IRRATIONALITY OF THE DILOGARITHM AT 1/3

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ABSTRACT. We prove that $\mathrm{Li}_2(1/3)$ is irrational. Irrationality of $\mathrm{Li}_2(1/s)$ was known for integers $s \le -5$ and $s \ge 6$ by work of Hata and of Rhin and Viola, and Prellberg recently proved it for $s = -4, 5, -3$. We define polynomials $G_m(T)$ with rational coefficients by an interpolation determinant. The rank of this determinant modulo primes p with $m/8 < p < 5m/4$ bounds the denominators of the coefficients. A concavity inequality for a logarithmic energy, combined with the inversion $v \mapsto 3/v$, bounds $G_m(\mathrm{Li}_2(1/3))$. Interval arithmetic is used only for the energy and the potentials of a printed reference measure, on a uniform dyadic subdivision. The integral of the local rank function is a rational sum over 76 printed affine pieces. The coefficients of G_4 and the 9×9 determinant modulo 17 are printed and can be checked by hand. A congruence modulo 17 shows that $G_m(\xi) \neq 0$ for every rational ξ and $m = 4 \cdot 17^t$, t large in terms of ξ.

1. INTRODUCTION

For $|z| \le 1$ let $\mathrm{Li}_2(z) = \sum_{k \ge 1} z^k/k^2$. Hata [5, Theorem 1.1] proved that $\mathrm{Li}_2(1/s)$ is irrational for every integer $s \le -5$ and every $s \ge 7$. Rhin and Viola settled $s = 6$ by their permutation group method [10, Section 6], and later treated negative arguments [11, Theorem 2.1 and Section 4]. Viola and Zudilin [13] introduced a six-parameter family of double integrals for dilogarithmic values. In 2024 Calegari, Dimitrov and Tang [2, Remark 2.8.2 and Section 15.2] listed the integers still open: $s \in \{-4, -3, -2, 2, 3, 4, 5\}$. Prellberg's 2026 preprint [9, Theorem 1.1] proves the cases $s = -4, 5, -3$. We prove the case $s = 3$. Our linear forms are values of an interpolation determinant of size $9n$, in place of the double integrals of [5, 9, 10, 13]. The rank of this determinant modulo primes controls the denominators of its coefficients.

Theorem 1. *The number* $\mathrm{Li}_2(1/3)$ *is irrational.*

By Landen's identity [8, (25.12.3)], $\mathrm{Li}_2(1/3) + \mathrm{Li}_2(-1/2) = -\frac{1}{2}\log^2(3/2)$, and $\log(3/2)$ is transcendental by the Hermite–Lindemann theorem, a special case of [1, Theorem 1.4]. So Landen's identity does not reduce the case $s = 3$ to the case $s = -2$.

Put $\theta = \mathrm{Li}_2(1/3)$ and $m = 4n$. Section 2 defines $G_m(T) \in \mathbb{Q}[T]$ by an interpolation determinant. Sections 3 and 4 give

$$\deg G_m \le m \text{ by Lemma 2.1}, \qquad K_m G_m(T) \in \mathbb{Z}[T], \qquad K_m \in \mathbb{Q}_{>0},$$

$$\limsup_{\substack{m \to \infty \\ 4 \mid m}} \frac{\log K_m}{m^2} \le \tau < 3.87273,$$

$$\log |G_m(\theta)| \le -3.8757\, m^2 + C\, m \log m \quad (m = 4n \ge 4). \tag{1.1}$$

The exact value of τ is in (3.12), and 3.8757 is a lower bound for the constant of (4.5). The finite constant C of Lemma 4.4 depends only on the reference measure. Since

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$3.87273 < 3.8757$, the quadratic exponent is negative. If $\theta = u/q_0$ with $q_0 > 0$, then $q_0^m K_m G_m(\theta) \in \mathbb{Z}$. Section 5 shows that it is nonzero for $m = 4 \cdot 17^t$ and t large, and (1.1) shows that it tends to zero.

The only numerical margin in the proof is $\alpha_0 - \tau > 0$. The size of C is not used. Lemma 4.2 is computer-assisted. It is proved by interval arithmetic on the reference measure of Tables C.1 and C.2, over the uniform subdivision of Lemma C.1. The integral (3.8) is an exact sum over the 76 affine pieces of Table A.2, each determined by the explicit lines of Table A.1. Tables 2.1, A.1 and A.2 and (5.2) can be checked by hand. Python scripts recheck Lemma 4.2 and part of the remaining proof (Appendix D). Apart from this lemma, the proof is complete without them.

The proof uses three classical inputs.

Fact 1 (prime number theorem [4, Theorems 6 and 420, §§1.8 and 22.4])**.** *$\theta_{\mathrm{Ch}}(y) = \sum_{p \le y} \log p$ satisfies $\theta_{\mathrm{Ch}}(y)/y \to 1$ as $y \to \infty$.*

Only this qualitative form is used.

Fact 2 ([12, Lemma I.1.8])**.** *If σ is a real compactly supported signed measure of total mass zero with $\iint |\log|u-w||\, d|\sigma|(u)\, d|\sigma|(w) < \infty$, then $\iint \log|u-w|\, d\sigma(u)\, d\sigma(w) \le 0$.*

A short proof is given in Appendix E.

Fact 3 (Cauchy–Vandermonde [7, Theorem 25], [3, Section 2])**.** *For $s = (s_1, \dots, s_m)$, $v = (v_1, \dots, v_L)$ and $h = L - m \ge 0$,*

$$\det \begin{pmatrix} ((s_{i'} + v_b)^{-1})_{1 \le i' \le m,\, 1 \le b \le L} \\ (v_b^{\ell-1})_{1 \le \ell \le h,\, 1 \le b \le L} \end{pmatrix} = \frac{\Delta(s)\Delta(v)}{\prod_{i',b}(s_{i'} + v_b)}, \qquad \Delta(u) = \prod_{i<j}(u_j - u_i).$$

A proof, including the sign, is given in the proof of Lemma 2.2.

The use of AI tools is described at the end of the paper.

2. The determinant

Put $a = 3$. For $n \ge 1$ let

$$m = 4n, \qquad \sigma = k = h = n, \qquad L = m + h,$$
$$I = \{\sigma + k, \dots, \sigma + k + m - 1\}, \quad J = \{\sigma, \dots, \sigma + 2m + h - 1\}, \quad \Lambda = \{1, \dots, h\},$$

all ordered increasingly. For $j \ge 0$ put

$$t_j = \sum_{u=1}^{j} \frac{a^{j-u}}{u}, \qquad b_j = \sum_{u=1}^{j} \frac{a^{j-u}}{u^2}; \qquad t_0 = b_0 = 0.$$

For $i \in I$, $j \in J$ and $\ell \in \Lambda$ let

$$C_{ij} = \begin{cases} (a^i - a^j)/(j-i), & i \ne j, \\ 0, & i = j, \end{cases} \quad B_{ij} = \begin{cases} (t_i - t_j)/(j-i), & i \ne j, \\ b_i, & i = j, \end{cases}$$

$$A_{ij} = a^i \delta_{ij}, \qquad V_{\ell j} = \frac{1}{j + \ell}, \qquad M_{ij} = \int_0^1 \int_0^1 \frac{x^i y^j}{xy - 3}\, dx\, dy,$$

and put

$$G_m(T) = \det \begin{pmatrix} C \\ B - TA \\ V \end{pmatrix}, \tag{2.1}$$

a determinant of size $2m + h = 9n$. For an indeterminate a the same formulas define

$$G_m(T;a) = \det \begin{pmatrix} C(a) \\ B(a) - TA(a) \\ V \end{pmatrix} \in \mathbb{Q}[a,T]; \qquad G_m(T) = G_m(T;3).$$

Lemma 2.1. *One has $G_m \in \mathbb{Q}[T]$, $\deg G_m \le m$, and*

$$M = B + \log(2/3)\, C - \theta A, \qquad G_m(\theta) = \det \begin{pmatrix} C \\ M \\ V \end{pmatrix}. \tag{2.2}$$

Proof. Write $\lambda = \log(2/3)$ and

$$\mathfrak{m}_j = \int_0^1 \frac{u^j}{u-3}\, du, \qquad \mathfrak{n}_j = \int_0^1 \frac{u^j \log(1/u)}{u-3}\, du.$$

Polynomial division and the geometric series give

$$\mathfrak{m}_0 = \lambda, \quad \mathfrak{n}_0 = -\theta, \qquad \mathfrak{m}_j = a\mathfrak{m}_{j-1} + \frac{1}{j}, \quad \mathfrak{n}_j = a\mathfrak{n}_{j-1} + \frac{1}{j^2} \quad (j \ge 1),$$
$$\mathfrak{m}_j = a^j\lambda + t_j, \qquad \mathfrak{n}_j = b_j - a^j\theta.$$

The substitution $u = xy$ gives

$$M_{ij} = \frac{\mathfrak{m}_i - \mathfrak{m}_j}{j-i} \quad (i \ne j), \qquad M_{ii} = \mathfrak{n}_i.$$

Hence the first identity in (2.2). Add λ times each C row to the corresponding row of $B - \theta A$. Only the m middle rows depend on T, and their coefficients are rational. □

Put $r = 2/5$ and $\gamma(\vartheta) = re^{i\vartheta}$ $(0 \le \vartheta \le 2\pi)$. Let $[3,1]$ be the segment from 3 to 1, and let

$$\Gamma = [3,1] + c_\Gamma \gamma, \qquad c_\Gamma = \frac{\log 3}{2\pi i},$$

a linear combination of paths with complex coefficients. For every integer d,

$$\int_\Gamma z^{d-1}\, dz = \begin{cases} (1-a^d)/d, & d \ne 0, \\ 0, & d = 0. \end{cases} \tag{2.3}$$

For $d \ne 0$ integrate z^d/d on each path. For $d = 0$ the segment gives $-\log 3$ and the circle gives $2\pi i c_\Gamma = \log 3$. The total variation is

$$T_r = 2 + r\log 3 = 2.43944\ldots < 3. \tag{2.4}$$

With Δ as in Fact 3, use variables $x = (x_1,\ldots,x_m)$, $v = (v_1,\ldots,v_L)$ and $z = (z_1,\ldots,z_m)$.

Lemma 2.2. *With product differentials $dx\, dv\, dz$,*

$$G_m(\theta) = \frac{3^{3m^2/2-m}}{(m!)^2 L!} \int_{[0,1]^m} \int_{[0,1]^L} \int_{\Gamma^m} \Phi(x,v,z)\, dz\, dv\, dx, \tag{2.5}$$

where

$$\Phi(x,v,z) = \frac{\Delta(x)^2\Delta(v)^2\Delta(z)^2 \prod_{d=1}^{m}\prod_{b=1}^{L}(z_d - v_b) \prod_{i'=1}^{m} x_{i'}^{\sigma+k+h} \prod_{b=1}^{L} v_b^{\sigma} \prod_{d=1}^{m} z_d^{-m-k}}{\prod_{i'=1}^{m}\prod_{b=1}^{L}(3 - x_{i'}v_b)}.$$

Proof. By (2.3), $C_{ij} = a^i \int_\Gamma z^{j-i-1}\,dz$. Put $f_i(v) = \int_0^1 x^i/(xv-3)\,dx$. Then

$$M_{ij} = \int_0^1 f_i(v) v^j\,dv, \qquad V_{\ell j} = \int_0^1 v^{\ell-1} v^j\,dv.$$

Set $e_\Delta = m(m-1)/2$ and $\mathbf{w} = (z_1,\dots,z_m,v_1,\dots,v_L)$. Multilinearity in the rows and antisymmetrization, applied to the L rows $[M;V]$, then to the m rows C, and then to the m functions f_i, give

$$G_m(\theta) = \frac{1}{(m!)^2 L!} \int_{[0,1]^m\times[0,1]^L\times\Gamma^m} D_z D_{zv} D_x D_{xv}\,dx\,dv\,dz,$$

where

$$D_z = \det(a^i z_d^{-i-1})_{i\in I,\,1\le d\le m} = (-1)^{e_\Delta} a^{m(\sigma+k)+e_\Delta} \Delta(z) \prod_d z_d^{-\sigma-k-m},$$

$$D_{zv} = \det(\mathbf{w}_b^j)_{1\le b\le 2m+h,\,j\in J} = (-1)^{mL} \Delta(z)\Delta(v) \prod_d z_d^\sigma \prod_b v_b^\sigma \prod_{d,b}(z_d - v_b),$$

$$D_x = \det(x_{i'}^i)_{i\in I,\,1\le i'\le m} = \Delta(x) \prod_{i'} x_{i'}^{\sigma+k},$$

$$D_{xv} = \det\begin{pmatrix} ((x_{i'}v_b-3)^{-1})_{1\le i'\le m,\,1\le b\le L} \\ (v_b^{\ell-1})_{1\le \ell\le h,\,1\le b\le L}\end{pmatrix} = \frac{(-1)^{mL} 3^{e_\Delta} \Delta(x)\Delta(v)\prod_{i'} x_{i'}^h}{\prod_{i',b}(3-x_{i'}v_b)}.$$

The last equality follows from the Cauchy–Vandermonde identity. Explicitly, for $s_{i'} = -3/x_{i'}$,

$$\det\begin{pmatrix} ((x_{i'}v_b-3)^{-1})_{i',b} \\ (v_b^{\ell-1})_{\ell,b}\end{pmatrix} = \prod_{i'} x_{i'}^{-1} \det\begin{pmatrix} ((s_{i'}+v_b)^{-1})_{i',b} \\ (v_b^{\ell-1})_{\ell,b}\end{pmatrix},$$

and $\Delta(s) = 3^{e_\Delta}\Delta(x)\prod_{i'} x_{i'}^{1-m}$, $\prod_{i',b}(s_{i'}+v_b) = (-1)^{mL}\prod_{i',b}(3-x_{i'}v_b)\prod_{i'} x_{i'}^{-L}$. Fact 3 and this row scaling give the formula for D_{xv}. Continuity covers $x_{i'} = 0$.

We prove Fact 3 as an identity of rational functions, with $s_{i'}, v_b$ independent indeterminates. We use induction on m, for all $h \ge 0$ at once. The case $m = 0$ is the Vandermonde identity. Let $m \ge 1$, let $v' = (v_1,\dots,v_{L-1})$, and let $\widehat{s}_{i'}$ denote s with $s_{i'}$ omitted. As a function of v_L, expansion along the last column writes the determinant as $\sum_{i'}(-1)^{i'+L} M_{i'}/(s_{i'}+v_L)$ plus a polynomial of degree at most $h-1$, where $M_{i'}$ is the minor without row i' and column L. The right side is a polynomial of degree $L-1 = m+h-1$ in v_L divided by $\prod_{i'}(s_{i'}+v_L)$, so it has the same form. By the induction hypothesis for $(\widehat{s}_{i'}, v')$, $M_{i'}$ is the right side of the fact for $(\widehat{s}_{i'}, v')$. We have $\Delta(s) = (-1)^{i'-1}\Delta(\widehat{s}_{i'})\prod_{j\ne i'}(s_j - s_{i'})$, and $\Delta(v) = (-1)^{L-1}\Delta(v')\prod_{b<L}(s_{i'}+v_b)$ at $v_L = -s_{i'}$. Hence, after the factors $s_j + v_L = s_j - s_{i'}$ $(j \ne i')$ and $s_{i'}+v_b$ $(b < L)$ cancel, the residue of the right side at $v_L = -s_{i'}$ is

$$(-1)^{i'+L} \frac{\Delta(\widehat{s}_{i'})\Delta(v')}{\prod_{j\ne i'}\prod_{b<L}(s_j+v_b)} = (-1)^{i'+L} M_{i'}.$$

So the residues agree, and the difference of the two sides is a polynomial in v_L of degree at most $h-1$, zero if $h = 0$. It vanishes at the $L-1 \ge h$ points $v_L = v_b$, $b < L$, where the determinant has two equal columns and $\Delta(v) = 0$. Hence it is zero.

The product of the four factors is $(-1)^{e_\Delta} a^{m(\sigma+k)+2e_\Delta}\Phi$. Since $m = 4n$, $e_\Delta = 2n(4n-1)$ is even and $m(\sigma+k)+2e_\Delta = 3m^2/2 - m$. All integrals converge absolutely: $3-xv \ge 2$ on $[0,1]^2$, $|z| \ge r$ on Γ, and Γ has finite variation. □

An exact calculation gives

$$G_4(T) = \frac{9}{67228000000000}\sum_{j=0}^{4} q_j T^j, \qquad 67228000000000 = 2^{11}5^9 7^5, \tag{2.6}$$

with the coefficients in Table 2.1.

TABLE 2.1. The integer coefficients in (2.6).

j	q_j
0	11390586457461030954563
1	−12440093793852170366 1570
2	50948627280770584998 3600
3	−92738228619136544640 0000
4	63301844929126809600 0000

3. The divisor

Put

$$N_{\mathrm{den}} = 10n-1, \qquad U = 11n-1, \qquad d_\ell = \mathrm{lcm}(1,\dots,\ell), \qquad \mathcal{D}_m = d_{N_{\mathrm{den}}}^{3m} d_U^n.$$

For $F \in \mathbb{Q}[T]$ let $v_p(F) = \min_j v_p([T^j]F)$. From now on k is a running index, and the parameter k of Section 2 equals n. Fix the seam $R = 8$. Primes $p \le m/R$ are cleared by $\mathcal{D}_m$ alone. For primes $m/R < p < 5m/4$ we use the rank bound of Lemma 3.2.

Lemma 3.1. *For an indeterminate a,*

$$P_m(a,T) := \mathcal{D}_m G_m(T;a) \in \mathbb{Z}[a,T]. \tag{3.1}$$

Proof. The denominators of a C entry divide $d_{N_{\mathrm{den}}}$. An off-diagonal B entry is a difference of harmonic sums divided by $j-i$, so its denominator divides $d^2_{N_{\mathrm{den}}}$, and so does that of b_i. The entries Ta^i are integral. The denominators of a V entry divide d_U. □

3.1. **Residue classes.** For $\rho > 0$ and $0 \le x < \rho$ let

$$\begin{aligned} K(x,\rho) &= \{k \ge 0 : 1/2 \le x + k\rho < 3/2\},\\ \mathcal{L}(x,\rho) &= \{l \ge 0 : 1/4 \le x + l\rho < 5/2\},\\ W(x,\rho) &= \{w \ge 1 : 0 < w\rho - x \le 1/4\}, \end{aligned} \tag{3.2}$$

and

$$\mathfrak{r}(x,\rho) = \min(2|K| + |W|, |\mathcal{L}|) - \mathbf{1}_{\{K=\{0\},\,\mathcal{L}=\{0,1\},\,W=\varnothing\}}, \qquad \mathfrak{c}(\rho) = 1 + \int_0^\rho \mathfrak{r}(x,\rho)\,dx. \tag{3.3}$$

Lemma 3.2. *Let p be a prime with $p^2 > U$, so that $p \ge 5$, and let $a = 3$. Then*

$$v_p(G_m) \ge -m - \sum_{u=0}^{p-1} \mathfrak{r}(u/m, p/m) = -m\mathfrak{c}(p/m). \tag{3.4}$$

Proof. Multiply the whole rows of C, $B - TA$ and V by p, p^2 and p. The resulting matrix $S_p(T)$ has $\det S_p(T) = p^{3m+n} G_m(T)$ and entries in $\mathbb{Z}_{(p)}[T]$, and $S_p(T) \equiv S_p(0) \pmod p$ because $p^2 TA \equiv 0$. Write $i = u + kp$, $j = u + lp$ with $0 \le u < p$. Only the harmonic terms with p in the denominator survive modulo p. Their quotients are p-units because all indices are less than p^2, and $a^p \equiv a \pmod p$, so

$$p t_{u+kp} \equiv a^u t_k, \qquad p^2 b_{u+kp} \equiv a^u b_k \pmod p.$$

The scaled C and B entries vanish modulo p between distinct residue classes modulo p. For a cross-class B entry, the numerator is p-integral after the row scaling and the remaining factor $p/(j-i)$ is zero modulo p. Within a class, for $j \neq i$, $j-i = p(l-k)$ with $0 < |l-k| < p$, so the quotient denominator is a unit. Diagonal entries are given by the separate cases $i = j$ in the definitions of C and B. Since $a^{u+kp} \equiv a^{u+k} \pmod p$, for $i \neq j$

$$pC_{ij} = \frac{a^i - a^j}{l-k} \equiv a^u \frac{a^k - a^l}{l-k} = a^u C_{kl}, \qquad p^2 B_{ij} = \frac{pt_i - pt_j}{l-k} \equiv a^u B_{kl} \pmod p.$$

A V row reduces to zero unless $\ell + u = pw$. Then its entry is $1/(l+w)$, without the factor a^u, and $l + w = (j+\ell)/p < p$. So, after permutations, the reduction has blocks with $2|K| + |W|$ rows and $|\mathcal{L}|$ columns, where $(x, \rho) = (u/m, p/m)$.

For $K = \{0\}$, $\mathcal{L} = \{0, 1\}$, $W = \varnothing$ the block, with its rows divided by the unit a^u, is

$$\begin{pmatrix} 0 & 1-a \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & -1 \end{pmatrix}, \tag{3.5}$$

of rank one for every p. Every other block has rank at most the minimum of its dimensions. So the rank of $S_p(0)$ modulo p is at most $\mathrm{rk}_p = \sum_u \mathfrak{r}(u/m, p/m)$. The matrix has $2m + n$ rows. Row operations over $\mathbb{Z}_{(p)}$, independent of T, make its last $2m + n - \mathrm{rk}_p$ rows divisible by p coefficientwise. Subtracting the scaling exponent $3m + n$ gives the first inequality in (3.4).

Every boundary in (3.2) has the form $x = c_0 + b\rho$ with $4c_0, b \in \mathbb{Z}$. At $\rho = p/m$, $m = 4n$, it lies in $m^{-1}\mathbb{Z}$. The $K, \mathcal{L}$ conditions are closed on the left and open on the right in x, and the W condition is $w\rho - 1/4 \leq x < w\rho$. So $\mathfrak{r}$ is constant on each $[u/m, (u+1)/m)$, and

$$\sum_{u=0}^{p-1} \mathfrak{r}(u/m, p/m) = m \int_0^{p/m} \mathfrak{r}(x, p/m)\, dx. \tag{3.6}$$ □

The lines that bound the sets (3.2) are $x = z_0 - k\rho$ ($z_0 \in \{1/4, 1/2, 3/2, 5/2\}$, $k \geq 0$) and $x = w\rho - 1/4$ ($w \geq 1$). The first meets the strip $0 < x < \rho$ exactly for $z_0/(k+1) < \rho < z_0/k$, the second exactly for $1/(4w) < \rho < 1/(4(w-1))$. For $k = 0$ and for $w = 1$ the upper endpoint is $+\infty$. So $x = z_0 - k\rho$ meets the strip for some $\rho \in (1/R, 5/4)$ if and only if $4z_0/5 - 1 < k < z_0 R$, and $x = w\rho - 1/4$ if and only if $1 \leq w < R/4 + 1$. Hence, for $1/R < \rho < 5/4$, the sets (3.2) change only on the 39 lines

$$\begin{gathered} x = 0, \quad x = \rho, \quad x = w\rho - 1/4 \quad (1 \leq w \leq 2), \\ x = 1/4 - k\rho \quad (0 \leq k \leq 1), \quad x = 1/2 - k\rho \quad (0 \leq k \leq 3), \\ x = 3/2 - k\rho \quad (1 \leq k \leq 11), \quad x = 5/2 - k\rho \quad (2 \leq k \leq 19). \end{gathered} \tag{3.7}$$

Cut $(1/R, 5/4)$ at the finitely many values of ρ where two of these lines meet in $0 \leq x \leq \rho$, and let $\mathcal{C}$ be the set of these cut points. For $\rho \in (1/R, 5/4) \setminus \mathcal{C}$ and each z_0 exactly one line $x = z_0 - k\rho$ meets $0 < x < \rho$, namely $k = \lfloor z_0/\rho \rfloor$, and exactly one line $x = w\rho - 1/4$ does, namely $w = \lfloor 1/(4\rho) \rfloor + 1$. These indices lie in the ranges of (3.7). So on each open interval of $(1/R, 5/4) \setminus \mathcal{C}$ the order of the lines in $0 < x < \rho$ is fixed, exactly five lines lie there, the sets (3.2) are constant on each of the six x-intervals between consecutive lines, and $\mathfrak{c}$ is affine. A change of ρ by δ moves each of these lines by $O(|\delta|)$ and the endpoint of the integral by δ. So the sets (3.2) change only on strips of total length $O(|\delta|)$. As $\mathfrak{r}$ is bounded, $|\mathfrak{c}(\rho) - \mathfrak{c}(\rho')| = O(|\rho - \rho'|)$, and $\mathfrak{c}$ is continuous. Table A.2 (Appendix A) lists 76 consecutive intervals with union $(1/R, 5/4)$ up to finitely many points, and the affine

function $\mathfrak{c}$ on each. Each row follows from (3.2)–(3.3) and (3.7). The affine functions of adjacent rows agree at all 75 internal joins. Summation of the rows gives

$$\int_{1/R}^{5/4} \mathfrak{c}(\rho)\, d\rho = \frac{580566971}{177365760}. \tag{3.8}$$

The configuration $K = \{0\}$, $\mathcal{L} = \{0, 1\}$, $W = \varnothing$ of (3.5) means $\max(1/2, 3/2 - \rho) \le x < 3/2$, $\max(1/4, 5/2 - 2\rho) \le x < 5/2 - \rho$ and $x < \rho - 1/4$. For $\rho < 5/4$ this is $\max(1/2, 3/2 - \rho, 5/2 - 2\rho) \le x < \rho - 1/4$, which is nonempty only for $\rho > 11/12$. So it occupies $5/2 - 2\rho \le x < \rho - 1/4$ for $11/12 < \rho < 1$, and $1/2 \le x < \rho - 1/4$ for $1 \le \rho < 5/4$. As $1/R < 11/12$, its contribution to (3.8) is

$$-\int_{11/12}^{1} (3\rho - 11/4)\, d\rho - \int_{1}^{5/4} (\rho - 3/4)\, d\rho = -\frac{5}{48}.$$

3.2. **Larger primes.** For $p > 5n$ define row potentials $\lambda_C = 0$, $\lambda_B(i)$, $\lambda_V(\ell)$ and column potentials $\chi(j)$ by

$$\begin{aligned}
5n < p < 6n: &\quad (\lambda_B, \lambda_V, \chi) = (1 + \mathbf{1}_{i \ge p},\ 1,\ \mathbf{1}_{j \ge p+2n}),\\
6n \le p < 8n: &\quad (\lambda_B, \lambda_V, \chi) = (\mathbf{1}_{i+p<10n},\ 1,\ \mathbf{1}_{j \ge p}),\\
8n \le p < 10n: &\quad (\lambda_B, \lambda_V, \chi) = (0,\ 1,\ \mathbf{1}_{j \ge p}),\\
10n \le p \le 11n - 1: &\quad (\lambda_B, \lambda_V, \chi) = (0,\ \mathbf{1}_{p-\ell \in J},\ 0),
\end{aligned}$$

and put

$$E_p^{\mathrm{out}} = \begin{cases} 19n - 2p, & 5n < p < 8n,\\ 11n - p, & 8n \le p \le U. \end{cases}$$

Lemma 3.3. *For a prime $5n < p \le U$ one has $v_p(G_m) \ge -E_p^{\mathrm{out}}$.*

Proof. A C entry has a pole only at $j = i + p$, of order at most one. A B entry has pole order at most

$$\mathbf{1}_{\max(i,j) \ge p} + \mathbf{1}_{\{\max(i,j) \ge p,\ i \equiv j \pmod p\}},$$

including the double pole of b_i for $i \ge p$. A V entry has a simple pole when $p \mid j + \ell$. As $p^2 > U$, there are no higher powers.

In the first range every B row has potential at least one, and two if $i \ge p$. A collision $j = i + p$ has $j \ge p + 2n$, and its column gives the second unit. In the second range all $i < p$. The columns $j \ge p$ cover the simple B poles and the C poles. A collision needs $i + p < 10n$, and its row gives the other unit. In the third range $i < p$ and $i + p \ge 10n$, so only simple B poles remain. In these three ranges every V row has potential one, which covers $j + \ell = p$ and $j + \ell = 2p$. In the fourth range $i, j < p$, only V poles remain, and the rows with $\lambda_V = 1$ are those containing one. So $\lambda_{\mathrm{row}} + \chi_j$ dominates every pole order. Multiply each row and column by p to the power of its potential. The potentials add up to $(10n - p) + n + (8n - p)$, $(8n - p) + n + (10n - p)$, $n + (10n - p)$ and $11n - p$ in the four ranges. □

3.3. **The factor at three.**

Lemma 3.4. *Let $N_m = \lceil 53m^2/48 \rceil - m$. Then*

$$P_m(a, T) \in a^{N_m} \mathbb{Z}[a, T]. \tag{3.9}$$

In particular $v_3(G_m) \ge N_m - v_3(\mathcal{D}_m)$.

Proof. Work in $\mathbb{Q}[[a]][T]$. Write

$$S_j(a)=\sum_{u\ge 1}\frac{a^u}{u^j},\qquad \mathsf{v}_k(i)=\begin{cases}(i-k)^{-1}, & i\neq k,\\ 0, & i=k,\end{cases}\quad \mathsf{z}_k(i)=\mathsf{v}_k(i)^2,\quad \mathsf{e}_k(i)=\mathbf{1}_{i=k}.$$

Subtract $S_1(a)$ times each C row from the paired row of $B-TA$. Comparison of coefficients, including the diagonal, gives

$$\begin{aligned}\begin{pmatrix}C\\ B-TA-S_1C\end{pmatrix}&=\sum_{k\ge 0}a^k\begin{pmatrix}\mathsf{v}_k(I)\\ -\mathsf{z}_k(I)-(T+S_2)\mathsf{e}_k(I)\end{pmatrix}\mathsf{e}_k(J)^t\\ &\quad+\sum_{k\ge 0}a^k\begin{pmatrix}\mathsf{e}_k(I)\\ \mathsf{v}_k(I)\end{pmatrix}\mathsf{v}_k(J)^t+\sum_{k\ge 0}a^k\begin{pmatrix}0\\ -\mathsf{e}_k(I)\end{pmatrix}\mathsf{z}_k(J)^t.\end{aligned}\tag{3.10}$$

For example, the off-diagonal lower entry is $\sum_{k\ge 0}a^k\mathsf{v}_k(i)\mathsf{v}_k(j)-(a^i+a^j)/(i-j)^2$, and the diagonal entry is $\sum_{k\ge 0}a^k\mathsf{v}_k(i)^2-a^i(T+S_2)=b_i-Ta^i$. Each coefficient of a involves finitely many terms.

Expand along the n rows of V, and expand each remaining $2m$-row minor by the rank-one summands in (3.10). A nonzero term selects $2m=8n$ distinct rank-one summands. At weight k at most

$$c_k=1+\mathbf{1}_{k\in J}+\mathbf{1}_{k\in I}$$

occur: the first summand needs $k\in J$, the third needs $k\in I$. All entries, including $T+S_2$, have nonnegative order in a. So every term has order at least the sum of the $8n$ smallest available weights. The infinite expansion is used coefficientwise. Modulo a^{N_m+1} only finitely many k occur, so Cauchy–Binet applies to a finite truncation.

There is one weight at each $0\le k<n$, two at each $n\le k<2n$, and three at each $2n\le k<6n$. Write $5n=3d+\epsilon'$, $0\le\epsilon'<3$. The minimum available-weight sum is

$$\begin{aligned}\sum_{k=0}^{n-1}k+2\sum_{k=n}^{2n-1}k+3\sum_{k=2n}^{2n+d-1}k+\epsilon'(2n+d)&=\frac{53n^2}{3}-4n+\frac{\epsilon'(3-\epsilon')}{6}\\ &=\left\lceil\frac{53n^2}{3}\right\rceil-4n=N_m.\end{aligned}$$

The minors of V do not depend on a,T, so $a^{N_m}\mid G_m(T;a)$ in $\mathbb{Q}[a,T]$. By (3.1), division by the monic a^{N_m} gives (3.9). Specialization at $a=3$ gives the last statement. □

3.4. **Summation over primes.** Put $e_0(p)=3m\lfloor\log_p N_{\mathrm{den}}\rfloor+n\lfloor\log_p U\rfloor=v_p(\mathcal{D}_m)$. With the first applicable line taking precedence, let

$$E_p=\begin{cases}e_0(3)-N_m, & p=3,\\ e_0(p), & p\le m/R \text{ or } p^2\le U \text{ or } p=5m/4,\\ m\mathfrak{c}(p/m), & m/R<p<5m/4,\\ E_p^{\mathrm{out}}, & 5m/4<p\le U,\\ 0, & p>U,\end{cases}\qquad K_m=\prod_p p^{E_p}\in\mathbb{Q}_{>0}.\tag{3.11}$$

Lemma 3.5. *One has $K_mG_m(T)\in\mathbb{Z}[T]$ for every $m=4n$, and $\log K_m\le\tau m^2+o(m^2)$ as $m\to\infty$, where*

$$\begin{aligned}A&=\frac{13}{4R}+\int_{1/R}^{5/4}\mathfrak{c}(\rho)\,d\rho+\frac{45}{32}=\frac{902042411}{177365760},\\ \tau&=A-\frac{53}{48}\log 3=3.872724286099666640\ldots,\end{aligned}\tag{3.12}$$

so that $\alpha_0 - \tau > 0.00305$ *for the constant* α_0 *defined after* (4.5) *in Section 4.*

Proof. By (3.6), $m\mathfrak{c}(p/m) \in \mathbb{Z}$. Lemmas 3.1, 3.2, 3.3 and 3.4 give $v_p(K_m G_m) \ge 0$ for every p. The exponent at three may be negative. Since $\mathfrak{r} \le |\mathcal{L}|$ and $\sum_u |\mathcal{L}(u/m, p/m)| = |J| = 9n$, every inner exponent is at most $13n$.

If $m \ge 11R^2/4$, then $p > m/R$ implies $p^2 > m^2/R^2 \ge 11m/4 > U$. Put

$$\eta_{\mathrm{PNT}}(X) = \sup_{y \ge X} |\theta_{\mathrm{Ch}}(y)/y - 1|.$$

By Fact 1, $\eta_{\mathrm{PNT}}(X) \to 0$. For an affine f on $[t_0, t_1] \subset [1/R, 11/4]$, Stieltjes integration by parts gives

$$m \sum_{t_0 m < p < t_1 m} f(p/m) \log p = m^2 \int_{t_0}^{t_1} f(\rho)\, d\rho + O_f\big(\eta_{\mathrm{PNT}}(m/R) m^2 + m \log m\big).$$

There are finitely many pieces, and the primes at the finitely many points of $\mathcal{C} \cup \{1/R, 5/4, 2, 11/4\}$, at most one for each point and each m, contribute $O(m \log m)$. For $p \le m/R$ the first powers contribute $(13m/4)\theta_{\mathrm{Ch}}(m/R) = \frac{13}{4R} m^2 + o(m^2)$, and the higher powers, including those of two and three,

$$O\Big(m \sum_{j=2}^{\lfloor \log_2 U \rfloor} \theta_{\mathrm{Ch}}(U^{1/j})\Big) = O(m^{3/2} \log(3m)).$$

Subtract $N_m \log 3$ once. For $p > 5m/4$ the exponent is $m\mathfrak{c}_{\mathrm{out}}(p/m)$ with

$$\mathfrak{c}_{\mathrm{out}}(\rho) = \begin{cases} 19/4 - 2\rho, & 5/4 < \rho < 2, \\ 11/4 - \rho, & 2 \le \rho \le 11/4, \end{cases} \qquad \int_{5/4}^{11/4} \mathfrak{c}_{\mathrm{out}}(\rho)\, d\rho = \frac{9}{8} + \frac{9}{32} = \frac{45}{32}.$$

With (3.8) this gives $A - \frac{53}{48} \log 3 = \tau$. The total error is $O(\eta_{\mathrm{PNT}}(m/R) m^2 + m^{3/2} \log(3m)) = o(m^2)$. The bounds $\log 3 > 2 \sum_{k=0}^{99} ((2k+1) 2^{2k+1})^{-1}$ and $\log 3 < 2 \sum_{k=0}^{99} ((2k+1) 2^{2k+1})^{-1} + 8/(3 \cdot 201 \cdot 2^{201})$ give the digits of τ and of $\alpha_0 - \tau$. □

4. Analytic estimate

In (2.5) there are m, $L = 5m/4$ and m variables x, v, z. Let

$$\mathcal{K}_x = \mathcal{K}_v = [0, 1], \qquad \mathcal{K}_z = [1, 3] \cup \{z : |z| = r\}.$$

For finite positive measures put

$$I(\sigma, \sigma') = \iint \log|u - w|\, d\sigma(u)\, d\sigma'(w), \qquad D(\sigma, \sigma') = \iint \log|3 - uw|\, d\sigma(u)\, d\sigma'(w),$$
$$\mathcal{Q}(\mu) = I(\mu_x, \mu_x) + I(\mu_v, \mu_v) + I(\mu_z, \mu_z) + I(\mu_v, \mu_z) - D(\mu_x, \mu_v).$$

For a fixed $\nu = (\nu_x, \nu_v, \nu_z)$ let $\mathcal{U}_i(q) = \int \log|q - t|\, d\nu_i(t)$, $D_i(q) = \int \log|3 - qt|\, d\nu_i(t)$ $(i = x, v)$,

$$\psi_x = 2\mathcal{U}_x - D_v, \qquad \psi_v = 2\mathcal{U}_v + \mathcal{U}_z - D_x, \qquad \psi_z = 2\mathcal{U}_z + \mathcal{U}_v,$$
$$F_x(q) = \psi_x(q) + \tfrac{3}{4} \log q, \quad F_v(q) = \psi_v(q) + \tfrac{1}{4} \log q \quad (0 < q \le 1),$$
$$F_z(q) = \psi_z(q) - \tfrac{5}{4} \log|q| \quad (q \in \mathcal{K}_z).$$

For a compact line segment or circle $\Gamma \subset \mathbb{C}$ let ℓ_Γ be arclength on Γ. A segment meets the open disk $B(q, t) = \{w : |w - q| < t\}$ in length at most $2t$. A circle meets it in arclength at most $2\pi t$. If t is at least the radius, this follows from the length of the circle. Otherwise the intersection is an arc of at most a semicircle, of length at most $\pi/2$ times its chord. So $\ell_\Gamma(B(q, t)) \le 2\pi t$ for all $q \in \mathbb{C}$, $t > 0$. A positive measure η is *tame* if $\eta \le H(\ell_{\Gamma_1} + \cdots + \ell_{\Gamma_N})$ for some $H \ge 0$ and compact segments or circles $\Gamma_1, \ldots, \Gamma_N$, that

is, if η is a finite sum of measures with arclength density at most H on finitely many segments and circles. Tame measures are finite and compactly supported, and sums of tame measures are tame. Let $\mathfrak{M}$ be the set of triples $\mu = (\mu_x, \mu_v, \mu_z)$ of tame measures of masses $(1, 5/4, 1)$ with $\operatorname{supp}\mu_x \subseteq \{|q| \le 19/16\}$ and $\operatorname{supp}\mu_v \subseteq \{\zeta \le |q| \le 19/16\}$ for some $\zeta > 0$. On these supports $|3 - uw| \ge 3 - (19/16)^2 > 1$.

Lemma 4.1. *For $\mu, \nu \in \mathfrak{M}$, with ψ_x, ψ_v, ψ_z formed from ν,*

$$\mathcal{Q}(\mu) \le -\mathcal{Q}(\nu) + \int \psi_x \, d\mu_x + \int \psi_v \, d\mu_v + \int \psi_z \, d\mu_z. \tag{4.1}$$

Proof. We first show that for tame η, η'

$$\iint |\log|u - w|| \, d\eta(u) \, d\eta'(w) < \infty.$$

By the definition of tame it suffices to take $\eta = \ell_\Gamma$, $\eta' = \ell_{\Gamma'}$. Let $\Lambda \ge 1$ bound $|u - w|$ on $\Gamma \times \Gamma'$. For $u \in \Gamma$ the layer-cake formula and the bound on $\ell_{\Gamma'}(B(u,t))$ give

$$\int |\log|u - w|| \, d\ell_{\Gamma'}(w) \le \int_0^\infty \ell_{\Gamma'}(B(u, e^{-s})) \, ds + \ell_{\Gamma'}(\mathbb{C}) \log\Lambda \le 2\pi + \ell_{\Gamma'}(\mathbb{C}) \log\Lambda,$$

and integration over u gives the bound.

Let V be the space of real signed measures $\eta - \eta'$ with η, η' tame and $\eta(\mathbb{C}) = \eta'(\mathbb{C})$. By this bound I is finite on $V \times V$ and on pairs of tame measures, so all pairings below are finite. For $\sigma = \eta - \eta' \in V$ one has $|\sigma| \le \eta + \eta'$, and Fact 2 gives

$$\|\sigma\|^2 = -I(\sigma, \sigma) \ge 0.$$

So $-I$ is a positive semidefinite symmetric form on V, and $|I(\sigma, \sigma')| \le \|\sigma\| \, \|\sigma'\|$.

Put $\sigma_i = \mu_i - \nu_i \in V$, $\iota(v) = 3/v$ and $\beta = \iota_* \sigma_v$. The v-parts are supported in $A = \{\zeta \le |q| \le 19/16\}$ (the smaller ζ). The map ι sends $\Gamma \setminus \{0\}$ into a line or circle and A into $\{|q| \le 3/\zeta\}$, and $|\iota'(v)| = 3/|v|^2 > 1$ on A. So $\iota_*(\ell_\Gamma|_A) \le \ell_{\Gamma'}$ for a compact segment or circle Γ'. Hence $\iota_*\mu_v$ and $\iota_*\nu_v$ are tame, $\beta \in V$, and $\log|v|$ is bounded on A. From

$$\log|\iota(v) - \iota(w)| = \log 3 + \log|v - w| - \log|v| - \log|w|,$$
$$\log|3 - xv| = \log|x - \iota(v)| + \log|v|,$$

and the zero masses,

$$I(\beta, \beta) = I(\sigma_v, \sigma_v), \qquad D(\sigma_x, \sigma_v) = I(\sigma_x, \beta).$$

With $s_x = \|\sigma_x\|$, $s_v = \|\sigma_v\| = \|\beta\|$ and $s_z = \|\sigma_z\|$,

$$\begin{aligned}\mathcal{Q}(\sigma) &= -s_x^2 - s_v^2 - s_z^2 + I(\sigma_v, \sigma_z) - I(\sigma_x, \beta) \le -s_x^2 - s_v^2 - s_z^2 + s_v s_z + s_x s_v \\ &= -(s_x - s_v/2)^2 - (s_z - s_v/2)^2 - s_v^2/2 \le 0.\end{aligned}$$

Let $\mathcal{Q}(\cdot,\cdot)$ be the symmetric bilinear form with $\mathcal{Q}(\mu,\mu) = \mathcal{Q}(\mu)$. Then $2\mathcal{Q}(\nu,\mu) = \int \psi_x \, d\mu_x + \int \psi_v \, d\mu_v + \int \psi_z \, d\mu_z$ and

$$\mathcal{Q}(\mu) = \mathcal{Q}(\nu) + 2\mathcal{Q}(\nu, \sigma) + \mathcal{Q}(\sigma) \le 2\mathcal{Q}(\nu, \mu) - \mathcal{Q}(\nu). \qquad \square$$

Lemma 4.2 (computer-assisted)**.** *The measure ν of Tables C.1 and C.2 lies in $\mathfrak{M}$,* $\operatorname{supp}\nu_i \subseteq \mathcal{K}_i$, *and*

$$\mathcal{Q}(\nu) \ge q_- := -5.193572, \tag{4.2}$$

$$F_x \le K_x := -4.83148, \qquad F_v \le K_v := -4.74318, \qquad F_z \le K_z := 0.04318 \tag{4.3}$$

on $(0,1]$, $(0,1]$ and $\mathcal{K}_z$. On $|z| = r$ even $F_z \le 0.04283$.

Proof. By Appendix C, ν_x, ν_v and the interval part ν_z^{int} of ν_z are sums of 131, 143 and 105 intervals $[t_0, t_1]$ of positive mass w, each carrying $w\,dt/(t_1-t_0)$. The intervals of each part form a chain of adjacent intervals, contained in $[v_0, 1]$ for ν_v, $v_0 = 0.00960735979838$. The circular part ν_z^{Poi} is a sum of 167 Poisson measures (c, w) with $w > 0$ and $|c| < r$,

$$\frac{w(r^2-|c|^2)}{2\pi|re^{i\vartheta}-c|^2}\,d\vartheta \qquad (0 \le \vartheta \le 2\pi).$$

Its arclength density is at most H_c in (4.7). The masses are 1, 5/4 and 1, and ν_z is invariant under conjugation. Let H_x, H_v, H_z be the largest values of $w/(t_1-t_0)$ over the intervals of the three parts. So the parts of ν are tame, $\operatorname{supp}\nu_v \subseteq [v_0, 1]$, and $\nu \in \mathfrak{M}$. An interval has potential

$$\frac{w}{t_1-t_0}\operatorname{Re}\{f(q-t_0)-f(q-t_1)\}, \qquad f(u) = u\log u - u,$$

with $u\log|u| - u$ on the real line and $0\log 0 = 0$. A Poisson measure has potential

$$w\begin{cases}\log|r^2 - q\bar{c}| - \log r, & |q| \le r,\\ \log|q-c|, & |q| \ge r.\end{cases} \tag{4.4}$$

Energy. Interval pairs use $g(u) = \frac{1}{2}u^2\log|u| - \frac{3}{4}u^2$, with $g''(u) = \log|u|$. A pair of Poisson measures (c, w), (c', w') gives $ww'(\log|r^2 - c\overline{c'}| - \log r)$, by (4.4) and harmonicity. Interval–Poisson pairs integrate (4.4) with the primitives of $\log|t-c|$ and $\log|r^2 - t\bar{c}|$, split at $t = r$. For the denominator, with $\mathsf{M}_j^i = \int t^j\,d\nu_i$ and $N = 150$,

$$D(\nu_x, \nu_v) = \frac{5}{4}\log 3 - \sum_{j=1}^{N}\frac{\mathsf{M}_j^x\mathsf{M}_j^v}{j3^j} - \mathcal{E}_N, \qquad 0 \le \mathcal{E}_N \le \frac{5/4}{2(N+1)3^N}.$$

Table B.1 gives the outward enclosures, at 320-bit precision. The enclosure of $D(\nu_x, \nu_v)$ contains the partial sum minus $[0, (5/4)/(2(N+1)3^N)]$. The interval sum of the rows gives $\mathcal{Q}(\nu) > q_- + 5\cdot 10^{-7}$.

Small arguments. Put $\varepsilon_0 = 10^{-12}$. For $0 < q \le \varepsilon_0$, $\mathcal{U}_x(q)$,$\mathcal{U}_v(q) \le 0$, $\mathcal{U}_z(q) \le \log(3+\varepsilon_0)$, $D_v(q) \ge \frac{5}{4}\log(3-\varepsilon_0)$ and $D_x(q) \ge \log(3-\varepsilon_0)$. Hence

$$F_x(q) \le -\tfrac{5}{4}\log(3-\varepsilon_0) + \tfrac{3}{4}\log\varepsilon_0 < -22.09,$$
$$F_v(q) \le \log(3+\varepsilon_0) - \log(3-\varepsilon_0) + \tfrac{1}{4}\log\varepsilon_0 < -6.90.$$

Both are below the caps.

Subdivision. The bounds on $[\varepsilon_0, 1]$, $[\varepsilon_0, 1]$, $[1, 3]$ and the upper half of $|z| = r$ are proved on the 126464 closed pieces (leaves) of Lemma C.1. The lower half of the circle follows by conjugation. All bounds are computed in Arb ball arithmetic [6] at 192-bit precision.

Let $[l, u]$ be a leaf on a real support, with centre c_0 and half-width h_0. The interval components of F come from the chains of ν_x on x-leaves and of ν_v, ν_z^{int} on v- and z-leaves. The endpoints of the chains of ν_x, ν_v, ν_z^{int} cut the initial pieces of the x-, v-, z-support. The chain of ν_z^{int} misses the v-support, and that of ν_v meets the z-support only at 1. So the open leaf meets at most one interval $[t_0, t_1]$ of each of these chains, and then lies in it. A v-leaf lies in $[0, r]$ or in $[r, 1]$. Write $F = F_C + F_N$, where F_C is the sum of the interval components of F, with their coefficient 2, whose interval contains $[l, u]$. Inside its interval a component is convex, since $\mathcal{U}'' = w/((q-t_0)(t_1-q)) > 0$ there, so F_C lies below its chord on $[l, u]$. With $B \ge \max\{0, \sup_{(l,u)} F_N''\}$, Taylor's formula with integral remainder for F_N at c_0 gives

$$\max_{[l,u]} F \le \max_{q\in\{l,u\}}\big(F_C(q) + F_N(c_0) + F_N'(c_0)(q-c_0)\big) + \tfrac{1}{2}h_0^2 B,$$

because the chord plus an affine function is affine. The other interval components have $\mathcal{U}'' = -w/((q-t_0)(q-t_1)) \le 0$ on (l,u). The terms $\frac{3}{4}\log q$ and $\frac{1}{4}\log q$ are concave, and $(-\frac{5}{4}\log q)'' \le \frac{5}{4}l^{-2}$ on $[l,u]$. The terms $-D_v$, $-D_x$ are expanded as $-M\log 3 + \sum_{j\ge 1} \mathsf{M}_j q^j/(j3^j)$ with the first $N' = 90$ moments. The omitted value is at most $M/(2(N'+1)3^{N'})$, the omitted derivative lies in $[0, M/(2\cdot 3^{N'})]$, and the second derivative is at most $M/4$. On the real line a Poisson measure has second derivative $w\,\mathrm{Re}(-\bar{c}^2/(r^2-q\bar{c})^2) \le w|c|^2/|r^2-q\bar{c}|^2$ for $|q|\le r$ and $w\,\mathrm{Re}(-(q-c)^{-2}) \le w/|q-c|^2$ for $|q|\ge r$; B uses enclosures of these second derivatives over the leaf.

On an arc $q = re^{i\vartheta}$, $|\vartheta-\vartheta_0| \le h_0$, of the circle we take the smaller of two bounds. The first is

$$F(\vartheta_0) + h_0|F'(\vartheta_0)| + \tfrac{1}{2}h_0^2 B, \qquad B \ge \max\{0, \sup F''\},$$

with derivatives in ϑ and

$$\frac{d}{d\vartheta}\log|q-s| = \mathrm{Re}\,\frac{iq}{q-s}, \qquad \frac{d^2}{d\vartheta^2}\log|q-s| = \mathrm{Re}\,\frac{qs}{(q-s)^2} \le \frac{r|s|}{|q-s|^2}.$$

Here B uses enclosures over the arc of these second derivatives for the Poisson terms ($s=c$). For an interval component of mass w and density H on $[t_0,t_1]$ it uses the smaller upper end of two enclosures: of the exact second derivative $H\,\mathrm{Re}\big(q\log\frac{q-t_1}{q-t_0} + \frac{q^2}{q-t_1} - \frac{q^2}{q-t_0}\big)$, and of $w\,\mathrm{Re}(qs/(q-s)^2)$ with s ranging over $[t_0,t_1]$. The second bound uses that the distance from $re^{i\vartheta}$ to a point $s\ge 0$ increases with $\vartheta\in[0,\pi]$. It bounds the real-interval terms by their value at the right end of the arc and applies the first bound to the Poisson terms alone. On the arc starting at $\vartheta = 0$ only the second bound is used.

On every leaf the upper end of the resulting bound is below the cap of its support in Table B.2: K_x, K_v, K_z on $[1,3]$, and $0.04283 < K_z$ on the circle. Table B.2 gives the largest upper end on each support. The least margin, attained on $[1,3]$, is at least $2\cdot 10^{-6}$. □

For this ν put

$$\alpha = q_- - \tfrac{3}{2}\log 3 - K_x - \tfrac{5}{4}K_v - K_z. \tag{4.5}$$

Since $\frac{3}{2}\log 3 < 1647919/10^6$,

$$\alpha > \alpha_0 := q_- - \frac{1647919}{10^6} - K_x - \frac{5}{4}K_v - K_z = 3.875784.$$

Lemma 4.3. *Let η be a compactly supported positive measure of mass M with $\eta(B(q,t)) \le At$ for all $q\in\mathbb{C}$, $t>0$. If $0 < d = |q-q'| \le 1/3$, then*

$$\Big|\int \log|q-t|\,d\eta(t) - \int \log|q'-t|\,d\eta(t)\Big| \le (6A+2M)\,d\,(1+\log(1/d)).$$

The components of the reference of Lemma 4.2 are compactly supported, and for them one may take

$$A_x = 2H_x, \qquad A_v = 2H_v, \qquad A_z = 2H_z + \sum_c \frac{w_c(r+|c|)}{r(r-|c|)}, \tag{4.6}$$

the sum over the 167 *Poisson measures (c,w_c). It is finite, since $|c|<r$ for every centre.*

Proof. The closed disk $\{w : |w-q|\le s\}$ is the intersection of the $B(q,s')$, $s'>s$, so its η-measure is also at most As. Let $N = \{t : |q-t|\le 2d\}$. On N both distances are at

most $3d \le 1$, and the layer-cake formula gives

$$\int_N \log \frac{1}{|q-t|}\, d\eta \le 2Ad \log \frac{1}{2d} + 2Ad,$$
$$\int_N \log \frac{1}{|q'-t|}\, d\eta \le 2Ad \log \frac{1}{3d} + 3Ad.$$

Off N, $|q'-t| \ge |q-t|/2$, the logarithmic difference is at most $2d/|q-t|$, and

$$\int_{|q-t|>2d} \frac{d\eta(t)}{|q-t|} \le \int_0^1 M\, ds + \int_1^{1/(2d)} \frac{A}{s}\, ds = M + A \log \frac{1}{2d}.$$

The sum is at most $6Ad \log(1/d) + 5Ad + 2Md$.

By the bounds on $\ell_\Gamma(B(q,t))$ before Lemma 4.1, an interval of density at most H_i gives $2H_i$, and a Poisson measure gives $2\pi H_c$: since $|re^{i\vartheta} - c| \ge r - |c|$ and arclength is $r\, d\vartheta$, its arclength density is at most

$$H_c = \frac{w(r+|c|)}{2\pi r(r-|c|)}. \tag{4.7}$$

□

Put $M_x = 1$, $M_v = 5/4$, $M_z = 1$, and

$$B_i = 6A_i + 2M_i, \qquad S = 2B_x + \tfrac{7}{2}B_v + \tfrac{13}{4}B_z, \qquad C = 10 + 3S. \tag{4.8}$$

S and C are finite and depend only on ν.

Lemma 4.4. *For every $m = 4n \ge 4$,*

$$\begin{aligned}\log|G_m(\theta)| \le\ & -\alpha m^2 + \tfrac{13}{2} m \log m + 3S\big(1 + \log(m^2/3)\big) + \tfrac{45}{4} \\ & + m \log(T_r/3) - \log\big((m!)^2 L!\big),\end{aligned} \tag{4.9}$$

and hence $\log|G_m(\theta)| \le -\alpha_0 m^2 + Cm \log m$ with C as in (4.8).

Proof. Expand Γ^m in (2.5) into products of its pieces and take absolute values. By (2.4) the paths contribute a factor at most T_r^m, and $[0,1]^m \times [0,1]^L$ has volume one. So it suffices to bound $\log|\Phi|$ for $x \in [0,1]^m$, $v \in [0,1]^L$, $z \in \mathcal{K}_z^m$. If $\Phi = 0$ there is nothing to prove. Otherwise $x_{i'}, v_b > 0$ and the points of each species are distinct. Let μ_x, μ_v, μ_z put mass $1/m$ at each $x_{i'}$, v_b, z_d. Since $\sigma + k + h = 3m/4$, $\sigma = m/4$ and $m + k = 5m/4$,

$$\frac{\log|\Phi|}{m^2} = \mathcal{Q}_{\mathrm{disc}}(\mu) + \frac{3}{4}\int \log x\, d\mu_x + \frac{1}{4}\int \log v\, d\mu_v - \frac{5}{4}\int \log|z|\, d\mu_z, \tag{4.10}$$

where $\mathcal{Q}_{\mathrm{disc}}$ omits the diagonal in the three self-energies.

Put $\epsilon = m^{-2} \le 1/16$. Replace every atom q by the uniform measure of mass $1/m$ on the circle $q + 2i\epsilon + \epsilon e^{i\vartheta}$, and let μ^ϵ be the result. Since $\frac{1}{2\pi}\int_0^{2\pi} \log|u + \epsilon e^{i\vartheta}|\, d\vartheta = \log\max(|u|, \epsilon)$ and the common translation cancels in differences, averaging does not decrease any off-diagonal positive pair. Each circle has energy $\log\epsilon$, and there are $13m/4$ atoms. The function $\log|3 - XV|$ is harmonic in each variable on the product of two smearing disks, since $|X|, |V| \le 1 + 3\epsilon \le 19/16$. Its mean is $\log|3 - (x+2i\epsilon)(v+2i\epsilon)| \le \log(3 - xv) + 3\epsilon$, because $|(x+2i\epsilon)(v+2i\epsilon) - xv| \le 4\epsilon + 4\epsilon^2$ and $3 - xv \ge 2$. With the masses $1 \cdot 5/4$,

$$\mathcal{Q}_{\mathrm{disc}}(\mu) \le \mathcal{Q}(\mu^\epsilon) + \frac{13}{4m}\log\frac{1}{\epsilon} + \frac{15}{4}\epsilon. \tag{4.11}$$

The measures μ^ϵ and ν lie in $\mathfrak{M}$: μ^ϵ is a sum of uniform measures on circles, inside $\{|q| \le 19/16\}$ for x and v, with $\operatorname{Im} q \ge \epsilon$ on the v-circles. For ν this is part of Lemma 4.2.

By Lemmas 4.1 and 4.2,

$$\mathcal{Q}(\mu^\epsilon) \le -q_- + \sum_i \int \psi_i \, d\mu_i^\epsilon.$$

Each point of a smearing circle lies within $d = 3\epsilon \le 3/16$ of the atom q it replaces. Since $s \mapsto s(1+\log(1/s))$ increases on $(0,1]$, Lemma 4.3 bounds the change of $\mathcal{U}_i$ by $B_i d(1+\log(1/d))$. For $q \in [0,1]$, $|p-q| \le d$ and $0 \le t \le 1$, $|\nabla_p \log|3-pt|| \le t/|3-pt| \le 1$, so D_i changes by at most $M_i d$. Counting masses, the coefficients of B_x, B_v, B_z are 2, $\frac{5}{4}\cdot 2 + 1 = \frac{7}{2}$ and $\frac{5}{4} + 2 = \frac{13}{4}$, and the D terms give $\frac{5}{4} + \frac{5}{4} = \frac{5}{2}$. Hence

$$\sum_i \int \psi_i \, d\mu_i^\epsilon \le \sum_i \int \psi_i \, d\mu_i + Sd(1+\log(1/d)) + \tfrac{5}{2}d.$$

The atoms lie in $(0,1]$, $(0,1]$ and $\mathcal{K}_z$. Adding the original external terms of (4.10) turns ψ_i into F_i, and (4.3) gives

$$\frac{\log|\Phi|}{m^2} \le -q_- + K_x + \tfrac{5}{4}K_v + K_z + \frac{13}{4m}\log\frac{1}{\epsilon} + 3S\epsilon\Big(1+\log\frac{1}{3\epsilon}\Big) + \frac{45}{4}\epsilon,$$

with $45/4 = 15/4 + 3\cdot 5/2$. Multiply by m^2, put $\epsilon = m^{-2}$, add the logarithm $(3m^2/2 - m)\log 3 - \log((m!)^2 L!)$ of the prefactor in (2.5) and $m\log T_r$. By (4.5) this is (4.9).

For the last statement drop $-\log((m!)^2 L!)$ and $m\log(T_r/3) < 0$. As $m \ge 4$, $3S(1 + 2\log m - \log 3) \le 6S\log m \le 3Sm\log m$ and $45/4 \le 3m\log m$. So the right-hand side is at most $-\alpha m^2 + (10+3S)m\log m = -\alpha m^2 + Cm\log m$. □

5. Nonvanishing

Let $\mathbb{Z}_{(17)} = \{u/v \in \mathbb{Q} : 17 \nmid v\}$, and let v_{17} be the 17-adic valuation, $v_{17}(0) = +\infty$.

Lemma 5.1. *For $m = 4$, coefficientwise in $\mathbb{Z}_{(17)}[T]$,*

$$G_4(T) \equiv g(T) := 11 + 5T^2 + 4T^4 \pmod{17}. \tag{5.1}$$

Proof. Here $I = \{2,\dots,5\}$, $J = \{1,\dots,9\}$, $\Lambda = \{1\}$, and all denominators divide $6350400 = 2520^2$, prime to 17. Modulo 17 the matrix (2.1) is

$$\begin{pmatrix}
11 & 0 & 16 & 15 & 7 & 7 & 3 & 13 & 9 \\
5 & 16 & 0 & 14 & 11 & 4 & 4 & 9 & 5 \\
8 & 15 & 14 & 0 & 8 & 16 & 12 & 12 & 10 \\
8 & 7 & 11 & 8 & 0 & 7 & 14 & 2 & 2 \\
6 & 16+8T & 4 & 13 & 10 & 3 & 15 & 14 & 8 \\
5 & 4 & 16+7T & 5 & 13 & 14 & 5 & 16 & 3 \\
5 & 13 & 5 & 13+4T & 4 & 10 & 5 & 6 & 6 \\
9 & 10 & 13 & 4 & 3+12T & 16 & 14 & 1 & 15 \\
9 & 6 & 13 & 7 & 3 & 5 & 15 & 2 & 12
\end{pmatrix}. \tag{5.2}$$

Its determinant is $g(T)$. Reduction of (2.6), where $9/67228000000000 \equiv 13 \pmod{17}$ and $(q_0,q_1,q_2,q_3,q_4) \equiv (10,0,3,0,16) \pmod{17}$, gives the same polynomial. □

Lemma 5.2. *Let $\xi \in \mathbb{Q}$. For every integer $t \ge 1$ with $2t + v_{17}(\xi) > 0$, $G_{4\cdot 17^t}(\xi) \ne 0$.*

Proof. Put $Q_t = 17^t$, $m_t = 4Q_t$ and $h = \sigma = k = Q_t$, and

$$P_t(W) = Q_t^{3m_t+h} G_{m_t}(W/Q_t^2). \tag{5.3}$$

This is the determinant obtained after $T = W/Q_t^2$ by multiplying the C, $B - TA$ and V rows by Q_t, Q_t^2 and Q_t. The middle rows become $Q_t^2 B - WA$. Every nonzero index,

difference and Cauchy denominator has absolute value less than $17Q_t$, hence 17-adic valuation at most t. So $Q_t t_i$ and $Q_t^2 b_i$ lie in $\mathbb{Z}_{(17)}$, and for $i \neq j$

$$Q_t^2 B_{ij} = \frac{Q_t}{j-i}\,(Q_t t_i - Q_t t_j), \qquad Q_t C_{ij} = \frac{Q_t}{j-i}\,(a^i - a^j), \qquad Q_t V_{\ell j} = \frac{Q_t}{j+\ell},$$

with $v_{17}(j-i), v_{17}(j+\ell) \le t$. So all entries of the scaled matrix lie in $\mathbb{Z}_{(17)}[W]$. Hence $P_t \in \mathbb{Z}_{(17)}[W]$.

Let $0 \le u < Q_t$. Only the summation indices divisible by Q_t survive, and $a^{Q_t} \equiv a \pmod{17}$, so

$$Q_t t_{Q_t j+u} \equiv a^u t_j, \qquad Q_t^2 b_{Q_t j+u} \equiv a^u b_j \pmod{17}.$$

The scaled C and B entries vanish modulo 17 between distinct residue classes modulo Q_t. In the class u write $i = Q_t i_0 + u$, $j = Q_t j_0 + u$. Then $2 \le i_0 \le 5$, $1 \le j_0 \le 9$, and $A_{ii} \equiv a^u A_{i_0 i_0}$. The only V row in this class is $\ell = Q_t - u$, with

$$\frac{Q_t}{j+\ell} = \frac{1}{j_0+1}.$$

Cross-class C and B entries have 17-adic valuation at least 1. Within a class the nonzero differences $j_0 - i_0$ satisfy $|j_0 - i_0| \le 7 < 17$ and are units. All other scaled V entries vanish modulo 17. So, after permutations, the matrix has Q_t diagonal blocks

$$\begin{bmatrix} a^u C_4 \\ a^u(B_4 - W A_4) \\ V_4 \end{bmatrix}, \qquad 0 \le u < Q_t,$$

where the subscript 4 refers to G_4. Each block has eight rows with the factor a^u. Hence

$$P_t(W) \equiv \varepsilon_t\, 3^{4Q_t(Q_t-1)} g(W)^{Q_t} \pmod{17}, \qquad \varepsilon_t \in \{1,-1\}, \tag{5.4}$$

coefficientwise. In particular $P_t(0) \equiv \pm 3^{4Q_t(Q_t-1)} 11^{Q_t} \not\equiv 0 \pmod{17}$.

Take $W = Q_t^2 \xi$. Then $v_{17}(W) = 2t + v_{17}(\xi) > 0$, and since $P_t \in \mathbb{Z}_{(17)}[W]$, $P_t(W) \equiv P_t(0) \not\equiv 0 \pmod{17}$. By (5.3), $G_{m_t}(\xi) \neq 0$. □

The restriction $2t + v_{17}(\xi) > 0$ is needed in this argument: g has the roots 7 and 10 in $\mathbb{F}_{17}$.

6. Proof of Theorem 1

Suppose $\theta = u/q_0$ with $u \in \mathbb{Z}$ and $q_0 \in \mathbb{Z}_{>0}$. By Lemmas 2.1 and 3.5,

$$Z_m := q_0^m K_m G_m(u/q_0) \in \mathbb{Z}.$$

By Lemma 5.2 with $\xi = \theta$, $Z_m \neq 0$ for $m = 4 \cdot 17^t$ and all large t. By Lemmas 3.5 and 4.4,

$$\log|Z_m| \le -(\alpha_0 - \tau)m^2 + Cm\log m + o(m^2) + m\log q_0.$$

As $\alpha_0 - \tau > 0.00305 > 0$, the right-hand side tends to $-\infty$ along this sequence. So $0 < |Z_m| < 1$ for large t, which is impossible. □

Appendix A. Arithmetic integration

Table A.1 lists the lines (3.7) for the seam $R = 8$. The index ranges are the integer solutions of $4z_0/5 - 1 < k < z_0 R$ and $1 \le w < R/4 + 1$ (Section 3).

Table A.2 gives, for $\rho_0 < \rho < \rho_1$, the function $\mathfrak{c}(\rho) = c_0 + c_1\rho$. Consecutive rows share an endpoint. The first row starts at $1/R$ and the last ends at $5/4$. The rows are the 76

open intervals of $(1/R, 5/4) \setminus \mathcal{C}$, so a row can be checked at one interior point ρ: the six cells of (3.2) at ρ give an affine function of ρ, which must equal $c_0 + c_1\rho$. Then

$$\int_{1/R}^{5/4} \mathfrak{c}(\rho)\, d\rho = \sum_{\text{rows}} (\rho_1 - \rho_0)\Big(c_0 + c_1 \frac{\rho_0 + \rho_1}{2}\Big).$$

TABLE A.1. Line families of (3.7).

family	lines	count
strip	$x = 0$, $x = \rho$	2
$\mathcal{L}$ lower	$x = 1/4 - k\rho$, $0 \le k \le 1$	2
K lower	$x = 1/2 - k\rho$, $0 \le k \le 3$	4
K upper	$x = 3/2 - k\rho$, $1 \le k \le 11$	11
$\mathcal{L}$ upper	$x = 5/2 - k\rho$, $2 \le k \le 19$	18
W	$x = w\rho - 1/4$, $1 \le w \le 2$	2
total		39

TABLE A.2. The affine pieces of $\mathfrak{c}$ on $(1/R, 5/4)$, $R = 8$: $\mathfrak{c}(\rho) = c_0 + c_1\rho$ for $\rho_0 < \rho < \rho_1$.

ρ_0	ρ_1	c_0	c_1	ρ_0	ρ_1	c_0	c_1	ρ_0	ρ_1	c_0	c_1
1/8	11/84	9/2	−10	2/11	11/60	11/4	2	1/3	11/32	9/2	−4
11/84	5/38	7/4	11	11/60	3/16	11/4	2	11/32	7/20	7/4	4
5/38	9/68	17/4	−8	3/16	5/26	1/2	14	7/20	5/14	7/2	−1
9/68	2/15	17/4	−8	5/26	7/36	3	1	5/14	3/8	6	−8
2/15	7/52	9/4	7	7/36	11/56	19/4	−8	3/8	11/28	15/4	−2
7/52	3/22	4	−6	11/56	1/5	2	6	11/28	2/5	15/4	−2
3/22	11/80	5/2	5	1/5	9/44	3	1	2/5	5/12	7/4	3
11/80	5/36	5/2	5	9/44	5/24	21/4	−10	5/12	7/16	1/2	6
5/36	9/64	5/4	14	5/24	11/52	4	−4	7/16	9/20	9/4	2
9/64	1/7	7/2	−2	11/52	3/14	4	−4	9/20	11/24	9/2	−3
1/7	11/76	9/2	−9	3/14	7/32	5/2	3	11/24	1/2	7/4	3
11/76	7/48	7/4	10	7/32	2/9	17/4	−5	1/2	11/20	19/4	−3
7/48	5/34	7/2	−2	2/9	9/40	9/4	4	11/20	9/16	2	2
5/34	3/20	6	−19	9/40	5/22	9/4	4	9/16	7/12	17/4	−2
3/20	11/72	15/4	−4	5/22	11/48	19/4	−7	7/12	5/8	6	−5
11/72	2/13	15/4	−4	11/48	1/4	2	5	5/8	2/3	19/4	−3
2/13	5/32	7/4	9	1/4	11/40	9/2	−5	2/3	11/16	11/4	0
5/32	7/44	1/2	17	11/40	5/18	7/4	5	11/16	3/4	11/4	0
7/44	9/56	9/4	6	5/18	9/32	17/4	−4	3/4	5/6	1/2	3
9/56	11/68	9/2	−8	9/32	2/7	17/4	−4	5/6	7/8	3	0
11/68	1/6	7/4	9	2/7	7/24	9/4	3	7/8	11/12	19/4	−2
1/6	11/64	19/4	−9	7/24	3/10	4	−3	11/12	1	19/4	−2
11/64	9/52	2	7	3/10	11/36	5/2	2	1	9/8	15/4	−1
9/52	7/40	17/4	−6	11/36	5/16	5/2	2	9/8	5/4	6	−3
7/40	5/28	6	−16	5/16	9/28	5/4	6				
5/28	2/11	19/4	−9	9/28	1/3	7/2	−1				

With (3.8),

$$A = \frac{13}{32} + \int_{1/R}^{5/4} \mathfrak{c}(\rho)\, d\rho + \frac{45}{32},$$

the constant of (3.12). The coefficient of $\log 3$ is $53/48$, and the block (3.5) contributes $-5/48$ to (3.8).

Appendix B. Certified values

Table B.1 gives the energy of ν. Here ν_z^{int} and ν_z^{Poi} are the interval and Poisson parts of ν_z, I_{int} is the energy of ν_z^{int}, $I_{\mathrm{mix}} = I(\nu_z^{\mathrm{int}}, \nu_z^{\mathrm{Poi}})$, I_{Poi} is the energy of ν_z^{Poi}, and

$$\mathcal{Q}(\nu) = I(\nu_x,\nu_x) + I(\nu_v,\nu_v) + I_{\mathrm{int}} + 2I_{\mathrm{mix}} + I_{\mathrm{Poi}} + I(\nu_v,\nu_z^{\mathrm{int}}) + I(\nu_v,\nu_z^{\mathrm{Poi}}) - D(\nu_x,\nu_v).$$

Every displayed interval is outward rounded, and the row of $D(\nu_x,\nu_v)$ contains the tail $\mathcal{E}_N$. The interval sum of the first eight rows, with these coefficients, exceeds q_-, which is (4.2).

Table B.1. The pair decomposition of $\mathcal{Q}(\nu)$ and its enclosures.

term	outward enclosure
$I(\nu_x,\nu_x)$	`[-1.604087582665193,-1.604087582665192]`
$I(\nu_v,\nu_v)$	`[-2.279694146532524,-2.279694146532523]`
I_{int}	`[-0.159655877431972,-0.159655877431971]`
I_{mix}	`[0.226965417324502,0.226965417324503]`
I_{Poi}	`[-0.495113146744583,-0.495113146744582]`
$I(\nu_v,\nu_z^{\mathrm{int}})$	`[0.278231287125003,0.278231287125004]`
$I(\nu_v,\nu_z^{\mathrm{Poi}})$	`[-0.191796788958401,-0.191796788958400]`
$D(\nu_x,\nu_v)$	`[1.195386024949843,1.195386024949844]`
$\mathcal{Q}(\nu)$	`[-5.193571445508505,-5.193571445508504]`

Table B.2. The caps (4.3): leaves of Lemma C.1, the largest upper bound over the leaves (rounded upward at 10^{-6}) and the cap. On $\mathcal{K}_z$ the cap is $K_z = \max(0.04318, 0.04283) = 0.04318$.

support	leaves	upper bound	cap
$x \in [\varepsilon_0, 1]$	67584	-4.831488	-4.83148
$v \in [\varepsilon_0, 1]$	37120	-4.743190	-4.74318
$z \in [1, 3]$	13568	0.043178	0.04318
$\lvert z\rvert = r$, $\operatorname{Im} z \geq 0$	8192	0.042823	0.04283

Appendix C. Reference measure and subdivision

Terminating decimals are exact. Other values are printed as fractions p/q.

C.1. **Intervals.** Table C.1 gives ν_x, ν_v and ν_z^{int}. Each is a chain of adjacent intervals. Its heading gives the start point b_0, and its row (k, b_k, w_k) is the interval $[b_{k-1}, b_k]$ carrying $w_k\,dt/(b_k - b_{k-1})$. The three chains have 131, 143 and 105 rows and end at 1, 1 and 3. Outside its chain a part vanishes. The masses are 1, 5/4 and 170340748021/500000000000.

TABLE C.1. Interval chains of the reference measure. A line gives the rows (k, b_k, w_k), $(k+1, b_{k+1}, w_{k+1})$ and $(k+2, b_{k+2}, w_{k+2})$.

k	b_k	w_k	b_{k+1}	w_{k+1}	b_{k+2}	w_{k+2}
ν_x: $b_0 = 0.08426519384873$						
1	0.09379426	0.004878668	0.1033233	0.005388618	0.1137017	0.006994053
4	0.1240801	0.007569004	0.1352634	0.008564016	0.1464466	0.008818097
7	0.1583869	0.009590098	0.1703271	0.009692372	0.1829732	0.0103232
10	0.1956193	0.01034485	0.2089171	0.01087642	0.2222149	0.0108577
13	0.2361074	0.01131139	0.25	0.01127106	0.2644278	0.01166085
16	0.2788557	0.01161365	0.293757	0.01194875	0.3086583	0.01190387
19	0.3239693	0.01219079	0.3392803	0.01215494	0.3549354	0.01239796
22	0.3705905	0.01237491	0.3865227	0.01257814	0.4024548	0.01257039
25	0.4185959	0.01273679	0.4347369	0.01274778	0.4510177	0.01287879
28	0.4672984	0.01290965	0.4836492	0.01300665	0.5	0.01305858
31	0.5163508	0.01312226	0.5327016	0.01319818	0.5489823	0.01322769
34	0.5652631	0.01332781	0.5814041	0.01332593	0.5975452	0.01345034
37	0.6134773	0.01341514	0.6294095	0.01356788	0.6450646	0.01349774
40	0.6607197	0.01367922	0.6760307	0.01357396	0.6913417	0.01378771
43	0.706243	0.01364148	0.7211443	0.01389263	0.7355722	0.01370571
46	0.75	0.01399346	0.7638926	0.01376403	0.7777851	0.01408874
49	0.7910829	0.01381373	0.8043807	0.01418846	0.8170268	0.01385703
52	0.8296729	0.01427975	0.8416131	0.01389719	0.8535534	0.01437142
55	0.8647366	0.01392421	0.8759199	0.0144671	0.8862983	0.01394331
58	0.8966767	0.0145547	0.9062057	0.01394895	0.9157348	0.0146563
61	0.9243738	0.01394465	0.9330127	0.01475476	0.9407245	0.01392136
64	0.9484364	0.01485533	0.9551881	0.01387691	0.9619398	0.01497938
67	0.9677024	0.01379006	0.9734651	0.01513013	0.978214	0.01364607
70	0.9829629	0.01534824607	0.9864815	0.01264469	0.99	0.01468005
73	0.9901963	0.0009304798	0.9903926	0.0008176946	0.9923846	0.009526088
76	0.9943766	0.01098753	0.9950495	0.004282778	0.9957224	0.004357794
79	0.9962801	0.003907897	0.9968377	0.004198966	0.9975297	0.005760908
82	0.9982217	0.006780532	0.9985756	0.003909506	0.9989295	0.004519428
85	0.9989498	0.0002978646	0.9989702	0.0002759106	0.9989851	0.0002057553
88	0.999	0.0002063948	0.9990431	0.0006106305	0.9990862	0.0006201
91	0.9991731	0.001304877	0.9992599	0.00137548	0.9993488	0.001492587
94	0.9994377	0.001607745	0.9994512	0.0002561717	0.9994647	0.0002462361
97	0.9995672	0.002058841	0.9996696	0.002355955	0.9996767	0.0001814995
100	0.9996838	0.0001633233	0.999753	0.00181967	0.9998222	0.002143887
103	0.9998327	0.0003687629	0.9998432	0.0003554893	0.9998716	0.001051055
106	0.9999	0.001186225	0.9999219	0.001022622	0.9999438	0.001204602
109	0.9999515	0.0004787153	0.9999593	0.0005156937	0.9999638	0.0003258994
112	0.9999684	0.000343669	0.9999753	0.000575319	0.9999822	0.000676004

k	b_k	w_k	b_{k+1}	w_{k+1}	b_{k+2}	w_{k+2}
115	0.9999861	0.0004312648	0.99999	0.0005066644	0.9999922	0.0003232857
118	0.9999944	0.0003798382	0.9999956	0.0002423703	0.9999968	0.0002847743
121	0.9999975	0.000181713	0.9999982	0.0002135055	0.9999986	0.0001362365
124	0.999999	0.0001600714	0.9999992	0.0001021401	0.9999994	0.0001200076
127	0.9999996	0.00007657429	0.9999997	0.00008996647	0.9999998	0.00005740427
130	0.9999999	0.0001104681	1	0.0003313966		
ν_v: $b_0 = 0.00960735979838$						
1	0.01332222	0.001656734	0.01703709	0.00542487	0.02178601	0.008072268
4	0.02653494	0.008280399	0.03229758	0.009887851	0.03806023	0.009561701
7	0.04481193	0.01077175	0.05156363	0.01030386	0.05927546	0.01128366
10	0.0669873	0.01079812	0.07562625	0.01163129	0.08426519	0.01116664
13	0.09379426	0.01188784	0.1033233	0.01146128	0.1137017	0.01209265
16	0.1240801	0.01171582	0.1352634	0.01227961	0.1464466	0.0119444
19	0.1583869	0.01244886	0.1703271	0.01216045	0.1829732	0.01261611
22	0.1956193	0.01236727	0.2089171	0.0127843	0.2222149	0.0125745
25	0.2361074	0.01295137	0.25	0.01278215	0.2644278	0.01312539
28	0.2788557	0.01298995	0.293757	0.01330075	0.3086583	0.01319874
31	0.3239693	0.01347729	0.3392803	0.01340707	0.3549354	0.01365223
34	0.3705905	0.01360987	0.3865227	0.01381909	0.4024548	0.01378893
37	0.4185959	0.01398487	0.4347369	0.01401875	0.4510177	0.01417834
40	0.4672984	0.014227	0.4836492	0.01434849	0.5	0.01442008
43	0.5163508	0.01450478	0.5327016	0.01460288	0.5489823	0.01464888
46	0.5652631	0.01477283	0.5814041	0.01478305	0.5975452	0.01493237
49	0.6134773	0.01490385	0.6294095	0.01508317	0.6450646	0.01501368
52	0.6607197	0.01522317	0.6760307	0.01511253	0.6913417	0.01535607
55	0.706243	0.01519763	0.7211443	0.01548108	0.7355722	0.01527532
58	0.75	0.01559771	0.7638926	0.01534276	0.7777851	0.0157047
61	0.7910829	0.01539736	0.8043807	0.01581345	0.8170268	0.01544195
64	0.8296729	0.01591008	0.8416131	0.0154806	0.8535534	0.01600481
67	0.8647366	0.01550264	0.8759199	0.01610211	0.8862983	0.01551437
70	0.8966767	0.01618908	0.9062057	0.01551011	0.9157348	0.01629058
73	0.9243738	0.01549424	0.9330127	0.01638817	0.9407245	0.01545718
76	0.9484364	0.0164881	0.9551881	0.01539698	0.9619398	0.0166144
79	0.9677024	0.01529057	0.9734651	0.01677105	0.978214	0.01512197
82	0.9829629	0.01700337185	0.9864815	0.01400531	0.99	0.01625595
85	0.9901963	0.001030202	0.9903926	0.0009054735	0.9923846	0.01054699
88	0.9943766	0.0121635	0.9950495	0.004740679	0.9957224	0.004823687
91	0.9962801	0.004325506	0.9968377	0.004647527	0.9975297	0.006376076
94	0.9982217	0.007504222	0.9985756	0.004326653	0.9989295	0.005001531
97	0.9989498	0.0003296327	0.9989702	0.0003053404	0.9989851	0.0002277013
100	0.999	0.0002284092	0.9990431	0.000675759	0.9990862	0.0006862373
103	0.9991731	0.001444043	0.9992599	0.001522167	0.9993488	0.001651755
106	0.9994377	0.001779182	0.9994512	0.0002834868	0.9994647	0.0002724926
109	0.9995672	0.002278363	0.9996696	0.002607138	0.9996767	0.0002008494
112	0.9996838	0.0001807361	0.999753	0.002013666	0.9998222	0.002372435
115	0.9998327	0.0004080737	0.9998432	0.0003933854	0.9998716	0.001163099
118	0.9999	0.001312675	0.9999219	0.001131631	0.9999438	0.001333007

k	b_k	w_k	b_{k+1}	w_{k+1}	b_{k+2}	w_{k+2}
121	0.9999515	0.0005297437	0.9999593	0.0005706635	0.9999638	0.0003606381
124	0.9999684	0.0003803018	0.9999753	0.0006366439	0.9999822	0.0007480609
127	0.9999861	0.0004772341	0.99999	0.0005606707	0.9999922	0.0003577452
130	0.9999944	0.0004203256	0.9999956	0.000268205	0.9999968	0.0003151286
133	0.9999975	0.000201082	0.9999982	0.0002362633	0.9999986	0.0001507581
136	0.999999	0.0001771335	0.9999992	0.0001130273	0.9999994	0.0001327994
139	0.9999996	0.00008473636	0.9999997	0.00009955619	0.9999998	0.0000635229
142	0.9999999	0.000122243	1	0.0003667206		
ν_z^{int}: $b_0 = 28303039933611/17500000000000$						
1	1.647939	0.000800039	1.678561	0.001634018	1.709871	0.002195722
4	1.741181	0.002569887	1.773045	0.002941359	1.80491	0.003206511
7	1.837192	0.003492292	1.869474	0.003707447	1.902035	0.003935108
10	1.934597	0.004118914	1.967298	0.00430909	2	0.004469524
13	2.032702	0.00462588	2.065403	0.004777443	2.097965	0.004901502
16	2.130526	0.005045769	2.162808	0.005145023	2.19509	0.005284974
19	2.226955	0.005358049	2.258819	0.005500629	2.290129	0.005547422
22	2.321439	0.005693562	2.352061	0.005716104	2.382683	0.005869112
25	2.412486	0.0058643	2.442289	0.006028879	2.471144	0.00599805
28	2.5	0.00617374	2.527785	0.0061167	2.55557	0.006304648
31	2.582166	0.00622033	2.608761	0.00642816	2.634054	0.006311412
34	2.659346	0.006538253	2.683226	0.006392705	2.707107	0.006640591
37	2.729473	0.006459666	2.75184	0.006737854	2.772597	0.006515946
40	2.793353	0.006824422	2.812411	0.006559483	2.83147	0.006912101
43	2.848748	0.006592625	2.866025	0.006992728	2.881449	0.006611407
46	2.896873	0.007069417	2.910376	0.006615346	2.92388	0.007152991
49	2.935405	0.006594439	2.94693	0.00724478	2.956428	0.006543148
52	2.965926	0.007358282	2.973356	0.00643243	2.980785	0.007538103528
55	2.985393	0.00532701	2.99	0.006421216	2.990722	0.001161462
58	2.991445	0.001127666	2.992911	0.002498185	2.994377	0.002795714
61	2.995607	0.00261479	2.996838	0.003080948	2.997348	0.001436316
64	2.997859	0.001592602	2.9979	0.0001394852	2.99794	0.000132549
67	2.998056	0.0003909291	2.998172	0.0004039291	2.998197	0.00008733582
70	2.998222	0.00008626531	2.998371	0.0005458644	2.99852	0.0005686528
73	2.998725	0.0008328935	2.998929	0.0009109751	2.998965	0.0001665063
76	2.999	0.0001617129	2.99917	0.0008451466	2.999339	0.0009489754
79	2.999388	0.0002982447	2.999438	0.0003002494	2.999561	0.0008270791
82	2.999684	0.0009730542	2.999685	0.00001266974	2.999686	0.000009434384
85	2.999754	0.0006105242	2.999822	0.0007153116	2.999861	0.0004647103
88	2.9999	0.0005473779	2.999909	0.0001464355	2.999919	0.0001487465
91	2.999931	0.0002195401	2.999944	0.000241353	2.999956	0.0002613112
94	2.999968	0.0003071303	2.999975	0.0001959502	2.999982	0.0002302158
97	2.999986	0.0001468951	2.99999	0.0001725934	2.999992	0.0001101306
100	2.999994	0.0001293995	2.999996	0.00008256952	2.999997	0.00009701663
103	2.999998	0.00006190621	2.999999	0.0001191514	3	0.0003570495

C.2. **Poisson measures.** Table C.2 gives ν_z^{Poi}. A row (c, w) with $\operatorname{Im} c > 0$ stands for the two Poisson measures on $|q| = r = 2/5$ with centres c and $\overline{c}$, each of mass w. The row $k = 0$, with c real, stands for one Poisson measure. The measures are those of the

proof of Lemma 4.2, and their logarithmic potential is (4.4). This gives $2 \cdot 83 + 1 = 167$ measures of total mass 329659251979/500000000000.

TABLE C.2. Poisson centres c and masses w: rows $k \geq 1$ have $\operatorname{Im} c > 0$, in increasing order of $\arg c$, and stand also for $\overline{c}$; row $k = 0$ is the real centre.

k	$\operatorname{Re} c$	$\operatorname{Im} c$	w	k	$\operatorname{Re} c$	$\operatorname{Im} c$	w
0	−0.394	0	0.006687968	42	−0.2046129	0.341376	0.0002611643
1	0.2535036	0.308895	0.0002724889	43	−0.2080587	0.3345857	0.004943139
2	0.2370883	0.3196766	0.001734768001	44	−0.2188947	0.327599	0.006091431004
3	0.221117	0.3309249	0.003160258	45	−0.2294962	0.3202616	0.004978119
4	0.2046129	0.341376	0.002355698	46	−0.2370883	0.3196766	0.0003172227
5	0.197	0.341214	0.00203316	47	−0.239852	0.3125812	0.004953756
6	0.1876159	0.3510047	0.001945593	48	−0.249951	0.3045661	0.006293054005
7	0.1742617	0.3533679	0.00333003	49	−0.2597823	0.2962249	0.00498701
8	0.1626066	0.3588803	0.003486625	50	−0.2672805	0.2948985	0.0003407953
9	0.152308	0.3677041	0.002375737	51	−0.2693354	0.2875664	0.005025951
10	0.1387865	0.3687469	0.003817593	52	−0.2786001	0.2786001	0.006379229002
11	0.1266471	0.3730905	0.003942579	53	−0.2875664	0.2693354	0.005074617
12	0.1155333	0.3808623	0.002775625	54	−0.2948985	0.2672805	0.0003451508
13	0.1019747	0.3805748	0.004181110002	55	−0.2962249	0.2597823	0.005068562
14	0.08946805	0.3837075	0.004331115	56	−0.3045661	0.249951	0.006456653
15	0.07764595	0.3903525	0.003082157	57	−0.3125812	0.239852	0.005132843
16	0.06418082	0.3887375	0.004515421	58	−0.3196766	0.2370883	0.0003275158
17	0.05142732	0.3906293	0.004618404	59	−0.3202616	0.2294962	0.00520169
18	0.03901082	0.3960835	0.003346546	60	−0.327599	0.2188947	0.006408025
19	0.02576883	0.3931564	0.004623723	61	−0.3345857	0.2080587	0.005265605
20	0.01289132	0.393789	0.005370324	62	−0.341376	0.2046129	0.0002748567
21	0	0.398	0.002954771	63	−0.341214	0.197	0.005351403
22	−0.01289132	0.393789	0.00558512	64	−0.347477	0.1857303	0.006322745001
23	−0.02576883	0.3931564	0.004147672	65	−0.3533679	0.1742617	0.00539497
24	−0.03861875	0.3921028	0.005377501	66	−0.3597877	0.1701669	0.0002230895
25	−0.05142732	0.3906293	0.004364202	67	−0.3588803	0.1626066	0.005494093
26	−0.05839873	0.3936923	0.0002681214	68	−0.3640085	0.1507773	0.006236832
27	−0.06418082	0.3887375	0.004435699	69	−0.3687469	0.1387865	0.005515612
28	−0.07686559	0.3864294	0.00549873	70	−0.3747345	0.1340822	0.0002252512
29	−0.08946805	0.3837075	0.004569928	71	−0.3730905	0.1266471	0.005431967
30	−0.09670611	0.3860724	0.0002359134	72	−0.3770345	0.1143722	0.006398725
31	−0.1019747	0.3805748	0.004673523	73	−0.3805748	0.1019747	0.005431584
32	−0.1143722	0.3770345	0.005552904	74	−0.3860724	0.09670611	0.0002813461
33	−0.1266471	0.3730905	0.004783363	75	−0.3837075	0.08946805	0.005361819
34	−0.1340822	0.3747345	0.0001976397	76	−0.3864294	0.07686559	0.00656059
35	−0.1387865	0.3687469	0.004891999	77	−0.3887375	0.06418082	0.005341411
36	−0.1507773	0.3640085	0.005599664001	78	−0.3936923	0.05839873	0.000338135
37	−0.1626066	0.3588803	0.00497547	79	−0.3906293	0.05142732	0.005290732
38	−0.1701669	0.3597877	0.0002060286	80	−0.3921028	0.03861875	0.00668558776
39	−0.1742617	0.3533679	0.004926876	81	−0.3931564	0.02576883	0.005271869
40	−0.1857303	0.347477	0.005847720003	82	−0.3975206	0.01952893	0.0003596048

k	$\operatorname{Re} c$	$\operatorname{Im} c$	w	k	$\operatorname{Re} c$	$\operatorname{Im} c$	w
41	-0.197	0.341214	0.004984704	83	-0.393789	0.01289132	0.005295008

C.3. **Dyadic subdivision.** The four supports are

$$s = x, v:\ [\varepsilon_0, 1], \qquad s = z:\ [1, 3], \qquad s = \circ:\ \{re^{i\pi t} : 0 \le t \le 1\},$$

the last parametrized by $t \in [0, 1]$. The *initial pieces* of a support are the closed intervals between consecutive points of the following sets, in increasing order:

x: ε_0, 1, and the endpoints of the ν_x-chain in $(\varepsilon_0, 1)$;
v: ε_0, 1, $r = 2/5$, and the endpoints of the ν_v-chain in $(\varepsilon_0, 1)$;
z: 1, 3, and the endpoints of the ν_z^{int}-chain in $(1, 3)$;
$\circ$: $t = k/32$, $0 \le k \le 32$.

Since the chains start at $0.0842\ldots$, v_0 and $1.617\ldots$ and end at 1, 1 and 3, and r is not an endpoint of the ν_v-chain, there are 132, 145, 106 and 32 initial pieces.

Lemma C.1. *Let $d_x = 9$, $d_v = 8$, $d_z = 7$, $d_\circ = 8$. Bisect every initial piece of the support s to depth d_s: a piece $[A, B]$ gives the 2^{d_s} leaves*

$$[A + (B - A)(j - 1)2^{-d_s},\ A + (B - A)j2^{-d_s}], \qquad 1 \le j \le 2^{d_s},$$

in q on the real supports and in t on the circle. The leaves of each support have rational endpoints, disjoint interiors and union equal to the support. There are $132 \cdot 2^9$, $145 \cdot 2^8$, $106 \cdot 2^7$ and $32 \cdot 2^8$ leaves, 126464 *in all (Table C.3).*

Proof. Within a piece the leaves are consecutive and their lengths add up to $B - A$. The pieces of a support are consecutive and cover it. The counts are products. □

The leaves are the closed pieces of the proof of Lemma 4.2. On each of them the bound of that proof is below the cap, so no piece needs a finer subdivision.

Table C.3. Uniform bisection depths.

support	initial pieces	depth d_s	leaves
x	132	9	67584
v	145	8	37120
z real	106	7	13568
$\circ$	32	8	8192
total	415		126464

Appendix D. Computer checks

The ancillary scripts recheck printed data and run finite tests (Table D.1). They use exact rational arithmetic and, for the analytic bounds, Arb ball arithmetic [6] through `python-flint` 0.9.0 under Python 3.12.3.

TABLE D.1. Computer checks.

check	statements checked
determinant	the exact seed determinant: (2.6) and Table 2.1. Finite tests: $C_{ij} = 3^i$ times the right side of (2.3) with $d = j - i$, for $0 \le i \le 16$ and $\lvert j - i\rvert \le 16$; the four determinant evaluations in Lemma 2.2 and their product for $m \le 3$ and three choices of (s, k, h).
arithmetic	recomputes the arrangement of (3.7) for $R = 8$ in exact rational arithmetic; checks the line families of Table A.1, every row of Table A.2 (by evaluating $\mathfrak{c}$ from (3.2)–(3.3) at two interior points), (3.8), A in (3.12), 45/32 and $-5/48$; rational bounds for τ and $\alpha_0 - \tau$; integrality of $K_m G_m$ for small n.
analytic	Lemmas 4.2 and C.1: the chains, masses, conjugation symmetry and $\lvert c\rvert < r$ of Tables C.1 and C.2; Tables B.1, B.2 and C.3, (4.2), (4.3) and α_0, with the bound of the proof of Lemma 4.2 on each of the 126464 leaves.
mod 17	(5.2), (5.1), and (5.4) for $t = 1$ at $W = 0, 1, 2$.

APPENDIX E. PROOF OF FACT 2

For $t > 0$ let $g_t(y) = (\pi t)^{-1} e^{-|y|^2/t}$ on $\mathbb{C}$, a centred Gaussian density. Then $g_s * g_t = g_{s+t}$ and $g_t(-y) = g_t(y)$. For $a, b > 0$, $\int_0^\infty (e^{-a/t} - e^{-b/t})\, dt/t = \log(b/a)$, and the integrand has constant sign. With $a = |y|^2$ and $b = 1$ this gives, for $y \neq 0$,

$$-\log|y| = \frac{\pi}{2} \int_0^\infty \big(g_t(y) - g_t(1)\big)\, dt, \qquad \int_0^\infty |g_t(y) - g_t(1)|\, dt = \frac{2}{\pi} \big|\log|y|\big|.$$

The hypothesis of Fact 2 makes the diagonal $|\sigma| \otimes |\sigma|$-null and, by the second identity, $g_t(u - w) - g_t(1)$ integrable for $|\sigma| \otimes |\sigma| \otimes dt$. By Fubini's theorem and $\sigma(\mathbb{C}) = 0$ (for each $t > 0$, g_t is bounded, so $\iint g_t(1)\, d\sigma\, d\sigma = g_t(1)\sigma(\mathbb{C})^2 = 0$),

$$-\iint \log|u - w|\, d\sigma(u)\, d\sigma(w) = \frac{\pi}{2} \int_0^\infty \iint g_t(u - w)\, d\sigma(u)\, d\sigma(w)\, dt.$$

For fixed t, $g_t(u - w) = \int g_{t/2}(x - u) g_{t/2}(x - w)\, dx$ (dx the Lebesgue measure on $\mathbb{C}$), and the corresponding triple integral against $|\sigma| \otimes |\sigma|$ is at most $|\sigma|(\mathbb{C})^2/(\pi t)$. So, by Fubini again,

$$\iint g_t(u - w)\, d\sigma(u)\, d\sigma(w) = \int_{\mathbb{C}} \Big(\int g_{t/2}(x - u)\, d\sigma(u) \Big)^2 dx \ge 0.$$

Hence $\iint \log|u - w|\, d\sigma(u)\, d\sigma(w) \le 0$. Compact support is not used.

USE OF AI TOOLS

The author devised the method of proof. Codex (OpenAI, GPT-6.1 Sol) computed the reference measure ν and the other printed data, and wrote the Python scripts in `anc/`. Claude Code (Anthropic, Claude Opus 5.5) checked the scripts, the data and the tables. The author checked every step of the proof, every table and every reference, and takes full responsibility for the content of the paper.

STATEMENTS AND DECLARATIONS

Competing interests. The author has no competing interests.

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Data availability. All data used in the proof are printed in Appendices A–C. The `anc/` attachment holds the scripts of Appendix D and a `README.md`. The

entry points are `check_determinant.py`, `check_arith.py`, `check_analytic.py` and `check_mod17.py`. The helpers are `arith.py`, `caps.py`, `combine.py`, `energy.py`, `indep_caps_params.py`, `mod17_replay.py`, `parse_tex.py`, `refmeasure.py`, `tex2refjson.py` and `tex_tables.py`. The scripts read the tables and constants from the TEX source of this manuscript. The analytic replay writes to a temporary directory. The reports in `anc/verification/` record the SHA-256 of the TEX source they checked. The file `anc/SCRIPTS_SHA256SUMS` lists the SHA-256 of the scripts and the `README.md`. Its own SHA-256 is

`890192149e125730cc0b9b0f85ddd7dab8e9ac391a6d14703849fb74d54f8cd7`.

References

[1] A. Baker, *Transcendental Number Theory*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990. MR1074572

[2] F. Calegari, V. Dimitrov, and Y. Tang, *The linear independence of* 1, $\zeta(2)$, *and* $L(2,\chi_{-3})$, preprint (2024), arXiv:2408.15403v2.

[3] G.-N. Han and C. Krattenthaler, *Rectangular Scott-type permanents*, Séminaire Lotharingien de Combinatoire **43** (1999), Article B43g, 25 pp., arXiv:math/0003072. MR1741888

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., revised by D. R. Heath-Brown and J. H. Silverman, Oxford University Press, Oxford, 2008. MR2445243

[5] M. Hata, *Rational approximations to the dilogarithm*, Trans. Amer. Math. Soc. **336** (1993), no. 1, 363–387, `https://doi.org/10.1090/S0002-9947-1993-1147401-5`. MR1147401

[6] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. **66** (2017), no. 8, 1281–1292, `https://doi.org/10.1109/TC.2017.2690633`. MR3681746

[7] C. Krattenthaler, *Advanced determinant calculus: a complement*, Linear Algebra Appl. **411** (2005), 68–166, `https://doi.org/10.1016/j.laa.2005.06.042`. MR2178686

[8] *NIST Digital Library of Mathematical Functions*, `https://dlmf.nist.gov/`, Release 1.2.8 of 2026-09-15, F. W. J. Olver et al., eds.

[9] T. Prellberg, *Three irrationality results for the dilogarithm: Rhin–Viola and Viola–Zudilin constructions at* $-1/4$, $1/5$, *and* $-1/3$, preprint (2026), arXiv:2609.06895v1.

[10] G. Rhin and C. Viola, *The permutation group method for the dilogarithm*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 3, 389–437, `https://doi.org/10.2422/2036-2145.2005.3.02`. MR2185863

[11] G. Rhin and C. Viola, *Linear independence of* 1, Li_1 *and* Li_2, Mosc. J. Comb. Number Theory **8** (2019), no. 1, 81–96, `https://doi.org/10.2140/moscow.2019.8.81`. MR3864310

[12] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der Mathematischen Wissenschaften, vol. 316, 1st ed., Springer, 1997, `https://doi.org/10.1007/978-3-662-03329-6`. MR1485778

[13] C. Viola and W. Zudilin, *Linear independence of dilogarithmic values*, J. Reine Angew. Math. **736** (2018), 193–223, `https://doi.org/10.1515/crelle-2015-0030`; preprint version: Max-Planck-Institut für Mathematik Preprint Series 2014 (26), `https://archive.mpim-bonn.mpg.de/id/eprint/1914/`. MR3769989

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