the energy function A. Section 7 proves

$$A(\log 67) \geq 6.2732861583, \qquad \tau(67) < 6.272107,$$
$$A(\log 70) \geq 6.3100053850, \qquad \tau(-70) < 6.307708,$$
$$\max\{\tau(67) - A(\log 67), \tau(-70) - A(\log 70)\} < 0.$$

These bounds give Theorem 1 for $a = 67$ and $a = -70$. Section 8 compares slopes in $\log|a|$ and extends the theorem to the full ranges.

The proof uses a computer at three points. The energy bounds of Section 7 use exact rational and directed interval arithmetic. They act on the density tables printed in Appendix A. The bounds for $\tau(67)$ and $\tau(-70)$ use exact rational series. The seed identity of Section 5 is an exact polynomial computation. Appendix B prints its coefficients. Appendix C lists these and the other exact checks. The use of AI tools is described at the end of the paper.

2. Determinant construction

Fix an integer a with $|a| \geq 2$. Let m be a positive multiple of 4. Put $g = m/2$, $I = \{0, \dots, m-1\}$, and $J = \{m, \dots, 2m-1\}$. Define

$$w(r) = \binom{r-1}{g}^3, \qquad F_{ij}(r) = \frac{w(r)}{(r+i)(r+j)}, \qquad K_{ij}(r) = F_{ij}''(r),$$

and

$$M_{ij}(a) = \sum_{r \geq 1} a^{-r} K_{ij}(r), \qquad i \in I, j \in J.$$

Polynomial division at the two poles gives a polynomial P_{ij} with

$$F_{ij} = P_{ij} + \frac{c_i}{r+i} + \frac{c_j}{r+j}, \quad c_i = \frac{w(-i)}{j-i}, \quad c_j = -\frac{w(-j)}{j-i}.$$

The residues c_i and c_j depend on the pair (i, j); we suppress the second index. Put $b = a - 1$. The letter b has this meaning throughout. For every integer a with $|a| \geq 2$ and every m with $4 \mid m$, define

$$E_{ij}(a) = \sum_{r \geq 1} a^{-r} P_{ij}''(r) - 2c_i a^i \sum_{n=1}^{i} \frac{a^{-n}}{n^3} - 2c_j a^j \sum_{n=1}^{j} \frac{a^{-n}}{n^3},$$
$$C_{ij}(a) = 2c_i a^i + 2c_j a^j.$$

Empty finite sums are zero. For $n \geq 0$ put $\mathsf{h}_3(n) = \sum_{s=1}^{n} a^{-s} s^{-3}$. The finite sums in E_{ij} are $\mathsf{h}_3(i)$ and $\mathsf{h}_3(j)$.

Lemma 2.1. *For every integer a with $|a| \geq 2$ and every m with $4 \mid m$, the matrices $E(a)$ and $C(a)$ satisfy*

$$M(a) = E(a) + \mathrm{Li}_3(1/a)C(a), \qquad G_{m,a}(T) = \det(E(a) + TC(a)).$$

Here the second formula defines $G_{m,a}$. Its degree is at most m.

Proof. Differentiating the partial fractions gives

$$F''_{ij} = P''_{ij} + 2c_i(r+i)^{-3} + 2c_j(r+j)^{-3}.$$

Also $\sum_{r\geq 1} a^{-r}(r+i)^{-3} = a^i(\mathrm{Li}_3(1/a) - \sum_{n=1}^{i} a^{-n}n^{-3})$. Since $|a| > 1$, the geometric series and all differentiated polynomial-weighted series converge absolutely. So the decomposition holds for positive and negative integers a in the stated domain. The entries of $E(a) + TC(a)$ are linear in T. Hence $G_{m,a}$ has degree at most m. This proves Lemma 2.1. □

3. Determinant criterion

Fact 1 is the determinant criterion of [8, Section 3]. Sections 4–8 check its Li_3-specific hypotheses.

Fact 1 (Determinant criterion)**.** *Let θ be a fixed real number. For a fixed base a, let $G_{m,a}(T)$ have degree at most m. Let m_k be a strictly increasing sequence. Suppose there are positive rationals $\Delta_m(a)$ with*

$$\Delta_m(a)G_{m,a}(T) \in \mathbb{Z}[T], \qquad \limsup_{k\to\infty} m_k^{-2} \log \Delta_{m_k}(a) \leq \tau(a),$$

$$\log|G_{m_k,a}(\theta)| \leq -\alpha(a)m_k^2 + O(m_k \log m_k), \qquad \alpha(a) > \tau(a),$$

Suppose also that each fixed rational ξ has a $k_0(a,\xi)$ with $G_{m_k,a}(\xi) \neq 0$ for $k \geq k_0(a,\xi)$. Then θ is irrational.

Proof. Suppose $\theta = u/v$ in lowest terms. For $k \geq k_0(a, u/v)$ and $m = m_k$, put

$$Z_m = v^m \Delta_m(a) G_{m,a}(u/v).$$

Since the degree is at most m, the factor v^m clears the denominator. Hence Z_m is a nonzero integer. The factor $|v|^{m_k} = e^{O(m_k)}$ is absorbed in the error term. By the two displayed bounds, $|Z_m| \to 0$. This is a contradiction. □

4. Coefficient integerization

Let $U = 2m-1$, $L = \mathrm{lcm}(1,\ldots,U)$, $N = m^2/8 - m/2$, $H_m = (3m/4-1)^2$, and $H = 3g-3$. For each prime p, let $d_p = v_p(L)$. For an integer $n \geq 0$, put $\kappa_n = 1$ if $n \bmod p \geq p - (g \bmod p)$, and $\kappa_n = 0$ otherwise. The flag depends on p; we suppress this. For $p > g$ the condition reads $n \bmod p \geq p - g$. Define the prime weights $W_m(p)$ by the first case that applies:

(1) $W_m(p) = 0$ for $p > U$;
(2) $W_m(p) = 4md_p$ for $p < 5$ or $p^2 \leq U + g$;
(3) for $p \geq 5$, $p^2 > U+g$, $p \leq g$, use $W_m(p) = 4m - \sum_{n=0}^{U} \kappa_n$;
(4) for $g < p < m$, put $u = p - g$. An index is active if it is at least p and has $\kappa = 0$. It is killed if $\kappa = 1$. A row below p with $\kappa = 0$ is a low row. Every column with $\kappa = 0$ is active. Let N_{act} be the total number of active rows and active columns, and set

$$c_I(z) = \#\{i \in I : i \equiv z \pmod p\}, \quad c_J(z) = \#\{j \in J : j \equiv z \pmod p\},$$

$$\Sigma_K = \sum_{z=p-g}^{p-1} \min(c_I(z), c_J(z)).$$

Use

$$W_m(p) = \min\{3N_{\mathrm{act}} + u + \Sigma_K,\ m + 2N_{\mathrm{act}} + u\}.$$

(5) for $m \le p \le U$, use $W_m(p) = 4(2m-p) - 3\max(0, 2m+g-2p)$.

Cases (3)–(5) are the moving-prime branches. They treat the primes $p \le U$ with $p \ge 5$ and $p^2 > U+g$.

4.1. **Entrywise bounds.**

Lemma 4.1. *For $i \in I$ and $j \in J$, the complete Newton expansion of P''_{ij} is*

$$P''_{ij}(r) = \sum_{\iota=0}^{3g-4} \gamma_{ij,\iota} \binom{r-1}{\iota}, \qquad L^4\gamma_{ij,\iota} \in \mathbb{Z}.$$

Proof. For an integer-valued polynomial f of degree at most d, the Newton expansion at an integer z is

$$f(z+y) = \sum_{\iota=0}^{d} \Delta^\iota f(z) \binom{y}{\iota}.$$

By the identity $\binom{y}{\iota}/y = \binom{y-1}{\iota-1}/\iota$, the polynomial $\operatorname{lcm}(1,\dots,d)(f(x)-f(z))/(x-z)$ is integer-valued, including its removable value. Since Δ is nilpotent on polynomials of degree at most d, $\partial = \log(1+\Delta)$ is a finite sum. Hence $\operatorname{lcm}(1,\dots,d)f'$ is integer-valued. The polynomial quotient P is exactly two divided differences of w. Apply each of these two bounds twice. The two divided-difference steps and the two derivative steps each cost one factor $\operatorname{lcm}(1,\dots,d)$, with $d \le \deg w = 3g$. So L^4P'' is integer-valued. Since $3g \le U$, each such factor divides L. This gives the polynomial part at every prime power. This proves Lemma 4.1. □

Lemma 4.2. *Let $p \ge 5$ be a prime with $p^2 > U+g$ and $p \le g$, and let κ_n be the carry flag defined above. Let $i \in I$, $j \in J$, and $\delta = j - i$. For every integer n with $0 \le n \le 3g-4$, including the removable poles $n = i$ and, when $j \le 3g-4$, $n = j$,*

$$v_p(P''_{ij}(-n)) \ge -B_{ij}, \qquad B_{ij} = 3 - \kappa_i\kappa_j + (1-\kappa_i)1_{i\equiv j\,(p)}.$$

Here and below, the subscript (p) means modulo p. The potentials $\phi_n = 2 - \kappa_n$, $n \in I \cup J$, satisfy $\phi_i + \phi_j \ge B_{ij}$, including the equal-residue case.

Proof. Since $p^2 > U+g$, the factorial valuation has at most one carry. Legendre's formula gives $v_p\binom{n+g}{g} = \kappa_n$. Deleting at most ℓ factors from the triple product gives

$$v_p(w^{(\ell)}(-n)) \ge 3\kappa_n - \ell, \qquad 0 \le \ell \le 3.$$

Let $n \notin \{i,j\}$, and put $d_i = i-n$, $d_j = j-n$. Since $0 < |d_i|, |d_j| < 4g < p^2$, we have $v_p(d_i) = \varepsilon_i$ and $v_p(d_j) = \varepsilon_j$, where $\varepsilon_i = 1_{n\equiv i\,(p)}$ and $\varepsilon_j = 1_{n\equiv j\,(p)}$. At $r = -n$, direct differentiation gives

$$F''_{ij} = \frac{w''}{d_id_j} - 2w'\left(\frac{1}{d_i^2d_j} + \frac{1}{d_id_j^2}\right) + 2w\left(\frac{1}{d_i^3d_j} + \frac{1}{d_i^2d_j^2} + \frac{1}{d_id_j^3}\right),$$

and $P''_{ij}(-n) = F''_{ij}(-n) - 2c_i/d_i^3 - 2c_j/d_j^3$. Each of the six terms of F''_{ij} is $w^{(\ell)}(-n)/(d_i^{e_1}d_j^{e_2})$ with $\ell + e_1 + e_2 = 4$ and $e_1, e_2 \ge 1$. Its valuation is at least $3\kappa_n - \ell - e_1\varepsilon_i - e_2\varepsilon_j$. Since $v_p(c_i) \ge 3\kappa_i - v_p(\delta)$ and $v_p(c_j) \ge 3\kappa_j - v_p(\delta)$, the two pole terms have valuations at least $3\kappa_i - v_p(\delta) - 3\varepsilon_i$ and $3\kappa_j - v_p(\delta) - 3\varepsilon_j$. There are four cases.

(1) If $\varepsilon_i = \varepsilon_j = 0$, every term is at least $-2 \ge -B_{ij}$.

(2) If $\varepsilon_i = 1$ and $\varepsilon_j = 0$, then $i \not\equiv j$, $v_p(\delta) = 0$ and $\kappa_n = \kappa_i$. Since $e_2 \geq 1$, every term is at least $3\kappa_i - 3 \geq -(3 - \kappa_i\kappa_j) = -B_{ij}$.
(3) The case $\varepsilon_i = 0$, $\varepsilon_j = 1$ is symmetric, with bound $3\kappa_j - 3 \geq -B_{ij}$.
(4) If $\varepsilon_i = \varepsilon_j = 1$, then $n \equiv i \equiv j$, $v_p(\delta) = 1$, and the three carry flags equal a common κ. Every term is at least $3\kappa - 4$. This is $-4 = -B_{ij}$ if $\kappa = 0$, and $-1 \geq -2 = -B_{ij}$ if $\kappa = 1$.

So $v_p(P''_{ij}(-n)) \geq -B_{ij}$ at every $n \notin \{i, j\}$. At $n = i$, Taylor expansion at $r = -i$ gives the corrected finite part

$$P''_{ij}(-i) = \frac{w'''(-i)}{3\delta} - \frac{w''(-i)}{\delta^2} + \frac{2w'(-i)}{\delta^3} + \frac{2(w(-j) - w(-i))}{\delta^4}.$$

At $n = j$ the symmetric formula is

$$P''_{ij}(-j) = -\frac{w'''(-j)}{3\delta} - \frac{w''(-j)}{\delta^2} - \frac{2w'(-j)}{\delta^3} + \frac{2(w(-i) - w(-j))}{\delta^4}.$$

Put $\varepsilon = v_p(\delta)$. In this branch $\varepsilon \leq 1$. The four terms of the formula at $n = i$ have lower bounds $3\kappa_i - 3 - \varepsilon$, $3\kappa_i - 2 - 2\varepsilon$, $3\kappa_i - 1 - 3\varepsilon$, $3\min(\kappa_i, \kappa_j) - 4\varepsilon$. If $\varepsilon = 0$, the last term is integral and the first three give the claimed bound. If $\varepsilon = 1$, then $i \equiv j \pmod p$, so the two carry flags agree. If their common value is 0, every term is at least -4. If it is 1, every term is at least $-1 \geq -B_{ij}$. Thus $v_p(P''_{ij}(-n)) \geq -B_{ij}$, including the equal-residue case. The same calculation with δ replaced by $-\delta$ applies at $n = j$. Finally $\phi_i + \phi_j - B_{ij} = (1 - \kappa_i)(1 - \kappa_j) \geq 0$ if $i \not\equiv j$, and $\phi_i + \phi_j - B_{ij} = \kappa_i^2 - \kappa_i = 0$ if $i \equiv j$. This proves Lemma 4.2. □

Lemma 4.3. *Let p be a prime and L' an integer. Put $d = 3g - 4$, and suppose a degree-at-most-d polynomial f satisfies $v_p(f(-n)) \geq L'$ for $0 \leq n \leq d$. Then every coefficient γ_ι of f in the basis $\binom{x-1}{\iota}$ has valuation at least L'.*

Proof. The Newton coefficients of f at $-d$ are

$$\alpha_s = \sum_{e=0}^{s} (-1)^{s-e} \binom{s}{e} f(-d + e), \qquad f(x) = \sum_{s=0}^{d} \alpha_s \binom{x + d}{s}.$$

They all have valuation at least L'. Changing to the basis at 1 gives

$$f(x) = \sum_{\iota=0}^{d} \gamma_\iota \binom{x - 1}{\iota}, \qquad \gamma_\iota = \sum_{s=\iota}^{d} \alpha_s \binom{d + 1}{s - \iota}.$$

These binomial coefficients are integers. So every γ_ι also has valuation at least L'. This proves Lemma 4.3. □

Lemma 4.4. *Let p be a prime, $i \in I$ and $j \in J$, and let $\gamma_{ij,\iota}$ be the Newton coefficients of Lemma 4.1. If $p \geq 5$, $p^2 > U + g$ and $p \leq g$, then $v_p(\gamma_{ij,\iota}) \geq -\phi_i - \phi_j$ for every ι. If $p > g$, every $\gamma_{ij,\iota}$ is p-integral.*

Proof. For $p \leq g$, apply Lemma 4.3 to $f = P''_{ij}$ with $L' = -\phi_i - \phi_j$. Its samples are the points $-n$, $0 \leq n \leq d$, of Lemma 4.2, including the removable poles among them. There $v_p \geq -B_{ij} \geq -\phi_i - \phi_j$. For $p > g$, the coefficients of $w(r) = \binom{r-1}{g}^3$ have denominators dividing $(g!)^3$, so they are p-integral. Division by the monic factor $(r + i)(r + j)$ keeps the quotient P_{ij} p-integral. So P''_{ij} is p-integral at every integer, and so is $\gamma_{ij,\iota} = (\Delta^\iota P''_{ij})(1)$. This proves Lemma 4.4. □

A potential is a function $\pi \geq 0$ on $I \cup J$. It covers a demand D if $\pi_i + \pi_j \geq D_{ij}$ for all $i \in I$, $j \in J$. Its total is $\sum_{n \in I \cup J} \pi_n$. For $p^2 > U + g$, $i \in I$ and $j \in J$, put $\varepsilon = 1_{i \equiv j\,(p)}$ and

$$D_{ij} = \max\{0,\ 3 \cdot 1_{i \geq p} - 3\kappa_i + \varepsilon,\ 3 \cdot 1_{j \geq p} - 3\kappa_j + \varepsilon\}. \tag{4.1}$$

Lemma 4.6 shows that D_{ij} bounds the p-adic denominators of the harmonic corrections and of C.

Lemma 4.5. *Let $g < p \leq U$ with $p^2 > U + g$. Use N_{act}, u and the active, low and killed indices of the definition of $W_m(p)$. Let D_{ij} be the demand* (4.1)*. For $g < p < m$, exact interval counting gives*

$$N_{\mathrm{act}} = \sum_{t=1}^{3} \max(0, \min(u, 2m - tp)), \qquad \Sigma_K = g + (5m/2 - 4p)_+.$$

Potential (I) assigns 3 to active rows and columns, 1 to each low row, and 0 to killed indices. It then assigns one additional unit to every copy on the smaller side of each killed residue; rows are chosen on a tie. Potential (II) assigns row weights 2, 1, 0 on active, low, killed rows, respectively, and column weights 3, 1 on active and killed columns. Both potentials cover D, and their totals are the two quantities in the definition of $W_m(p)$.

For $m \leq p \leq U$, the row weight $1_{i<2m-p}$ and the column weight $3 \cdot 1_{j \geq p, \kappa_j = 0}$ cover D, with total $W_m(p)$.

Proof. Potential (I) has costs 3 on active rows and columns, 1 on low rows, and one extra cost on the smaller side of each killed residue. Hence $W_{\mathrm{I}} = 3N_{\mathrm{act}} + u + \Sigma_K$. Potential (II) has row costs 2,1,0 on active, low, killed rows and column costs 3,1 on active, killed columns. Hence $W_{\mathrm{II}} = m + 2N_{\mathrm{act}} + u$.

Active-active edges demand at most 4 and receive at least 5. Active-killed edges demand 3, because equal residues cannot have different carry flags. Low-active edges demand at most 4. Low-killed edges demand 0. Killed-killed edges demand ε. They are covered by the selected smaller side in (I) and by the killed-column unit in (II). Hence both totals are feasible for every residue configuration, including endpoints. So $\min(W_{\mathrm{I}}, W_{\mathrm{II}})$ is valid coefficientwise.

Now let $m \leq p \leq U$. An active column demands $3 + \varepsilon$. If $\varepsilon = 1$, then $j = i + p$ and the row weight supplies the fourth unit. Every other edge demands at most ε. The row weight covers it when $\varepsilon = 1$. Thus row cost $1_{i<2m-p}$ and column cost $3 \cdot 1_{j \geq p, \kappa_j = 0}$ cover every edge. The column count is $2m - p - \max(0, 2m + g - 2p)$, giving the stated total. This proves Lemma 4.5. □

Lemma 4.6. *Let p be a prime, $i \in I$ and $j \in J$. Consider the finite harmonic corrections in E_{ij} and the entry C_{ij}. In case (2) of the definition of $W_m(p)$, each of their coefficients has p-adic valuation at least $-4d_p$. In case (3) it is at least $-B_{ij}$, with B_{ij} as in Lemma 4.2. In cases (4) and (5) it is at least $-D_{ij}$, with D_{ij} as in* (4.1)*, and Lemma 4.5 applies to this demand. For $p > U$ these coefficients are p-integral. In each case the weight $W_m(p)$ covers the demand.*

Proof. With h_3 as in Section 2,

$$a^n \mathsf{h}_3(n) = \sum_{s=1}^{n} \frac{a^{n-s}}{s^3}, \qquad n \in \{i, j\}.$$

The integer $w(-n) = (-1)^{3g}\binom{n+g}{g}^3$ has valuation $3\kappa_n$ in the one-carry branches. Also $v_p(j-i) = \varepsilon$ is at most one. Thus $v_p(c_n) \geq 3\kappa_n - \varepsilon$. Since $n \leq U < p^2$, each coefficient of $a^n \mathsf{h}_3(n)$ has valuation at least $-3 \cdot 1_{n \geq p}$. So the entry demand is bounded by D_{ij} of (4.1). The coefficients of C have no harmonic denominator and satisfy this bound as well.

Let $p \leq g$. Then $D_{ij} \leq B_{ij}$. When $\varepsilon = 1$, the residues and carry flags agree. The worst case $\kappa_i = \kappa_j = 0$ is bounded by the extra equal-residue unit in B_{ij}. When $\varepsilon = 0$, $D_{ij} \leq 3$. If both carry flags are one, then $D_{ij} = 0 \leq B_{ij}$.

For $g < p < m$ and $m \leq p \leq U$, this is the edge demand covered in Lemma 4.5. If $p^2 \leq U + g$, each harmonic coefficient costs at most $3d_p$ from s^3 and at most d_p from $j - i$. The coarse weight $4md_p$ covers every determinant term. Primes $p > U$ divide none of these denominators. This proves Lemma 4.6. □

4.2. **Formal divisors.** In Lemmas 4.7 and 4.8, a is an indeterminate and $\mathsf{b} = \mathsf{a} - 1$. Put

$$\mathcal{Q}_m(\mathsf{a}, T) = L^{4m} \mathsf{b}^{\,mH} G_{m,\mathsf{a}}(T).$$

By Lemma 4.1, $\sum_{r \geq 1} \mathsf{a}^{-r}\binom{r-1}{\iota} = \mathsf{b}^{-\iota-1}$ and $\iota + 1 \leq H$. The harmonic corrections and the entries of C are polynomials in a with denominators dividing L^4. So every entry of $L^4 \mathsf{b}^H (E + TC)$ lies in $\mathbb{Z}[\mathsf{a}, T]$, and $\mathcal{Q}_m \in \mathbb{Z}[\mathsf{a}, T]$.

Lemma 4.7. *Let $\mathcal{Q}_m$ be as above. Every nonzero coefficient of the determinant has total a-order at least*

$$\sum_{j=0}^{m-1} \lfloor j/4 \rfloor = \frac{m^2}{8} - \frac{m}{2} = N.$$

Hence a^N divides $\mathcal{Q}_m$ in $\mathbb{Z}[\mathsf{a}, T]$.

Proof. For a polynomial P in the polynomial part, the rational continuation of $\sum_{r \geq 1} a^{-r} P(r)$ has the formal expansion $-\sum_{s \geq 0} a^s P(-s)$ at $a = 0$. This identity is used only for polynomial parts. The cubic-pole corrections are handled by their finite harmonic sums and by the following exceptional-row decomposition.

Put $\varrho(a) = \sum_{s \geq 1} a^s / s^3$ and consider $E + (X + \varrho(a))C$ in $\mathbb{Q}[X][[a]]$. For $k \notin I \cup J$, the coefficient matrix of a^k is a sum of three outer products in the row features $(i-k)^{-1}, (i-k)^{-2}, (i-k)^{-3}$. For $k \in I$, use the four row features $(v_i, v_i^2, v_i^3, [i = k])$ and four column features $(v_j, v_j^2, v_j^3, v_j^4)$, where $v_i = (i-k)^{-1}$, $v_k = 0$, and $v_j = (j-k)^{-1}$. Write w_1, w_2, w_3 for w', w'', w''' at $-k$, and put $\delta' = j - k$. The middle matrix is

$$\begin{pmatrix} -w_2 & 2w_1 & -2\varpi_k & 0 \\ 2w_1 & -2\varpi_k & 0 & 0 \\ -2\varpi_k & 0 & 0 & 0 \\ -w_3/3 + 2X\varpi_k & w_2 & -2w_1 & 2\varpi_k \end{pmatrix}, \qquad \varpi_k := w(-k).$$

The fourth row comes from expanding at the removable pole. Explicitly, the coefficient used in the formal rank calculation is

$$[a^k](E + (X + \varrho(a))C)_{kj} = -w_3/(3\delta') + w_2/\delta'^2 - 2w_1/\delta'^3 + 2\varpi_k/\delta'^4 + 2X\varpi_k/\delta'.$$

The case $k \in J$ is the transpose. So each coefficient matrix has rank at most four. Through order s, the coefficient columns lie in a space of dimension at most $4(s+1)$. In a determinant coefficient, at most four columns can be selected from any one coefficient matrix. Expand the determinant multilinearly in its columns,

each written as a power series in a. In a nonzero term at most four of the m columns come from each power a^k, $k \geq 0$. So its total a-order is at least $\sum_{j=0}^{m-1} \lfloor j/4 \rfloor = N$. Equality holds when four columns come from each a^k, $0 \leq k \leq m/4-1$. This gives the stated minimum a-weight.

The rank statement holds over the formal series ring. It is not a statement about finitely many values of a. Since ϱ has zero constant term, substituting $X = T - \varrho(a)$ preserves the a-adic order. So a^N divides $\mathcal{Q}_m$. This proves Lemma 4.7. □

Lemma 4.8. *Let $\mathcal{Q}_m$ be as above. Then b^{mH-H_m} divides $\mathcal{Q}_m$ in $\mathbb{Z}[\mathsf{a}, T]$.*

Proof. By the formulas for c_i and c_j, $P_{ij}(r) = w[r, -i, -j]$ is the second divided difference of w at the nodes $r, -i, -j$. Since $\deg w = 3g$, it is a symmetric polynomial in $(r, -i, -j)$ of total degree at most $3g-2$. So $[r^d]P''_{ij} = (d+1)(d+2)[r^{d+2}]P_{ij}$ is a polynomial in (i, j) of degree at most $3g-4-d$. With $x = 1/a$,

$$\sum_{r \geq 1} a^{-r} r^d = \Big(x \frac{d}{dx}\Big)^d \frac{x}{1-x}.$$

This rational function has a pole of order $d+1$ at $x = 1$. Since $1 - x = b/a$, this is a pole of order $d+1$ at $b = 0$. Hence, for $1 \leq s \leq H$, the coefficient of b^{-s} in E_{ij} is a polynomial in (i, j) of degree at most $3g-3-s = H-s$. The harmonic corrections and the entries of C are Laurent polynomials in a. They have no pole at $b = 0$.

Order I and J increasingly. For $0 \leq \alpha < m$, replace row α by the α-th forward difference of rows $0, \ldots, \alpha$. Do the same for columns. These operations are unimodular. A forward difference of order α lowers the degree of a polynomial by α. So the new entry (α, α') has pole order at most $(H - \alpha - \alpha')_+$ at $b = 0$. Suppose k entries of a determinant term have a positive pole bound. Their row orders are distinct, and so are their column orders. Each set sums to at least $k(k-1)/2$. So the pole order of the term is at most

$$kH - k(k-1) = k(H+1-k) \leq \Big(\frac{3m}{4} - 1\Big)^2 = H_m.$$

Each row of $E + TC$ has pole order at most H at $b = 0$. So $b^{mH} G_{m,a}(T)$ is regular at $b = 0$ and divisible by b^{mH-H_m}, coefficientwise in T. This proves Lemma 4.8. □

Since a and b are monic and coprime in $\mathbb{Z}[\mathsf{a}]$, Lemmas 4.7 and 4.8 give

$$\mathcal{R}_m(\mathsf{a}, T) = \frac{\mathcal{Q}_m(\mathsf{a}, T)}{\mathsf{a}^N \mathsf{b}^{mH-H_m}} \in \mathbb{Z}[\mathsf{a}, T].$$

For a fixed integer a with $|a| \geq 2$,

$$G_{m,a}(T) = \frac{a^N}{L^{4m}(a-1)^{H_m}} \mathcal{R}_m(a, T).$$

4.3. **The common denominator.**

Proposition 4.9. *Set*

$$F_p = 4md_p + H_m v_p(a-1) - N v_p(a), \quad e_p = \min\{F_p, W_m(p) + mH v_p(a-1)\},$$

and $\Delta_m(a) = \prod_p p^{e_p}$. Then $\Delta_m(a)$ is a positive rational number. For every m with $4 \mid m$ and every fixed integer a with $|a| \geq 2$, we have $\Delta_m(a) G_{m,a}(T) \in \mathbb{Z}[T]$.

Proof. Negative e_p are allowed. Fix a prime p. It suffices to show that each of the two exponents in e_p makes $p^e G_{m,a}(T)$ p-integral.

First alternative. By Lemmas 4.7 and 4.8,

$$G_{m,a}(T) = a^N \mathcal{R}_m(a,T)/(L^{4m}(a-1)^{H_m})$$

with $\mathcal{R}_m \in \mathbb{Z}[\mathsf{a},T]$. Since a and b are monic and coprime, division by both factors stays in $\mathbb{Z}[\mathsf{a},T]$. So every coefficient of $G_{m,a}(T)$ has $v_p \geq -F_p$. Fixed primes dividing a or $a-1$ are retained here.

Second alternative. Choose a potential π on $I \cup J$ by the case of $W_m(p)$. Take $\pi = 0$ for $p > U$, $\pi = 2d_p$ in case (2), and $\pi_n = \phi_n = 2 - \kappa_n$ in case (3). In case (4) take potential (I) or (II) of Lemma 4.5, whichever has the smaller total. In case (5) take the weights of Lemma 4.5. The total of π is $W_m(p)$. In case (2) it is $2m \cdot 2d_p$. In case (3) it is $\sum_{n=0}^{U}(2-\kappa_n) = 4m - \sum_n \kappa_n$. In cases (4) and (5) Lemma 4.5 gives it.

Expand $\mathsf{b}^H(E+TC)_{ij}$ in a and T. Its polynomial part is $\sum_\iota \gamma_{ij,\iota}\mathsf{b}^{H-\iota-1}$, by Lemma 4.1. Its harmonic part and C are polynomials in a. Every coefficient has $v_p \geq -(\pi_i + \pi_j)$, by Lemmas 4.1, 4.2, 4.4 and 4.6. In case (2) this is the bound L^4 of Lemmas 4.1 and 4.6. For $p > U$ no denominator contains p. Now expand

$$\mathsf{b}^{mH} G_{m,\mathsf{a}}(T) = \det\bigl(\mathsf{b}^H(E+TC)\bigr)$$

over permutations $\varsigma\colon I \to J$. The term of ς has $v_p \geq -\sum_{i\in I}(\pi_i + \pi_{\varsigma(i)}) = -\sum_{n\in I\cup J}\pi_n = -W_m(p)$. At the integer a divide by b^{mH}, with $b = a-1 \neq 0$. So every coefficient of $G_{m,a}(T)$ has $v_p \geq -W_m(p) - mHv_p(a-1)$.

Hence $p^{e_p}G_{m,a}(T) \in \mathbb{Z}_{(p)}[T]$ for every p. For $p > U$ with $p \nmid a(a-1)$, $F_p = W_m(p) = 0$, so $e_p = 0$. Thus $\Delta_m(a)$ is a positive rational and $\Delta_m(a)G_{m,a}(T) \in \mathbb{Z}[T]$. The divisibilities by a^N and $\mathsf{b}^{\,mH-H_m}$ hold coefficientwise, so no finite-grid extrapolation enters. The rational $\Delta_m(a)$ is a common denominator and need not equal the primitive content. This proves Proposition 4.9. □

4.4. **The arithmetic rate.**

Lemma 4.10. *Let $x = p/m$. For $g < p < m$ the normalized weight satisfies $W_m(p)/m = \omega(x) + O(1/m)$, where*

$$\omega(x) = \begin{cases} 6x-2, & 1/2 < x < 5/8, \\ 3-2x, & 5/8 < x < 2/3, \\ 7x-3, & 2/3 < x < 3/4, \\ 5x-3/2, & 3/4 < x < 5/6, \\ 7/2-x, & 5/6 < x < 1. \end{cases}$$

The five integrals, in order, are $11/64, 41/576, 47/288, 59/288, 31/72$; their sum is $25/24$. For $m \leq p \leq U$, up to $O(1/m)$, the normalized profile is $1/2+2x$ on $(1,5/4)$ and $8-4x$ on $(5/4,2)$. Their integrals are $11/16$ and $9/8$, so this range contributes $29/16$.

Proof. Since $u/m = x - 1/2$, the counts of Lemma 4.5 give, up to $O(1/m)$,

$$\frac{N_{\mathrm{act}}}{m} = \sum_{t=1}^{3}\bigl(\min(x-\tfrac{1}{2},\,2-tx)\bigr)_+, \qquad \frac{\Sigma_K}{m} = \frac{1}{2} + \Bigl(\frac{5}{2}-4x\Bigr)_+.$$

The minima switch at $x = 5/8$ for $t = 3$ and at $x = 5/6$ for $t = 2$. The term $t = 3$ vanishes for $x \geq 2/3$. Put $W_{\mathrm{I}} = 3N_{\mathrm{act}} + u + \Sigma_K$ and $W_{\mathrm{II}} = m + 2N_{\mathrm{act}} + u$. On the four intervals cut at $5/8, 2/3, 5/6$, the triple $(N_{\mathrm{act}}, W_{\mathrm{I}}, W_{\mathrm{II}})/m$ is

$$(3x - \tfrac{3}{2},\, 6x - 2,\, 7x - \tfrac{5}{2}), \qquad (1 - x,\, 3 - 2x,\, \tfrac{5}{2} - x),$$
$$(2x - 1,\, 7x - 3,\, 5x - \tfrac{3}{2}), \qquad (\tfrac{3}{2} - x,\, \tfrac{9}{2} - 2x,\, \tfrac{7}{2} - x).$$

The minimum is W_{I} on $(1/2, 3/4)$ and W_{II} on $(3/4, 1)$. The table uses four intervals, while ω has five pieces: inside $(2/3, 5/6)$ the minimum switches from W_{I} to W_{II} at $x = 3/4$, where $7x - 3 = 5x - 3/2$. This gives ω. In units of $1/576$, the five integrals are $99, 41, 94, 118, 248$, with sum 600. For $m \leq p \leq U$, the weight is $4(2m - p) - 3(2m + g - 2p)_+$. Normalized, this is $8 - 4x - 3(5/2 - 2x)_+$, which gives the two stated pieces. In total,

$$\frac{25}{24} + \frac{29}{16} = \frac{50 + 87}{48} = \frac{137}{48}.$$

This proves Lemma 4.10. □

Lemma 4.11. *Set*

$$\begin{aligned}\mathcal{B}_m &= \prod_{n=0}^{2m-1} \binom{n+g}{g},\\ \mathcal{T} &= \lim_{m\to\infty} \frac{\log \mathcal{B}_m}{m^2}\\ &= \int_0^2 \big((y + 1/2)\log(y + 1/2) - y \log y - (1/2)\log(1/2)\big)\, dy.\end{aligned}$$

Then

$$\mathcal{T} = \frac{25}{8}\log 5 - 4\log 2 - \frac{1}{2}.$$

Let $m \geq 8$ and $p > g$. Then $p^2 > U+g$, since $p^2-(U+g) \geq (m/2+1)^2-(5m/2-1) = (m-2)(m-4)/4$. The exact carry profile of $v_p(\mathcal{B}_m)/m$, with $x = p/m$, is

$$k(x) = \frac{1}{2}\left\lfloor \frac{2}{x} \right\rfloor + \left(\frac{5}{2} - (\lfloor 2/x \rfloor + 1)x\right)_+, \qquad \frac{1}{2} < x \leq \frac{5}{2}.$$

Its integral is $\int_{1/2}^{5/2} k(x)\, dx = 43/32$.

Proof. By Stirling's formula, $\log \binom{n+g}{g}$ is m times the integrand of $\mathcal{T}$ at $y = n/m$, up to $O(\log m)$. This holds uniformly for $0 \leq n < 2m$. Summing over n gives the integral. Since

$$\int_{1/2}^{5/2} y \log y\, dy = \frac{25}{8}\log 5 - 3\log 2 - \frac{3}{2}, \qquad \int_0^2 y \log y\, dy = 2\log 2 - 1,$$

we get $\mathcal{T} = \frac{25}{8}\log 5 - 3\log 2 - \frac{3}{2} - (2\log 2 - 1) + \log 2$, the stated value.

Let $m \geq 8$ and $p > g$, so $p^2 > U + g$. Legendre's formula has at most one carry. So $v_p\binom{n+g}{g} = 1$ exactly when $n \bmod p \geq p - g$, and 0 otherwise. These n form the windows $[tp - g, tp)$, $t \geq 1$. The windows with $tp \leq 2m$ lie in $[0, 2m)$, and there are $\lfloor 2m/p \rfloor$ of them. The next window meets $[0, 2m)$ in length $(2m+g-(\lfloor 2m/p \rfloor+1)p)_+$. Dividing by m gives $k(x)$. On the intervals cut at $5/8, 2/3, 5/6, 1, 5/4, 2$, k equals

$$4 - 4x,\ \tfrac{3}{2},\ \tfrac{7}{2} - 3x,\ 1,\ 3 - 2x,\ \tfrac{1}{2},\ \tfrac{5}{2} - x,$$

with integrals $7/32, 1/16, 5/24, 1/6, 3/16, 3/8, 1/8$. Their sum is $129/96 = 43/32$. This proves Lemma 4.11. □

Proposition 4.12. *For every fixed integer a with $|a| \geq 2$,*

$$\limsup_{4|m} \frac{\log \Delta_m(a)}{m^2} \leq \tau(a) := S + \frac{9}{16}\log|a-1| - \frac{1}{8}\log|a|,$$

where

$$S = \frac{643}{96} + 4\log 2 - \frac{25}{8}\log 5 = 4.44101\ldots.$$

Proof. By Lemma 4.11, which applies for $m \geq 8$, the prime number theorem and partial summation,

$$\lim \frac{1}{m^2}\sum_{p\leq g} v_p(\mathcal{B}_m)\log p = \mathcal{T} - \frac{43}{32}.$$

Since $4m\vartheta(g)/m^2 \to 2$, with Chebyshev's function $\vartheta(x) = \sum_{p\leq x}\log p$, the $p \leq g$ moving-prime contribution is $2-(\mathcal{T}-43/32) = 123/32+4\log 2-(25/8)\log 5$. The profile of Lemma 4.10 is continuous on $(1/2, 2)$, so $W_m(p)/m$ equals it at $x = p/m$ up to $O(1/m)$, uniformly in p. Applying the prime number theorem on each linear piece, the primes $g < p \leq U$ contribute $25/24 + 29/16$ by Lemma 4.10.

The case $p^2 \leq U + g$, which precedes the moving-prime branches, contributes at most

$$4m \sum_{p\leq\sqrt{U+g}} d_p \log p \leq 4m\lfloor\sqrt{U+g}\rfloor \log U = O(m^{3/2}\log m) = o(m^2).$$

It includes every prime power through $d_p = \lfloor \log_p U\rfloor$. The same bound controls the difference between this coarse branch and the one-carry model for these primes. Each of the $2m$ binomial factors in $\mathcal{B}_m$ has valuation at most $\lfloor\log_p(U+g)\rfloor$. Their total over $p \leq \sqrt{U+g}$ is $O(m^{3/2}\log m)$. For the remaining primes $p^2 > U + g$, Legendre's formula has at most one carry. Partial summation against $\vartheta(x) = x+o(x)$ gives the piecewise-affine profile integrals above. The finitely many rational profile walls contribute $o(m^2)$. Moving the lower wall $\sqrt{U+g}/m$ to zero changes the normalized sum by $o(1)$, by the same bound.

Fixed primes dividing a or $a-1$ are retained in the first alternative of e_p. They give $-N/m^2 \to -1/8$ and $H_m/m^2 \to 9/16$. Finally $123/32 + 25/24 + 29/16 = 643/96$, which gives the displayed S. This proves Proposition 4.12. □

5. Nonvanishing along a prime-power subsequence

Proposition 5.5 below uses a prime p chosen from a. It reduces $G_{4p^k,a}$ modulo p to a power of the seed determinant $G_{4,a}$.

Lemma 5.1. *Let H_0 be the degree-22 integer polynomial whose coefficients are listed in Appendix B. Then*

$$G_{4,a}(0) = \frac{H_0(a)}{2^{18}\cdot 3^9\cdot 5^{12}\cdot 7^4(a-1)^4}.$$

The residues of H_0 at $a = 0, \ldots, 12$ modulo 13 are

$$(10, 4, 11, 4, 4, 11, 1, 3, 11, 6, 7, 9, 10).$$

So $H_0(a)$ is nonzero for every integer a.

Proof. For $m = 4$ the entries of $(a-1)^3 E(a)$ are polynomials in a by Lemma 2.1, since $\deg P_{ij} \le 4$ for $m = 4$. Expanding the 4×4 determinant exactly over $\mathbb{Q}[a]$ gives the identity (Appendix C). Evaluate the coefficients of Table B.1 at $a = 0, \dots, 12$ modulo 13. This gives the listed residues. None is zero, so $13 \nmid H_0(a)$ for every integer a. This proves Lemma 5.1. □

Fix an integer a with $|a| \ge 2$, and choose a prime divisor p of

$$12!\,|a(a-1)H_0(a)| + 1.$$

Then $p > 12$, and p divides none of a, $a-1$, $H_0(a)$, or the seed denominator. Put $Q = p^k$, $m = 4Q$, $g = 2Q$, and work in $R = \mathbb{Z}_{(p)}$. All congruences below are coefficientwise. The superscript seed marks the objects of Section 2 that define the seed determinant, with row index i_0 and column index j_0. The tower row of Appendix C is a finite check; the proof for all k does not use it.

5.1. **The local tower.**

Lemma 5.2. *For $0 \le s < Q$, set*

$$\Phi_s(x) = \binom{Qx - s - 1}{2Q}.$$

Then $\Phi_s(x) \in R[x]$ and

$$\Phi_s(x) \equiv \binom{x-1}{2} \pmod p$$

coefficientwise.

Proof. Write the numerator of Φ_s as the product of the $2Q$ linear factors $Qx-s-1-j$, $0 \le j < 2Q$. For each $e \le k$, exactly $2Q/p^e$ of their constant terms are divisible by p^e. So the p-adic content of the product has valuation $\sum_{e=1}^{k} 2Q/p^e = v_p((2Q)!)$. Division by $(2Q)!$ proves $\Phi_s(x) \in R[x]$. After this content is removed, only the two factors whose constant terms are divisible by Q can contribute degree in the reduction. So $\deg(\Phi_s \bmod p) \le 2$. At $x = 0$, the identity $\binom{-s-1}{2Q} = \binom{s+2Q}{2Q}$ and Lucas' theorem give residue 1. At $x = 1$ and $x = 2$ the binomial coefficients are zero. These are the values of $\binom{x-1}{2}$ at 0, 1, 2. Since $p > 2$ and the reduced degree is at most two, this proves Lemma 5.2. □

Lemma 5.3. *Let $i \in I$ and $j \in J$. Write $i = Qi_0 + s_i$, $j = Qj_0 + s_j$, with $0 \le i_0 \le 3$, $4 \le j_0 \le 7$, and $0 \le s_i, s_j < Q$. Fix $s \in \{0, \dots, Q-1\}$, and put*

$$\Xi_i(x) = \frac{Q}{Qx + i - s}, \quad \Xi_j(x) = \frac{Q}{Qx + j - s},$$

with Φ_s as in Lemma 5.2 and c_i, c_j as in Section 2. Then

$$Q^2 P_{ij}(Qx - s) = \Phi_s(x)^3 \Xi_i(x)\Xi_j(x) - Qc_i\Xi_i(x) - Qc_j\Xi_j(x), \tag{5.1}$$

$Q^2 P_{ij}(Qx - s) \in R[x]$, and

$$Q^4 P''_{ij}(Qx - s) \equiv 1_{s_i = s_j = s}\,(P^{\mathrm{seed}}_{i_0,j_0})''(x) \pmod p. \tag{5.2}$$

Proof. Identity (5.1) is the partial-fraction identity of Section 2 at $r = Qx - s$, multiplied by Q^2. If both offsets are nonzero, $|i - s|, |j - s| < 8Q$ implies $v_p(i - s), v_p(j - s) \le k$. Hence each $\Xi_n = (Q/d)/(1 + (Q/d)x)$, with offset $d = n - s$, is in $R[[x]]$, and its reduction is zero unless $Q|d$. There cannot be two zero offsets,

because I and J are disjoint. If, for example, $i = s$, then $\Xi_i = 1/x$. Its only possible negative-power coefficient on the right side of (5.1) is

$$\Phi_s(0)^3\Xi_j(0) - Qc_i = \Phi_s(0)^3\frac{Q}{j-i} - Q\frac{w(-i)}{j-i} = 0.$$

So the Laurent principal part cancels exactly, and the remaining Taylor series is p-integral. Gauss integrality gives $Q^2P_{ij}(Qx-s) \in R[x]$. The argument divides only by monic polynomials.

Modulo p, (5.1) is zero unless $s_i = s_j = s$. In that matching case, by Lemma 5.2, it is the seed polynomial part of $\binom{x-1}{2}^3/((x+i_0)(x+j_0))$. The reduced residues are $w^{\mathrm{seed}}(-i_0)/(j_0-i_0)$ and $-w^{\mathrm{seed}}(-j_0)/(j_0-i_0)$, since $0 < |j_0-i_0| \le 7 < p$. The same cancellation covers $i_0 = 0$, with no separate evaluation at a pole. Differentiating the polynomial identity twice gives (5.2). This proves Lemma 5.3. □

Lemma 5.4. *For i_0, j_0 as in Lemma 5.3 and $0 \le s, s' < Q$, in R,*

$$Q^4E_{Qi_0+s,Qj_0+s'} \equiv 1_{s=s'}a^sE^{\mathrm{seed}}_{i_0,j_0}, \qquad QC_{Qi_0+s,Qj_0+s'} \equiv 1_{s=s'}a^sC^{\mathrm{seed}}_{i_0,j_0} \pmod p.$$

Proof. The polynomial $f(r) = Q^4P''_{ij}(r)$ is p-integral at every integer. Every integer has a unique form $r = Qx - s$, with $0 \le s < Q$, and (5.1)–(5.2) have p-integral coefficients. So its complete Newton expansion is

$$f(r) = \sum_{\iota=0}^{3g-4}\gamma_\iota\binom{r-1}{\iota}, \qquad \gamma_\iota \in R, \quad \sum_{r\ge1}z^rf(r) = \sum_{\iota=0}^{3g-4}\gamma_\iota\frac{z^{\iota+1}}{(1-z)^{\iota+1}}. \tag{5.3}$$

The last identity is a finite identity of rational functions. Now partition the positive integers as $r = Qx - s$ ($x \ge 1$, $0 \le s < Q$) in (5.2). Explicitly, for each s put $f_s(x) = f(Qx-s) = Q^4P''_{ij}(Qx-s) \in R[x]$. Its values at integers lie in R, so its Newton coefficients $\gamma_\iota(s) = (\Delta^\iota f_s)(1)$ lie in R, and

$$\sum_{x\ge1}a^{-Qx}f_s(x) = \sum_{\iota=0}^{3g-4}\frac{\gamma_\iota(s)}{(a^Q-1)^{\iota+1}}, \qquad Q^4\sum_{r\ge1}a^{-r}P''_{ij}(r) = \sum_{s=0}^{Q-1}a^s\sum_{x\ge1}a^{-Qx}f_s(x).$$

Here $a^Q - 1 \equiv a - 1 \pmod p$ is a unit of R. By (5.2), $\gamma_\iota(s)$ reduces to zero unless $s_i = s_j = s$, and in that case to the corresponding Newton coefficient of the seed polynomial part. Only the matching class survives. Its generating function is a^s times the seed polynomial-part generating function at a^Q. Here $a^Q = a$ in $\mathbb{F}_p$, and both $1-a^Q$ and $1-a$ are units. So (5.3) gives the polynomial-part entry congruence for every k, including the coefficients that the finite check of Appendix C does not reach.

For the pole contribution use h_3 of Section 2. By Lemma 2.1, $E_{\mathrm{pole}} = -2c_ia^i\mathsf{h}_3(i) - 2c_ja^j\mathsf{h}_3(j)$ and $C = 2c_ia^i + 2c_ja^j$. Since $i, j < 8Q$, every summation index $e \le j$ of h_3 has $v_p(e) \le k$. Hence Q^3/e^3 is p-integral and reduces to zero unless $Q \mid e$. Also Qc_i and Qc_j are p-integral and reduce to zero unless the two index residues agree. In the matching block, $i = Qi_0 + s$ and $j = Qj_0 + s$. Then $Qc_i \equiv c^{\mathrm{seed}}_{i_0}$, $Qc_j \equiv c^{\mathrm{seed}}_{j_0}$, $a^{Qi_0+s} \equiv a^{i_0+s}$, and $Q^3\mathsf{h}_3(Qi_0+s) \equiv \mathsf{h}_3(i_0)$ modulo p; the same holds for j. This covers the polynomial part, both poles, both derivatives, and the Newton sums. This proves Lemma 5.4. □

Proposition 5.5. *Let a, p, $Q = p^k$ and $m = 4Q$ be as above. Coefficientwise in $\mathbb{Z}_{(p)}[X]$,*

$$Q^{4m}G_{4Q,a}(X/Q^3) \equiv G_{4,a}(X)^Q \pmod p.$$

Proof. Group rows and columns by residues in the determinant, and apply Lemma 5.4. The permutation signs occur once for rows and once for columns and cancel. The block with residue s is a^s times a 4×4 seed block. So the remaining scalar is $\prod_{s=0}^{Q-1} a^{4s} = a^{2Q(Q-1)}$. Since $p \nmid a$ and $(p-1) \mid (Q-1)$, Fermat's theorem gives $a^{Q-1} \equiv 1 \pmod{p}$. Hence $a^{2Q(Q-1)} = (a^{Q-1})^{2Q} \equiv 1 \pmod{p}$. This proves Proposition 5.5. □

Proposition 5.6. *Let a and p be as above. For every fixed rational ξ,*

$$G_{4p^k,a}(\xi) \neq 0$$

for all sufficiently large k. The selected prime depends on a, not on ξ, while the starting index may depend on both.

Proof. Choose k so that $3k + v_p(\xi) \geq 1$, with $v_p(0) = \infty$. Then $Q^3\xi \in pR$. The whole normalized polynomial is p-integral. So Proposition 5.5 at $X = Q^3\xi$ gives

$$Q^{4m} G_{4Q,a}(\xi) \equiv G_{4,a}(0)^Q \not\equiv 0 \pmod{p}.$$

The right side is nonzero by Lemma 5.1 and the choice of p. This proves Proposition 5.6. □

6. Analytic determinant estimate

For $q = |a|$, $\lambda = \log q$, and probability measures μ on $[1/2, \infty)$ with density $0 \leq f \leq 3$ and finite first moment, define

$$h(t) = t \log t - (t - 1/2) \log(t - 1/2) + (1/2) \log 2, \qquad B(t) = \int_0^2 \log(t + s)\, ds,$$

$$\mathcal{E}_\lambda(\mu) = \frac{3}{2} + \int (B - 3h + \lambda t)\, d\mu(t) - \iint \log|t - s|\, d\mu(t) d\mu(s), \qquad A(\lambda) = \inf_\mu \mathcal{E}_\lambda(\mu).$$

Here $0 \log 0 = 0$. These measures form the energy domain.

6.1. **Andreief expansion and local estimates.** Let $f_i(z) = (z + i)^{-1}$, $k_j(z) = (z + j)^{-1}$, and

$$\mathcal{F}(z_1, \ldots, z_m) = \prod_{\ell=1}^{m} w(z_\ell) \det[f_i(z_\ell)]_{i \in I, \ell} \det[k_j(z_\ell)]_{j \in J, \ell}.$$

Lemma 6.1. *For a finite cutoff N_0, let $\mathcal{I}_{N_0}(\psi) = \sum_{r=1}^{N_0} a^{-r} \psi''(r)$. Then*

$$\det[\mathcal{I}_{N_0}(w f_i k_j)] = \frac{1}{m!} \mathcal{I}_{N_0}^{\otimes m}(\mathcal{F}).$$

The identity persists as $N_0 \to \infty$.

Proof. This is the algebraic Andreief identity. Expanding the permutations proves it directly. Each derivative has polynomial growth, and the factor $|a|^{-\sum r_\ell}$ is geometric. So the limit $N_0 \to \infty$ exists by absolute convergence. For negative a, the exact factor $(-1)^{\sum r_\ell}$ is retained until the final absolute value. This proves Lemma 6.1. □

Lemma 6.2. *Up to a fixed sign, the summand in Lemma 6.1 equals*

$$\frac{\left(\prod_\ell \prod_{s=1}^{g} (z_\ell - s)^3\right) \left(\prod_{\ell < \ell'} (z_\ell - z_{\ell'})^2\right)}{(g!)^{3m} \prod_\ell \prod_{i=0}^{2m-1} (z_\ell + i)} \left(\prod_{i < i'} (i' - i)\right) \left(\prod_{j < j'} (j' - j)\right). \tag{6.1}$$

Every nonzero term of the multiple sum has occupancy at most three; triple occupancy is allowed.

Proof. The formula is Cauchy's double alternant [4, (2.7)]. At $1 \le r_\ell \le g$, the triple zero of w leaves only two derivatives in that variable, and the summand vanishes. If ν variables have the same integer value, their internal Vandermonde square has order $\nu(\nu-1)$. The available derivatives have total order 2ν. So every nonzero term has occupancy at most three. This proves Lemma 6.2. □

Use independent circles $|z_\ell - r_\ell| \le 1/4$. All poles are outside. The multivariate Cauchy estimate contributes 32^m. Write $t_\ell = r_\ell/m$. Replace an ordered tuple by the density μ whose value on each cell $[r/m, (r+1)/m)$ is the occupancy of r. By Lemma 6.2, this is a probability density, bounded by 3, supported on $[1/2, \infty)$. In this section the cost of a step is what it adds to the upper bound for $\log|\cdot|$ of a summand.

Lemma 6.3. *Put*

$$e_d = \log d - \int_0^1 \int_0^1 \log|d + u - v|\, du\, dv.$$

Then $e_1 = 3/2 - 2\log 2 < 1/8$, *and for* $d \ge 2$,

$$e_d = \sum_{j\ge1} \frac{1}{j(2j+1)(2j+2)d^{2j}} \le \frac{1}{12d(d-1)}.$$

With occupancy at most 3, the distinct-site logarithms obey

$$\sum_{\ell\ne\ell'} \log\bigl(\max(1, |r_\ell - r_{\ell'}|)/m\bigr) \le m^2 L(\mu,\mu) + m\log m + 6m, \tag{6.2}$$

where $L(\mu,\mu) = \iint \log|t-s|\, d\mu(t) d\mu(s)$.

Proof. Direct integration gives $\int_0^1\int_0^1 \log|1+u-v|\, du\, dv = 2\log 2 - 3/2$. Hence

$$e_1 = \frac{3}{2} - 2\log 2 < \frac{1}{8}.$$

The inequality holds since $\log 2 = 2\,\mathrm{artanh}(1/3) > 2(1/3 + 1/81) = 56/81 > 11/16$. For $d \ge 2$, expand the logarithm in $|u-v|/d$. Odd powers integrate to zero, and

$$\int_0^1 \int_0^1 (u-v)^{2j}\, du\, dv = \frac{2}{(2j+1)(2j+2)}.$$

This gives the series for e_d. Since $1/[j(2j+1)(2j+2)] \le 1/12$,

$$e_d \le \frac{1}{12} \sum_{j\ge1} d^{-2j} = \frac{1}{12(d^2-1)} \le \frac{1}{12d(d-1)}.$$

Also $\sum_{d\ge2}[12d(d-1)]^{-1} = 1/12$. Hence

$$\sum_{d\ge1} e_d < \frac{1}{8} + \frac{1}{12} = \frac{5}{24}. \tag{6.3}$$

Let $\nu_r \in \{0,1,2,3\}$ be the occupancy of the integer center r. Then $\sum_r \nu_r = m$, and $\mu = \nu_r$ on $\mathcal{C}_r = [r/m, (r+1)/m)$. So $\int \mu = 1$. Since $\int_0^1\int_0^1 \log|u-v|\, du\, dv =$

$-3/2$, a cell-by-cell comparison gives

$$\begin{aligned} &m^2 L(\mu,\mu) \\ &\qquad = \sum_{r\neq s} \nu_r\nu_s \left(\log \frac{|r-s|}{m} - e_{|r-s|} \right) + \sum_r \nu_r^2 \left(-\log m - \frac{3}{2} \right), \\ &\sum_{\ell\neq\ell'} \log\bigl(\max(1,|r_\ell - r_{\ell'}|)/m\bigr) \\ &\qquad = \sum_{r\neq s} \nu_r\nu_s \log\frac{|r-s|}{m} + \sum_r \nu_r(\nu_r-1)(-\log m). \end{aligned}$$

The first sum runs over ordered pairs of distinct cells. For $r = s$, the left side contains only the $\nu_r(\nu_r - 1)$ distinct particle pairs. So the self-cell part of the difference is $\nu_r \log m + (3/2)\nu_r^2$. Subtracting the two identities gives

$$\begin{aligned} &\sum_{\ell\neq\ell'} \log\bigl(\max(1,|r_\ell - r_{\ell'}|)/m\bigr) - m^2 L(\mu,\mu) \\ &\qquad = \sum_{r\neq s} \nu_r\nu_s e_{|r-s|} + m\log m + \frac{3}{2}\sum_r \nu_r^2. \end{aligned} \tag{6.4}$$

For fixed r and $d \geq 1$, at most two cells lie at distance d, each with occupancy at most three. By (6.3),

$$\sum_{r\neq s} \nu_r\nu_s e_{|r-s|} \leq 6m \sum_{d\geq 1} e_d < \frac{5}{4}m, \qquad \frac{3}{2}\sum_r \nu_r^2 \leq \frac{9}{2}m.$$

Put these bounds into (6.4). This gives (6.2). This proves Lemma 6.3. □

Lemma 6.4. *The logarithm of the product of the fixed row and column Vandermonde factors is at most*

$$m(m-1)\log m - \frac{3}{2}m^2 + m\log m + 2m. \tag{6.5}$$

The moving complex Vandermonde satisfies

$$\log|\Delta(z)^2| \leq m(m-1)\log m + m^2 L(\mu,\mu) + 4m\log m + 9m. \tag{6.6}$$

Proof. Put

$$\mathcal{D}_m = \prod_{0\leq i<i'<m} (i'-i).$$

Each difference d occurs $m-d$ times. Hence

$$\mathcal{D}_m = \prod_{d=1}^{m-1} d^{\,m-d} = \prod_{k=1}^{m-1} k!.$$

The shift $j \mapsto j - m$ maps J onto I. So both fixed factors $\prod_{i<i'}(i'-i)$ and $\prod_{j<j'}(j'-j)$ in (6.1) equal $\mathcal{D}_m$. Put

$$\zeta_m(x) = (m-x)\log x \qquad (1 \leq x \leq m).$$

For $m \geq 2$,

$$\zeta_m''(x) = -\frac{x+m}{x^2} < 0, \qquad \zeta_m(1) = \zeta_m(m) = 0.$$

By concavity, the trapezoid inequality on each interval $[k, k+1]$ gives

$$\int_k^{k+1} \zeta_m(x)\,dx \geq \frac{\zeta_m(k)+\zeta_m(k+1)}{2}.$$

Sum over $k = 1, \dots, m-1$. The endpoint values vanish. Hence

$$\log \mathcal{D}_m = \sum_{k=1}^{m-1} \zeta_m(k) \leq \int_1^m (m-x)\log x\,dx = \frac{1}{2}m^2\log m - \frac{3}{4}m^2 + m - \frac{1}{4}.$$

Therefore

$$2\log \mathcal{D}_m \leq m^2\log m - \frac{3}{2}m^2 + 2m - \frac{1}{2}.$$

Since $m(m-1)\log m + m\log m = m^2\log m$, this is stronger than (6.5).

For the moving complex Vandermonde, unequal centers at distance d satisfy $\log(d+1/2) \leq \log d + 1/(2d)$. Equal centers contribute at most $\log(1/2) \leq 0$. At each positive distance at most six particles occur. So the reciprocal-distance sum is at most $6\mathcal{S}_m$ per particle, where $\mathcal{S}_n = \sum_{d=1}^n d^{-1} \leq 1 + \log n$ for $n \geq 1$. Combining this with (6.2) gives (6.6). This proves Lemma 6.4. □

Lemma 6.5. *For $r \geq g+1$ and z on the circle around r used above,*

$$|w(z)| \leq \binom{r-1}{g}^3 \exp\left(\frac{3}{4}\mathcal{S}_g\right) \leq \exp\left(3mh(r/m) + \frac{3}{4}(1+\log m)\right), \tag{6.7}$$

with $\mathcal{S}_g \leq \mathcal{S}_m$ as in Lemma 6.4. For each particle, the $2m$ denominator factors cost at most $4/3$. Over all particles the cost is at most $4m/3 \leq 2m$. Also, for $t \geq 1/2$,

$$\sum_{i=0}^{2m-1} \log(t+i/m) \geq mB(t) - \log 5, \qquad B(t) = \int_0^2 \log(t+s)\,ds. \tag{6.8}$$

Proof. The first inequality in (6.7) follows from $\log(1+y) \leq y$ on each factor of the triple product. The binomial entropy inequality proves the second one. For integers $n \geq 1$ and $0 < k < n$, put $x = k/n$. The binomial probability of its k-th term gives

$$1 \geq \binom{n}{k} x^k (1-x)^{n-k}.$$

Hence

$$\binom{n}{k} \leq \exp\left(n\log n - k\log k - (n-k)\log(n-k)\right), \tag{6.9}$$

with $0\log 0 = 0$. The cases $k = 0$ and $k = n$ are immediate. Take $n = r$, $k = g = m/2$, and $t = r/m$. Since $r \geq g+1$, $\binom{r-1}{g} \leq \binom{r}{g}$, and (6.9) gives

$$\binom{r-1}{g} \leq \exp\left(r\log r - g\log g - (r-g)\log(r-g)\right).$$

Substitute $r = mt$, $g = m/2$, and $r - g = m(t-1/2)$. The exponent becomes

$$m\left[t\log t - (t-\tfrac{1}{2})\log(t-\tfrac{1}{2}) + \tfrac{1}{2}\log 2\right] = mh(t).$$

Therefore $\binom{r-1}{g}^3 \leq \exp(3mh(t))$. The disk estimate supplies the separate factor $\exp(3\mathcal{S}_g/4)$, and $\mathcal{S}_g \leq 1 + \log m$. For the denominators, $|z+i| \geq (r+i)(1-y_i)$ with

$y_i = 1/[4(r+i)] \le 1/4$. Since $-\log(1-y) \le y/(1-y) \le 4y/3$ for $0 \le y \le 1/4$, and $r \ge g+1 > m/2$, the cost $\sum_i \log((r+i)/|z+i|)$ of one particle is at most

$$\frac{4}{3}\sum_{i=0}^{2m-1} y_i = \frac{1}{3}\sum_{i=0}^{2m-1}\frac{1}{r+i} \le \frac{2m}{3r} < \frac{4}{3}.$$

Over the m particles this is at most $4m/3 \le 2m$. For (6.8), the logarithm is increasing. So the right Riemann sum satisfies

$$\sum_{i=1}^{2m}\log\Big(t+\frac{i}{m}\Big) \ge m\int_0^2 \log(t+s)\,ds = mB(t).$$

The left sum in (6.8) equals this right sum minus $\log((t+2)/t)$. For $t \ge 1/2$, $(t+2)/t \le 5$. This gives (6.8). This proves Lemma 6.5. □

Proposition 6.6. *Let a be an integer with $|a| \ge 2$, and let $4 \mid m$. Then*

$$\log|\det M(a)| \le -m^2 A(\log|a| - 1/m) + 4m\log m + (23 + \tfrac{1}{2}\log|a|)m. \tag{6.10}$$

Proof. By Lemma 6.1, write the differentiated Andreief summand with the two Cauchy determinants of (6.1) and the weight w. Set $q = |a|$, $\lambda = \log q$, and $\beta = \lambda - 1/m > 0$. Reserve $\exp(-\sum r_\ell/m)$ from $q^{-\sum r_\ell}$.

Cauchy's estimate on circles of radius $1/4$ contributes 32^m. Real-cell smoothing, Lemma 6.3, enters only through the moving Vandermonde bound below; its $m\log m + 6m$ is part of the $4m\log m + 9m$ there. The two fixed Vandermonde factors contribute $m(m-1)\log m - (3/2)m^2 + m\log m + 2m$. The moving complex Vandermonde, including self-energy and the six nearest-distance classes, contributes at most $m(m-1)\log m + m^2 L(\mu,\mu) + 4m\log m + 9m$. The binomial weight contributes $3mh(t) + (3/4)(1+\log m)$ per particle. The function h is increasing on the support. Hence $h(r/m) \le m\int_{r/m}^{(r+1)/m} h(t)\,dt$. So the cell average of h replaces $h(t_\ell)$ at no cost. The denominator Riemann sum contributes $mB(t) - \log 5$, with an additional $2m$ pole loss. Since B is increasing and $B'(t) = \log((t+2)/t) \le \log 5$, the cell average of B exceeds $B(r/m)$ by at most $(\log 5)/m$. Over the m particles this costs at most $m\log 5$. The fixed and the moving Vandermonde bounds each contain the term $m(m-1)\log m$. The $2m^2$ denominator factors $|z_\ell + i|$ give $-2m^2\log m$ when each is normalized by m, as in (6.8). The net homogeneous factor is therefore exactly m^{-2m}.

With the occupancies ν_r of the proof of Lemma 6.3,

$$\begin{aligned}\int t\,d\mu(t) &= \sum_r \nu_r \int_{r/m}^{(r+1)/m} t\,dt \\ &= \frac{1}{m^2}\sum_r \nu_r\left(r+\frac{1}{2}\right) = \frac{1}{m^2}\left(\sum_{\ell=1}^m r_\ell + \frac{m}{2}\right).\end{aligned}$$

Equivalently,

$$\frac{1}{m}\sum_{\ell=1}^m r_\ell = m\int t\,d\mu(t) - \frac{1}{2}. \tag{6.11}$$

The reserved factor splits $q^{-\sum_\ell r_\ell}$ as

$$q^{-\sum_\ell r_\ell} = \exp\left(-\beta \sum_\ell r_\ell\right) \exp\left(-\frac{1}{m} \sum_\ell r_\ell\right).$$

By (6.11), the first factor is exactly

$$\exp\left(-\beta m^2 \int t \, d\mu(t) + \frac{\beta m}{2}\right).$$

Since $\beta < \lambda$, its positive prefactor satisfies

$$\exp\left(\frac{\beta m}{2}\right) \le \exp\left(\frac{\lambda m}{2}\right).$$

The negative quadratic term is the term $\beta \int t \, d\mu$ of $\mathcal{E}_\beta$. The right side gives a linear term $(\lambda/2)m$. The cell rounding of the mean causes no other error.

Combining (6.2)–(6.8), the energy expression for each tuple is bounded above by

$$-m^2 \mathcal{E}_\beta(\mu) + \frac{15}{4} m \log m + \left(2 + 9 + \frac{3}{4} + 2 + 2\log 5 + \log 32 + \frac{\lambda}{2}\right) m. \tag{6.12}$$

The coefficient of $m \log m$ is the sum

$$\underbrace{4}_{(6.6)} + \underbrace{1}_{(6.5)} + \underbrace{3/4}_{(6.7)} - \underbrace{2}_{m^{-2m}} = \frac{15}{4}.$$

All $m^2 \log m$ terms cancel in this count. Here $\mathcal{E}_\beta$ is $\mathcal{E}_\lambda$ with λ replaced by β, and $A(\beta) = \inf_\mu \mathcal{E}_\beta(\mu)$. The term $\lambda m/2$ is the bound on $\exp(\beta m/2)$ above.

It remains to sum the reserved factor over all tuples. The sum over all tuples, with no cutoff, satisfies

$$\frac{1}{m!} \sum_{\boldsymbol{r}} \exp(-\sum r_\ell/m) \le \frac{1}{m!} \left(\frac{1}{e^{1/m} - 1}\right)^m \le \frac{m^m}{m!} \le e^m. \tag{6.13}$$

After adding this e^m factor to (6.12), the linear term before rounding is

$$\left(\frac{15}{4} m \log m + \left(2 + 9 + \frac{3}{4} + 2 + 2\log 5 + \log 32 + \frac{\lambda}{2} + 1\right) m\right).$$

The constant part after separating $\lambda/2$ is below 22. We use the larger constant 23, which leaves room for rounding. Since $15/4 < 4$, this yields (6.10). Occupancy at most three is used in the real-cell error. The proof uses no cutoff and no positivity of the determinant. For negative a, the alternating signs are discarded only after taking the absolute value of the full multiple sum. This proves Proposition 6.6. □

Lemma 6.7. *The function A is finite and concave on $(0, \infty)$, hence continuous. In particular $A(\lambda - 1/m) \to A(\lambda)$. Every density μ in the energy domain satisfies*

$$\int t \, d\mu(t) \ge 2/3,$$

with equality for density $3\,1_{[1/2,5/6]}$. For bounded zero-mass densities ρ with finite first moment, $-L(\rho, \rho) \ge 0$. Let $\lambda > 0$, and let σ be a compactly supported density in the energy domain. Put $U^\sigma(t) = \int \log|t - s| \, d\sigma(s)$, $V_\lambda(t) = B(t) - 3h(t) + \lambda t$, and $\Psi = V_\lambda - 2U^\sigma$. Then for every real Λ,

$$A(\lambda) \ge \frac{3}{2} + L(\sigma, \sigma) + \Lambda - 3 \int_{1/2}^{\infty} (\Lambda - \Psi(t))_+ \, dt. \tag{6.14}$$

Proof. Let $\beta > 0$. The density $3\,1_{[1/2,5/6]}$ is compactly supported and bounded. So its energy is finite, and $A(\beta) < \infty$. For the lower bound, $h(t) = \frac{1}{2}\log 2 + \int_{t-1/2}^{t}(1+\log x)\,dx \le \frac{1}{2} + \frac{1}{2}\log(2t)$. Also $B(t) \ge 2\log t$, and $\log|t-s| \le \log(1+t)+\log(1+s)$ for $t, s \ge 0$. Hence every μ in the energy domain satisfies

$$\mathcal{E}_\beta(\mu) \ge \frac{3}{2} + \int c_\beta\,d\mu, \qquad c_\beta(t) = 2\log t - \frac{3}{2} - \frac{3}{2}\log(2t) + \beta t - 2\log(1+t).$$

The function c_β is continuous on $[1/2,\infty)$ and $c_\beta(t) \to \infty$ as $t \to \infty$. So c_β is bounded below, and $A(\beta) \ge \frac{3}{2} + \inf_{t\ge 1/2} c_\beta(t) > -\infty$. The energy $L(\mu,\mu)$ converges absolutely by the diagonal estimate below. The domain is independent of β, and $\mathcal{E}_\beta$ is affine in β. So A is concave. A finite concave function on the open interval is continuous.

The mean bound is the bathtub inequality for the cap three. Let Ψ be measurable, $\Lambda \in \mathbb{R}$, and $0 \le f \le 3$ with $\int f = 1$. Then

$$\int \Psi f \ge \Lambda - 3\int_{1/2}^{\infty} (\Lambda - \Psi)_+\,dt. \tag{6.15}$$

Pointwise $(\Psi - \Lambda)f \ge -3(\Lambda - \Psi)_+$. Integrate and use $\int f = 1$. With $\Psi(t) = t$ and $\Lambda = 5/6$, (6.15) gives $\int t\,d\mu \ge 5/6 - 3\int_{1/2}^{5/6}(5/6 - t)\,dt = 2/3$. The density $3\,1_{[1/2,5/6]}$ attains this value.

Now let $d\rho(t) = v(t)\,dt$ be a bounded density on $[1/2,\infty)$ with $\int v = 0$ and $\int(1+t)|v(t)|\,dt < \infty$. Its energy converges absolutely. On $|t-s| < 1$, boundedness of v gives

$$\iint_{|t-s|<1} |v(t)v(s)|\,|\log|t-s||\,dt\,ds \le \|v\|_1\|v\|_\infty \int_{-1}^{1} |\log|u||\,du < \infty.$$

On $|t-s| \ge 1$, since $t, s \ge 1/2 > 0$,

$$0 \le \log|t-s| \le \log(1+t+s) \le t+s.$$

This is integrable against $|v(t)v(s)|\,dt\,ds$ by the first moment. Extend v by zero to $\mathbb{R}$, and put $\widehat{v}(u) = \int_{\mathbb{R}} e^{iut}v(t)\,dt$. Integrate the regularized cosine identity

$$\int_0^\infty e^{-\varepsilon u}\frac{\cos(ud) - \cos u}{u}\,du = \frac{1}{2}\log\frac{1+\varepsilon^2}{d^2+\varepsilon^2},$$

with $d = x - y$, against $v(x)v(y)\,dx\,dy$. For $0 < u \le 1$ the interchange follows from $|\cos(u(x-y)) - \cos u| \le u(1+|x-y|)$ and the first moment. For $u \ge 1$ the exponential factor gives absolute integrability. Since $\int v = 0$, the constant term disappears. Hence

$$\int_0^\infty e^{-\varepsilon u}\frac{|\widehat{v}(u)|^2}{u}\,du = -\frac{1}{2}\iint \log\bigl((x-y)^2+\varepsilon^2\bigr)v(x)v(y)\,dx\,dy.$$

The first-moment and zero-mass conditions give $|\widehat{v}(u)| = O(u)$ at the origin. For $0 < \varepsilon \le 1$ and $x \ne y$,

$$\left|\frac{1}{2}\log\bigl((x-y)^2+\varepsilon^2\bigr) - \log|x-y|\right| \le \frac{1}{2}\log\left(1 + \frac{1}{(x-y)^2}\right).$$

The right side is integrable against $|v(x)v(y)|\,dx\,dy$. It has only a logarithmic singularity on the diagonal and is bounded away from it. Dominated convergence

on the right and monotone convergence on the nonnegative left give

$$-L(\rho,\rho)=\int_0^\infty \frac{|\widehat{v}(u)|^2}{u}\,du\ge 0.$$

Put $L(\mu_1,\mu_2)=\iint \log|t-s|\,d\mu_1(t)\,d\mu_2(s)$. Let σ, U^σ and V_λ be as in the statement, and let μ be in the energy domain. Then the density of $\rho=\mu-\sigma$ is bounded, has zero mass and has finite first moment. The diagonal estimate above shows that $L(\mu,\mu)$ converges absolutely. With the compact support of σ, it also makes the mixed energy finite. So the zero-mass bound applies to ρ. Expanding the quadratic term gives

$$\begin{aligned}\mathcal{E}_\lambda(\mu)-\mathcal{E}_\lambda(\sigma)&=\int V_\lambda\,d\rho-2L(\sigma,\rho)-L(\rho,\rho)\\&=\int\big(V_\lambda-2U^\sigma\big)\,d(\mu-\sigma)-L(\rho,\rho).\end{aligned}$$

By $-L(\rho,\rho)\ge 0$,

$$\mathcal{E}_\lambda(\mu)\ge\mathcal{E}_\lambda(\sigma)+\int\Psi\,d(\mu-\sigma),\qquad \Psi=V_\lambda-2U^\sigma. \tag{6.16}$$

Apply (6.15) to the μ-integral. Since $\mathcal{E}_\lambda(\sigma)-\int\Psi\,d\sigma=3/2+L(\sigma,\sigma)$, this gives (6.14). The level Λ is a chosen certificate level and need not be the exact Euler–Lagrange threshold. This proves Lemma 6.7. □

7. Energy bounds

Proposition 7.1.

$$A(\log 67)\ge 6.2732861583,\qquad A(\log 70)\ge 6.3100053850.$$

Proof. Use a density σ constant on 256 cells of width $1/128$ on $[1/2,5/2]$. Its exact density on cell i is

$$\sigma_i=\frac{n_{i,q}}{2^{40}},\qquad i=0,\dots,255,$$

where the full integer arrays $n_{i,q}$ for $q=67$ and $q=70$ are printed in Appendix A. For each q, the exact checks are

$$\sum_{i=0}^{255}n_{i,q}=128\cdot 2^{40},\qquad 0\le n_{i,q}\le 3\cdot 2^{40}.$$

These are exactly the cap and mass conditions, and σ vanishes outside $[1/2,5/2]$. The reference levels are the exact rationals

$$\Lambda_{67}=3451654071/536870912,\qquad \Lambda_{70}=3477398357/536870912.$$

The directed interval calculation applies Lemma 6.7 to $\mu-\sigma$. It uses (6.14) with $\lambda=\log q$ and $\Lambda=\Lambda_q$. Each cell is split into 256 subcells of length 2^{-15}, and the bound is formed on each subcell. Let y_k be the cell walls where σ jumps, with jumps J_k. With $\chi_0(x)=x\log|x|-x$, we have $U^\sigma(t)=\sum_k J_k\chi_0(t-y_k)$ and $\chi_0''(x)=1/x$. This follows from $\int_y^{y'}\log|t-s|\,ds=\chi_0(t-y)-\chi_0(t-y')$. Put

$\chi_1(x) = \frac{1}{2}x^2 \log|x| - \frac{3}{4}x^2$, with $\chi_1(0) = 0$. Then $\chi_1' = \chi_0$ and $\chi_1''(x) = \log|x|$. Integrating twice over two cells at distance $d = |i-k|$ gives

$$L(\sigma,\sigma) = \frac{1}{128^2} \sum_{i,k=0}^{255} \sigma_i\sigma_k\big(-\log 128 + \chi_1(d+1) - 2\chi_1(d) + \chi_1(d-1)\big).$$

Also

$$V_\lambda(t) = (t+2)\log(t+2) - 4t\log t + 3(t-\tfrac{1}{2})\log(t-\tfrac{1}{2}) + \lambda t - 2 - \tfrac{3}{2}\log 2.$$

On a subcell $[x_0, x_1]$ with midpoint $\bar{x}$, no wall lies inside. Let Ψ_1 collect the terms $(t+2)\log(t+2)$, $3(t-\frac{1}{2})\log(t-\frac{1}{2})$, λt, the constant, and the terms $-2J_k\chi_0(t-y_k)$ with $J_k(t-y_k) < 0$ on the subcell. Let Ψ_2 collect $-4t\log t$ and the remaining terms $-2J_k\chi_0(t-y_k)$. Each term of Ψ_1 is convex there, and each term of Ψ_2 is concave. Then

$$\Psi(t) \geq \Psi_2(x_0) + \frac{\Psi_2(x_1) - \Psi_2(x_0)}{x_1 - x_0}(t - x_0) + \Psi_1(\bar{x}) + \Psi_1'(\bar{x})(t - \bar{x}) =: \underline{\Psi}(t).$$

The chord inequality for Ψ_2 and the midpoint tangent inequality for Ψ_1 prove this bound. Hence $(\Lambda - \Psi)_+ \leq (\Lambda - \underline{\Psi})_+$. Its integral is evaluated exactly from the two endpoint values of the affine function.

The calculation also checks the monotone tail. For $t \geq 5/2$, the support of σ lies in $[1/2, 5/2]$. Hence $U^\sigma(t) \leq \log(t - 1/2)$. Since $\Psi = V_\lambda - 2U^\sigma$, this upper bound gives the lower bound

$$\Psi(t) \geq \varphi_\infty(t) := V_\lambda(t) - 2\log(t - 1/2).$$

Here

$$B'(t) = \int_0^2 \frac{ds}{t+s} = \log\frac{t+2}{t} > 0,$$

$$h'(t) = \log\frac{t}{t-1/2} = \log\left(1 + \frac{1/2}{t-1/2}\right) \leq \frac{1/2}{t-1/2}.$$

Consequently

$$\varphi_\infty'(t) = B'(t) - 3h'(t) + \lambda - \frac{2}{t-1/2} \geq \lambda - \frac{7/2}{t-1/2}.$$

The tail test checks $\lambda(5/2 - 1/2) > 7/2$. So φ_∞ is strictly increasing on $[5/2, \infty)$. It also checks $\varphi_\infty(5/2) > \Lambda$. Then the tail part of the integral in (6.14) is zero.

The calculation uses the printed integers and the rational levels above. The outputs are rounded downward. Appendix C records the interval conventions. This proves Proposition 7.1. □

Lemma 7.2.

$$\tau(67) < 6.272107, \qquad \tau(-70) < 6.307708.$$

Proof. Here τ is as in Proposition 4.12. Both bounds follow from exact rational series enclosures of $\log 2$, $\log 5$, $\log 66$, $\log 67$, $\log 70$ and $\log 71$ (Appendix C). For $1 \leq x \leq 2$ and $z = (x-1)/(x+1)$, the script uses the first 32 terms of $\log x = 2\sum_{k\geq 0} z^{2k+1}/(2k+1)$ as a lower bound and adds $2z^{65}/(65(1-z^2))$ for an upper

bound. It applies this to $x = 2$, $x = 5/4$ and $x = n/64$, with $\log 5 = 2\log 2 + \log(5/4)$ and $\log n = 6\log 2 + \log(n/64)$. Rounding outward to twelve decimals gives

$$0.693147180559 \le \log 2 \le 0.693147180560,$$
$$1.609437912434 \le \log 5 \le 1.609437912435,$$
$$4.189654742026 \le \log 66 \le 4.189654742027,$$
$$4.204692619390 \le \log 67 \le 4.204692619391,$$
$$4.248495242049 \le \log 70 \le 4.248495242050,$$
$$4.262679877041 \le \log 71 \le 4.262679877042.$$

Insert the upper bounds for $\log 2$, $\log 66$, $\log 71$ and the lower bounds for $\log 5$, $\log 67$, $\log 70$ into $\tau(67) = S + \frac{9}{16}\log 66 - \frac{1}{8}\log 67$ and $\tau(-70) = S + \frac{9}{16}\log 71 - \frac{1}{8}\log 70$. This gives $\tau(67) < 6.2721062$ and $\tau(-70) < 6.3077075$. No decimal approximation is used as input to the script. This proves Lemma 7.2. □

By Proposition 7.1 and Lemma 7.2, the endpoint margins $A(\log|a|) - \tau(a)$ exceed 0.0011791583 and 0.0022973850, respectively.

8. Extension to both rays

Lemma 8.1. *For every integer $a \ge 67$ and every integer $a \le -70$, $\tau(a) < A(\log|a|)$.*

Proof. By Lemma 6.7, every density in the energy domain has mean at least $2/3$. Thus, for $\lambda_2 \ge \lambda_1$,

$$A(\lambda_2) - A(\lambda_1) \ge \frac{2}{3}(\lambda_2 - \lambda_1).$$

For $q \ge 67$ and $q \ge 70$, respectively,

$$\frac{d\tau_+(q)}{d\log q} = \frac{7}{16} + \frac{9}{16(q-1)} < \frac{2}{3}, \quad \frac{d\tau_-(q)}{d\log q} = \frac{7}{16} - \frac{9}{16(q+1)} < \frac{2}{3},$$

where $\tau_+(q) = S + (9/16)\log(q-1) - (1/8)\log q$ and $\tau_-(q) = S + (9/16)\log(q+1) - (1/8)\log q$. So the endpoint margins of Section 7 persist for all $a \ge 67$ and all $a \le -70$. This proves Lemma 8.1. □

9. Proof of Theorem 1

Fix an integer $a \ge 67$ or $a \le -70$. Put $\theta = \mathrm{Li}_3(1/a)$ and $\eta = A(\log|a|) - \tau(a)$. By Lemma 8.1, $\eta > 0$. Use the prime p of Section 5 and the sequence $m_k = 4p^k$.

We check the hypotheses of Fact 1 along m_k. Propositions 4.9 and 4.12 give the common denominator and its rate $\tau(a)$. Proposition 5.6 gives the nonvanishing. By Lemma 2.1, $G_{m,a}(\theta) = \det M(a)$. By Lemma 6.7, A is continuous, so $A(\log|a| - 1/m) \ge A(\log|a|) - \eta/2$ for large m. Then Proposition 6.6 gives

$$\log|G_{m_k,a}(\theta)| \le -\alpha(a)m_k^2 + O(m_k \log m_k), \qquad \alpha(a) = \tau(a) + \eta/2 > \tau(a).$$

By Fact 1, θ is irrational. Hence $\mathrm{Li}_3(1/a)$ is irrational for every integer $a \ge 67$ and every integer $a \le -70$.

Appendix A. Energy density data

Tables A.1 and A.2 list the exact integers of Proposition 7.1. The cell indexed by i is $[1/2 + i/128, 1/2 + (i+1)/128)$, and its density is $n_{i,q}/2^{40}$. Table A.1 gives $0 \le i \le 145$. For $146 \le i \le 255$, $n_{i,67} = n_{i,70} = 0$ except at the indices of Table A.2. Thin spaces group the digits in threes.

Table A.1. Exact density numerators, $0 \le i \le 145$.

i	$n_{i,67}$	$n_{i,70}$	i	$n_{i,67}$	$n_{i,70}$
0	3 298 534 883 327	3 298 534 883 327	73	743 218 245 060	743 511 482 073
1	3 298 534 883 328	3 298 534 883 326	74	733 586 957 103	733 767 821 479
2	3 298 534 883 328	3 298 534 883 327	75	724 063 979 600	724 130 743 460
3	3 298 534 883 328	3 298 534 883 327	76	714 633 804 543	714 593 289 424
4	3 298 534 883 328	3 298 534 883 327	77	705 308 750 924	705 153 775 130
5	3 298 534 883 327	3 298 534 883 328	78	696 066 324 313	695 805 840 165
6	3 298 534 883 328	3 298 534 883 327	79	686 922 964 168	686 544 817 400
7	2 837 740 076 112	2 929 507 514 113	80	677 857 175 716	677 374 129 835
8	2 579 830 538 424	2 611 327 406 754	81	668 876 902 316	668 276 999 478
9	2 420 350 848 703	2 449 121 767 729	82	659 974 539 219	659 263 335 684
10	2 294 701 993 027	2 318 020 934 797	83	651 142 927 803	650 321 381 950
11	2 192 096 474 768	2 212 493 695 329	84	642 391 628 006	641 453 846 118
12	2 104 830 956 933	2 123 027 534 365	85	633 704 219 837	632 653 649 882
13	2 028 842 799 377	2 045 379 653 516	86	625 080 483 510	623 912 675 091
14	1 961 473 219 432	1 976 652 892 798	87	616 521 521 312	615 240 699 720
15	1 900 927 502 170	1 915 015 957 026	88	608 019 131 905	606 625 153 563
16	1 845 934 087 324	1 859 082 459 821	89	599 575 574 097	598 061 446 150
17	1 795 535 515 486	1 807 861 656 432	90	591 185 928 237	589 549 766 310
18	1 749 010 864 383	1 760 639 654 259	91	582 848 535 025	581 100 692 187
19	1 705 798 199 911	1 716 803 774 837	92	574 557 530 595	572 688 805 669
20	1 665 449 974 681	1 675 878 497 878	93	566 312 373 091	564 312 299 860
21	1 627 601 319 628	1 637 527 753 883	94	558 104 866 998	555 985 724 845
22	1 591 957 568 799	1 601 418 725 634	95	549 936 942 805	547 695 163 261
23	1 558 268 103 429	1 567 299 451 837	96	541 808 525 406	539 442 069 156
24	1 526 320 584 358	1 534 958 961 513	97	533 713 846 910	531 219 581 633
25	1 495 951 949 651	1 504 224 975 802	98	525 646 641 070	523 024 911 727
26	1 467 005 274 918	1 474 941 261 234	99	517 613 032 258	514 850 639 023
27	1 439 352 411 825	1 446 956 609 275	100	509 604 272 948	506 705 211 717
28	1 412 866 625 887	1 420 174 760 856	101	501 611 184 173	498 580 651 667
29	1 387 465 778 549	1 394 488 509 386	102	493 641 522 146	490 470 433 716
30	1 363 058 541 086	1 369 813 935 750	103	485 680 445 204	482 373 025 645
31	1 339 562 218 678	1 346 060 868 220	104	477 737 566 559	474 283 381 510
32	1 316 910 439 071	1 323 161 144 637	105	469 808 416 003	466 203 539 235
33	1 295 052 317 437	1 301 071 918 389	106	461 883 680 884	458 124 976 063
34	1 273 921 017 813	1 279 716 705 206	107	453 963 450 933	450 049 162 369
35	1 253 469 800 685	1 259 057 340 936	108	446 038 949 108	441 968 015 039
36	1 233 662 027 062	1 239 031 295 744	109	438 109 998 046	433 874 159 402
37	1 214 444 257 538	1 219 619 623 647	110	430 173 202 823	425 771 932 843

i	$n_{i,67}$	$n_{i,70}$	i	$n_{i,67}$	$n_{i,70}$
38	1 195 781 292 404	1 200 769 050 550	111	422 225 237 858	417 648 576 104
39	1 177 655 977 549	1 182 453 637 751	112	414 263 161 675	409 510 245 335
40	1 160 022 716 275	1 164 643 621 899	113	406 281 503 958	401 347 210 061
41	1 142 854 413 729	1 147 299 899 432	114	398 277 869 721	393 149 925 051
42	1 126 132 555 707	1 130 413 478 835	115	390 242 177 196	384 914 683 880
43	1 109 826 747 486	1 113 937 917 264	116	382 170 159 759	376 639 440 886
44	1 093 914 731 233	1 097 864 291 468	117	374 058 105 221	368 313 025 791
45	1 078 379 453 115	1 082 173 058 218	118	365 901 334 013	359 939 628 731
46	1 063 196 481 520	1 066 845 728 286	119	357 689 075 425	351 501 518 313
47	1 048 361 885 565	1 051 854 361 760	120	349 425 425 359	342 991 622 755
48	1 033 842 814 171	1 037 190 665 847	121	341 091 151 314	334 403 410 713
49	1 019 631 483 916	1 022 837 185 572	122	332 685 882 194	325 728 691 544
50	1 005 713 829 553	1 008 780 972 902	123	324 194 572 447	316 955 017 593
51	992 075 334 790	995 003 374 441	124	315 611 287 347	308 077 451 459
52	978 706 838 780	981 494 813 545	125	306 919 583 928	299 072 081 975
53	965 585 937 121	968 245 797 513	126	298 111 860 995	289 929 969 617
54	952 708 561 847	955 242 631 847	127	289 179 240 754	280 634 653 157
55	940 073 288 207	942 473 944 193	128	280 101 948 147	271 169 346 841
56	927 656 029 308	929 927 854 857	129	270 861 978 998	261 506 685 525
57	915 451 600 997	917 600 361 123	130	261 435 465 903	251 626 042 707
58	903 452 952 463	905 475 184 724	131	251 799 593 167	241 497 299 128
59	891 652 586 642	893 558 048 776	132	241 932 466 883	231 079 622 233
60	880 034 697 345	881 819 050 375	133	231 795 533 278	220 324 018 894
61	868 606 778 811	870 269 251 598	134	221 350 513 342	209 197 790 185
62	857 350 646 483	858 895 616 896	135	210 548 734 743	197 609 810 721
63	846 256 331 106	847 673 557 743	136	199 324 638 055	185 484 583 517
64	835 322 671 871	836 638 048 384	137	187 608 721 688	172 698 700 639
65	824 547 349 521	825 740 663 935	138	175 288 089 867	159 096 648 491
66	813 920 621 765	815 001 854 700	139	162 223 414 127	144 438 578 346
67	803 433 680 585	804 401 390 536	140	148 219 665 899	128 367 873 920
68	793 084 619 741	793 934 955 833	141	132 963 147 892	110 356 816 428
69	782 871 016 033	783 603 957 520	142	116 030 425 593	88 695 358 111
70	772 772 717 523	773 407 088 892	143	96 134 925 979	64 384 502 640
71	762 807 632 126	763 318 213 555	144	73 807 739 715	0
72	752 955 771 076	753 361 338 835	145	25 028 165 403	0

TABLE A.2. The nonzero numerators with $146 \le i \le 255$.

i	$n_{i,67}$	$n_{i,70}$	i	$n_{i,67}$	$n_{i,70}$	i	$n_{i,67}$	$n_{i,70}$	i	$n_{i,67}$	$n_{i,70}$
203	0	1	218	0	1	231	15	1	244	1	1
206	0	1	219	0	1	232	0	1	245	0	2
207	0	1	220	0	1	233	0	1	246	1	2
208	0	1	221	0	1	234	0	1	247	1	1
209	0	1	222	0	1	235	0	1	248	1	2
210	0	1	223	0	1	236	0	1	249	1	1
211	0	1	224	0	1	237	0	1	250	1	2
212	0	1	225	0	1	238	0	1	251	0	1
213	0	1	226	0	1	239	0	2	252	1	3
214	0	1	227	0	1	240	0	1	253	0	3
215	0	1	228	0	1	241	1	1	254	1	1
216	0	1	229	0	1	242	1	1	255	2	1
217	0	1	230	0	1	243	1	2			

For both columns, $\sum_{i=0}^{255} n_{i,q} = 128 \cdot 2^{40}$ and $0 \le n_{i,q} \le 3 \cdot 2^{40}$. The numerators at most 15 in rows 203–255 are a mass correction. They absorb the rounding remainder, so that each column sums to exactly $128 \cdot 2^{40}$.

Appendix B. Seed polynomial data

Write $H_0(a) = \sum_{j=0}^{22} h_j a^j$. Table B.1 lists the exact coefficients, with digits grouped in threes.

Table B.1. Coefficients of H_0.

j	h_j
0	10 800 000 000
1	11 740 514 605 033 125
2	−2 234 728 309 789 895 625
3	12 503 964 285 688 089 124 039
4	−900 666 890 754 909 256 765 480
5	51 469 139 172 706 466 372 928 924
6	456 466 256 397 198 727 523 441 512
7	−33 207 248 244 575 743 619 207 381 680
8	565 491 299 064 578 808 379 816 924 560
9	−5 678 999 386 036 812 519 180 515 606 000
10	41 111 570 114 445 539 975 570 912 133 500
11	−228 978 443 522 344 205 691 600 197 760 000
12	992 547 355 262 121 306 502 065 534 960 000
13	−3 342 309 579 645 701 636 111 080 752 000 000
14	8 728 191 540 790 483 230 703 335 528 000 000
15	−17 641 091 042 330 545 125 131 806 080 000 000
16	27 452 159 949 368 316 719 117 877 120 000 000
17	−32 526 862 932 461 704 482 025 728 000 000 000
18	28 765 994 116 122 156 561 189 984 000 000 000
19	−18 360 677 750 742 784 090 690 560 000 000 000
20	7 979 736 698 623 281 827 450 880 000 000 000
21	−2 110 819 684 443 007 426 560 000 000 000 000
22	256 144 411 145 893 036 032 000 000 000 000

The seed identity and the residues of H_0 modulo 13 are stated in Lemma 5.1.

Appendix C. Computer checks

Table C.1 lists the scripts. They do the exact and directed-interval computations of the paper. They and their outputs accompany the paper as ancillary files. They run under Python 3.12.3. `cert.py` and `check_energy.py` need python-flint 0.9.0, which provides the Arb interval arithmetic; the other scripts use only the standard library. The prime-power row checks the pole parts at $p = 13$. This prime is admissible at $a = 67$ and $a = -70$. It is not the prime chosen in Section 5. Lemma 4.2 proves the pole finite parts. The proof uses neither of these two rows; they are diagnostic checks.

For Proposition 7.1, `cert.py` computes the certificate with 256 cells and 256 subcells per cell. The script `check_energy.py` reads the tables of Appendix A from the source of this paper. It then recomputes the bounds with the routine of `cert.py`. The subcells have length $1/(128 \cdot 256)$. On each subcell it splits Ψ into the convex part Ψ_1 and the concave part Ψ_2 of that proof. It evaluates $\Psi_2(x_0)$,

$\Psi_2(x_1)$, $\Psi_1(\bar{x})$ and $\Psi_1'(\bar{x})$ as Arb balls. These give the two endpoint values of the affine minorant $\underline{\Psi}$ as balls. It takes the upper endpoints of $\Lambda - \underline{\Psi}(x_0)$ and $\Lambda - \underline{\Psi}(x_1)$. The positive part of an affine function has an exact integral in terms of its two endpoint values. This integral increases with them. The upper endpoints of these subcell contributions are summed, so the result bounds the positive-part integral from above.

TABLE C.1. Exact or directed-interval computations.

Script	Statement checked; result
`check_energy.py`	printed energy tables (routine of `cert.py`): 256 cells, 256 subcells per cell, at each endpoint; exact mass and cap; directed interval arithmetic. PASS, about 5 min.
`check_seed.py`	seed identity: exact 4×4 determinant over $\mathbb{Q}[a]$ gives $G_{4,a}(0) = H_0(a)/(2^{18} \cdot 3^9 \cdot 5^{12} \cdot 7^4(a-1)^4)$; printed Table B.1 and residues mod 13. PASS, exact, under 1 s.
`check_tau.py`	arithmetic rates: $\tau(67) < 6.272107$ and $\tau(-70) < 6.307708$. PASS, exact.
`check_pole_parts.py`	pole finite parts: $m = 4, 8, \ldots, 32$; 6,528 equalities. PASS, about 7 s. Diagnostic, not used in the proof.
`check_tower.py`	prime-power diagnostic: $p = 13$, $Q = 169$, $m = 676$, at $a = 67, -70$; pole parts only. PASS, about 5 s. Diagnostic, not used in the proof.

USE OF AI TOOLS

The author devised the method of proof. Codex (OpenAI, GPT-5.6 Sol) wrote the Python code and the certificate tables. Claude Code (Anthropic, Claude Fable 5) checked the code, the parameters and the tables. The author checked every step of the proof, every table and every reference, and takes full responsibility for the content of the paper.

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Data availability. The scripts of Appendix C and their outputs accompany the paper as ancillary files.

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