IRRATIONALITY OF THE DILOGARITHM AT 1/4

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Abstract. We prove that $\mathrm{Li}_2(1/4)$ is irrational. Irrationality of $\mathrm{Li}_2(1/s)$ was known for integers $s \le -5$ and $s \ge 6$ by work of Hata and of Rhin and Viola, and Prellberg recently proved it for $s = -4, 5, -3$. The linear forms in 1 and $\mathrm{Li}_2(1/4)$ come from one double integral of Viola and Zudilin at $z = 4$. We expand it into Rhin–Viola integrals in three ways and apply their factorial transformations to each term. This gives a common factor of the coefficients of the Nth form. The third expansion adds the primes between $265N$ and $322N$ to this factor, which is too small without them. We bound the integral by the weighted arithmetic-geometric mean inequality and by concavity on circular arcs.

1. Introduction

For $|w| \le 1$ let $\mathrm{Li}_2(w) = \sum_{n\ge1} w^n/n^2$. Hata [3] proved that $\mathrm{Li}_2(1/s)$ is irrational for every integer $s \le -5$ and every $s \ge 7$. Rhin and Viola adapted their permutation group method [5] to the dilogarithm and settled $s = 6$ [6]. They later treated negative arguments [7]. Viola and Zudilin [8] introduced a six-parameter family of double integrals. In 2024 Calegari, Dimitrov and Tang [1, Remark 2.8.2 and Section 15.2] listed the integers still open: $s \in \{-4, -3, -2, 2, 3, 4, 5\}$. Prellberg's 2026 preprint [4] treats $s = -4, 5, -3$. We prove the case $s = 4$.

Theorem 1. *The number* $\mathrm{Li}_2(1/4)$ *is irrational.*

By Landen's identity [2, (25.12.3)], $\mathrm{Li}_2(1/4) + \mathrm{Li}_2(-1/3) = -\frac{1}{2}\log^2(3/4)$, so Theorem 1 does not follow directly from the case $s = -3$.

Put $\theta = \mathrm{Li}_2(1/4)$, $d_m = \mathrm{lcm}(1, \dots, m)$, $d_0 = 1$, and for $N \ge 1$

$$\begin{aligned} G_N &= J_4(240N, 240N, 190N, 157N, 215N, 165N), \\ D_N &= d_{323N}\, d_{322N}\, 4^{132N}\, 3^{107N}, \end{aligned} \tag{1.1}$$

with J_4 as in Section 2. By [8, Lemma 2.1], $D_N G_N \in \mathbb{Z} + \mathbb{Z}\theta$. Sections 3 and 4 give $\Delta_N \mid D_N$ with

$$(D_N/\Delta_N)\, G_N \in \mathbb{Z} + \mathbb{Z}\theta,$$

$$\lim_{N\to\infty} \frac{\log \Delta_N}{40N} = \delta > 4.7670, \qquad \lim_{N\to\infty} \frac{\log D_N}{40N} < 23.63856,$$

and Sections 5 and 6 give

$$G_N > 0 \text{ for large } N, \qquad \limsup_{N\to\infty} \frac{\log G_N}{40N} \le -18.87333.$$

Since $23.63856 - 4.7670 - 18.87333 < 0$, Theorem 1 follows.

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The arithmetic follows [4, Section 3.2]. Identity (2.1) for G_N and for its image under (F2) gives the analogues of [4, (3.11)–(3.12)]. The factorial transformations of [6] act on each term. For $265N < p < 322N$ these expansions give only $M = 0$ in Lemma 3.2. These primes account for $57/40$ of δ, and the rest of δ is below $3.35 < 23.63856 - 18.87333$. Expansion (2.2) gives one power of each such p. The bounds of Lemma 3.2 depend only on N/p, so we need not exclude the exceptional primes of [4, (3.6)–(3.7)]. One linear form suffices, since $G_N > 0$ for large N.

Two Python scripts recheck only part of the proof (Appendix C). The proof is complete without them. The use of AI tools is described at the end of the paper.

2. Integrals and known results

Let $z > 1$. For integers $h, j, k, l, m, q \geq 0$ with $j + k - m,\ j + q - m,\ h + m - k,\ h + q - k \geq 0$ put

$$R_z(x, y) = z^{k-l-q}\, \frac{x^j (1-x)^h\, y^k (1-y)^l\, (1 - y + zy)^{j+q-m}}{\big(x(1-y) + zy\big)^{j+k-m+1}},$$

and, with small positively oriented circles,

$$J_z^{(0)} = \int_0^1 \int_0^1 R_z\, dy\, dx, \qquad J_z^{(1)} = \int_0^1 \frac{1}{2\pi i} \oint_{|y - \frac{x}{x-z}| = \varepsilon} R_z\, dy\, dx,$$

$$J_z^{(2)} = \frac{1}{2\pi i} \oint_{|x - z| = \varepsilon'} \frac{1}{2\pi i} \oint_{|y - \frac{x}{x-z}| = \varepsilon} R_z\, dy\, dx, \qquad J_z = J_z^{(0)} - (\log z) J_z^{(1)}.$$

For $t = (h, j, k, l, m) \in \mathbb{Z}^5_{\geq 0}$ define $I_z^{(0)}(t)$, $I_z^{(1)}(t)$, $I_z^{(2)}(t)$ and $I_z(t)$ in the same way from

$$z^{-l-m}\, \frac{x^j (1-x)^h\, y^k (1-y)^l}{\big(x(1-y) + zy\big)^{j+k-m+1}} \qquad \text{[6, (2.1)–(2.4)]}.$$

For $\mathcal{J} = J_z$ or $\mathcal{J} = I_z(t)$, with components $\mathcal{J}^{(0)}, \mathcal{J}^{(1)}, \mathcal{J}^{(2)}$, put

$$c(\mathcal{J}) = \big(c_1(\mathcal{J}), c_2(\mathcal{J})\big) = \big(\mathcal{J} + \mathrm{Li}_2(1/z)\, \mathcal{J}^{(2)},\ \mathcal{J}^{(2)}\big).$$

For $t = (h, j, k, l, m)$ let

$$u(t) = (l + m - j,\ h + m - k,\ h + j - l,\ j + k - m), \qquad T(t) = h!\, j!\, k!\, l!\, m!,$$
$$H(t) = \max_i u_i, \qquad K(t) = \max\{u_1, \min(u_2, u_3), u_4\},$$

and call t *admissible* if $u(t) \in \mathbb{Z}^4_{\geq 0}$. Put

$$\rho(t) = (h, m, l, k, j), \qquad \sigma(t) = (h + m - k,\ j + k - m,\ m,\ l,\ k),$$

and let $O(t)$ be the orbit of t under the group generated by ρ and σ. Both maps permute the multiset $\{h, j, k, l, m, u_1, u_2, u_3, u_4\}$ [6, (3.2), (3.15)], so every $s \in O(t)$ is admissible when t is.

(F1) [8, Lemma 2.1] Let $H_1 = \max\{l+q,\ h+m-k,\ h+j-l,\ j+k-m\}$, $H_2 = \max\{l+q,\ \min(h+m-k, h+j-l),\ j+k-m\}$, $\alpha = \max\{0, l+q-k, l+q-m\}$ and $\beta = \max\{0, k + l - h\}$. There are $P, Q \in \mathbb{Z}[z]$ with

$$d_{H_1} d_{H_2}\, z^\alpha (1-z)^\beta\, c(J_z) = (P(z), Q(z)).$$

(F2) [8, Lemma 3.1] $c(J_z(h, j, k, l, m, q)) = c(J_z(j, h, m, l, k, q))$.

(F3) [6, Theorem 2.1 and (2.9)] Let $\alpha(t) = \max\{j+k, k+l, l+m\}$ and $\beta(t) = \max\{0, k+l-h\}$. There are $P_t, Q_t \in \mathbb{Z}[z]$ with

$$d_{H(t)} d_{K(t)}\, z^{\alpha(t)} (z-1)^{\beta(t)}\, c(I_z(t)) = (P_t(z), Q_t(z)).$$

(F4) [6, (3.10)–(3.16)] If t is admissible and $s \in O(t)$, then $c(I_z(t))/T(t) = c(I_z(s))/T(s)$.

In the sources, (F1) and (F3) read $\varphi\mathcal{J} = P - Q\,\mathrm{Li}_2(1/z)$ and $\varphi\mathcal{J}^{(2)} = Q$, with φ the displayed multiplier; (F2) and (F4) are stated there for each component. Numbering in [8] refers to the preprint version.

Lemma 2.1. *Let $a = j+q-m$. Then*

$$c(J_z(h,j,k,l,m,q)) = \sum_{\lambda=0}^{a} \binom{a}{\lambda} z^{j+k+\lambda}\, c(I_z(h,\, j,\, k+\lambda,\, l+a-\lambda,\, m+\lambda)), \tag{2.1}$$

$$c(J_z(h,j,k,l,m,q)) = \sum_{\lambda=0}^{a} \binom{a}{\lambda} z^{j+k} (1-z)^{\lambda}\, c(I_z(h,\, j,\, k,\, l+\lambda,\, m)). \tag{2.2}$$

Proof. Expand $(1-y+zy)^a$ as $((1-y)+zy)^a$ and as $(z+(1-z)(1-y))^a$. Each term has the denominator and the pole $y = x/(x-z)$ of the corresponding I_z-integrand. The powers of z are $z^{k-l-q+\lambda}\cdot z^{(l+a-\lambda)+(m+\lambda)} = z^{j+k+\lambda}$ and $z^{k-l-q+a-\lambda}\cdot z^{(l+\lambda)+m} = z^{j+k}$. The identities hold for each component, hence for c. □

Identity (2.1) is used in the proof of [8, Lemma 2.1] and in [4, (3.11)–(3.12)].

3. The divisor

3.1. **Local bounds.** For the tuple in (1.1), (F1) gives $H_1 = 323N$, $H_2 = 322N$, $\alpha = 132N$, $\beta = 107N$. Since $(1-z)^{\beta} = \pm 3^{107N}$ at $z = 4$,

$$D_N\, c(G_N) \in \mathbb{Z}^2, \qquad G_N = c_1(G_N) - \theta\, c_2(G_N). \tag{3.1}$$

Fix $N \geq 1$ and a prime $p > 3$ with $p^2 > 430N$. Put $x = N/p$ and $F_a = \lfloor ax \rfloor$, and let $[\,\cdot\,]$ be the Iverson bracket. For $0 \leq n \leq 430N$, $v_p(n!) = \lfloor n/p \rfloor$ and $v_p(d_n) = [\, n \geq p\,]$.

Lemma 3.1. *Let t be admissible with all entries of t and $u(t)$ in $[0, 430N]$, and let $s \in O(t)$. Then, for $i = 1, 2$,*

$$v_p\big(c_i(I_4(t))\big) \;\geq\; \sum_{\nu=1}^{5} \Big(\lfloor \frac{t_\nu}{p} \rfloor - \lfloor \frac{s_\nu}{p} \rfloor \Big) - [\, H(s) \geq p\,] - [\, K(s) \geq p\,].$$

Proof. By (F4), $c(I_4(t)) = T(t)T(s)^{-1} c(I_4(s))$. The entries of s and $u(s)$ are those of t and $u(t)$ in some order. Apply (F3) to s, where $z = 4$ and $z-1 = 3$ are units at p. □

Lemma 3.2. *Suppose $p < 322N$. For $b \in \{190, 215\}$ put $e_b = 2b - 165$ and*

$$\begin{aligned} C &= 2F_{240} - F_{265} - F_{215}, \\ U_b &= 2F_{240} + 2F_b - F_{e_b} - F_{323} - F_{322}, \\ V_b &= 2F_b + F_{157} - F_{e_b} - F_{322}. \end{aligned}$$

Then, for $i = 1, 2$,

$$v_p\big(D_N\, c_i(G_N)\big) \;\geq\; M := \max\{0,\, C,\, U_{190},\, U_{215},\, V_{190},\, V_{215}\}. \tag{3.2}$$

If moreover $p > 265N$, then $v_p(D_N\, c_i(G_N)) \geq 1$.

Proof. Here $v_p(D_N) = 2$. By (F2), $c(G_N)$ is unchanged if (1.1) is replaced by $(240N, 240N, 215N, 157N, 190N, 165N)$. So (2.1), applied to (1.1) for $b = 190$ and to this tuple for $b = 215$, gives

$$c(G_N) = \sum_{\lambda=0}^{bN} \binom{bN}{\lambda} 4^{(240+b)N+\lambda}\, c(I_4(t)),$$

where $t = (240, 240, b, 157+b, 405-b)N + (0,0,1,-1,1)\lambda$. Here $u(t) = (322,\ 645-2b,\ 323-b,\ e_b)N + (0,0,1,0)\lambda$; all entries of t and $u(t)$ lie in $[0, 430N]$. The tuples

$$\begin{aligned} s_1 &= \sigma t = (645-2b,\ e_b,\ 405-b,\ 157+b,\ b)N + (0,0,1,-1,1)\lambda,\\ s_2 &= \sigma\rho\sigma t = (323-b,\ 322,\ e_b,\ 405-b,\ 157+b)N + (1,0,0,1,-1)\lambda,\\ s_3 &= \sigma\rho\sigma\rho\sigma t = (240,\ 240,\ 322,\ e_b,\ 405-b)N + (0,0,0,0,1)\lambda \end{aligned}$$

lie in $O(t)$. Put $\tau = \lambda/p$. Then $v_p\binom{bN}{\lambda} = F_b - \lfloor\tau\rfloor - \lfloor bx-\tau\rfloor$. Also $[\,H(s) \ge p\,] + [\,K(s) \ge p\,] \le 2 = v_p(D_N)$ for each $s \in O(t)$, so Lemma 3.1 shows that the λ-th term of $D_N c_i(G_N)$ has valuation at least

$$g(s) = \sum_{\nu} \big(\lfloor t_\nu/p\rfloor - \lfloor s_\nu/p\rfloor\big) + F_b - \lfloor\tau\rfloor - \lfloor bx-\tau\rfloor.$$

Direct subtraction gives

$$\begin{aligned} g(s_1) &= C + \big(F_b - \lfloor\tau\rfloor - \lfloor bx-\tau\rfloor\big),\\ g(s_2) &= 2F_{240} + F_b - F_{e_b} - F_{322} + \big(\lfloor bx+\tau\rfloor - \lfloor\tau\rfloor\big)\\ &\qquad - \big(\lfloor bx-\tau\rfloor + \lfloor(323-b)x+\tau\rfloor\big),\\ g(s_3) &= F_b - F_{e_b} - F_{322} + \big(\lfloor bx+\tau\rfloor - \lfloor\tau\rfloor\big)\\ &\qquad + \big(\lfloor(157+b)x-\tau\rfloor - \lfloor bx-\tau\rfloor\big). \end{aligned}$$

By $\lfloor y\rfloor + \lfloor w\rfloor \le \lfloor y+w\rfloor$,

$$\begin{aligned} F_b - \lfloor\tau\rfloor - \lfloor bx-\tau\rfloor \ge 0, &\qquad \lfloor bx+\tau\rfloor - \lfloor\tau\rfloor \ge F_b,\\ \lfloor(157+b)x-\tau\rfloor - \lfloor bx-\tau\rfloor \ge F_{157}, &\qquad \lfloor bx-\tau\rfloor + \lfloor(323-b)x+\tau\rfloor \le F_{323}. \end{aligned}$$

Hence $g(s_1) \ge C$, $g(s_2) \ge U_b$ and $g(s_3) \ge V_b$ for every λ. The ultrametric inequality and (3.1) give (3.2).

Now let $p > 265N$. By (2.2),

$$c(G_N) = \sum_{\lambda=0}^{190N} \binom{190N}{\lambda} 4^{430N}(-3)^\lambda\, c(I_4(t')),$$

where

$$\begin{aligned} t' &= (240, 240, 190, 157, 215)N + (0,0,0,1,0)\lambda,\\ u(t') &= (132, 265, 323, 215)N + (1,0,-1,0)\lambda,\\ s' = \sigma\rho\sigma t' &= (323, 132, 215, 215, 157)N + (-1,1,0,0,1)\lambda,\\ u(s') &= (240, 265, 240, 190)N. \end{aligned}$$

So $[\,H(s') \ge p\,] = [\,K(s') \ge p\,] = 0$. Also $v_p\binom{190N}{\lambda} = 0$ and $F_{240} = F_{215} = F_{190} = 0$. By Lemma 3.1 and $v_p(D_N) = 2$, each term of $D_N c_i(G_N)$ has valuation at least $2 - \lfloor 323x-\tau\rfloor - \lfloor 132x+\tau\rfloor \ge 2 - \lfloor 455x\rfloor \ge 1$, as $455x < 2$. □

3.2. **Periodic intervals.** Since 5 divides 240, 265 and 215, and $2\cdot 240 = 265+215$, we get $C(x+1/5) = C(x)$. For $1 \le k \le 47$ put $j_k = \lfloor 5k/48 \rfloor$, $\varrho_k = 5k - 48j_k$, $a_k = k/240$ and

$$b_k = \begin{cases} (k-j_k)/215, & \varrho_k \le 21, \\ (k+j_k+1)/265, & \varrho_k \ge 22. \end{cases}$$

Lemma 3.3. *For $1 \le k \le 47$ one has $a_k < b_k \le (k+1)/240$ and $C = 1$ on (a_k, b_k).*

Proof. Here $0 < \varrho_k < 48$, $(k-j_k)/215 - a_k = \varrho_k/10320$, $(k+j_k+1)/265 - a_k = (48-\varrho_k)/12720$ and $265(k-j_k) - 215(k+j_k+1) = 10\varrho_k - 215$. So $b_k = \min\{(k-j_k)/215,\ (k+j_k+1)/265\} > a_k$, and $48j_k \le 5k$ gives $b_k \le (k+j_k+1)/265 \le (k+1)/240$. For $a_k < x < b_k$,

$$k - j_k - 1 < k - \tfrac{5k}{48} < 215x < k - j_k, \qquad k + j_k \le k + \tfrac{5k}{48} < 265x < k + j_k + 1,$$

and $F_{240} = k$. Hence $C = 2k - (k - j_k - 1) - (k + j_k) = 1$. □

Put $\mathcal{C} = \bigcup_{r=0}^{9} \bigcup_{k=1}^{47} (r/5 + a_k,\ r/5 + b_k) \subset (1/240, 2)$; by Lemma 3.3 these 470 intervals are pairwise disjoint. Let $\mathcal{E}_1, \dots, \mathcal{E}_{18}$ be the open intervals

(1/322, 1/265)	(1/215, 2/323)	(2/215, 3/322)	(3/215, 4/265),
(4/190, 7/323)	(7/265, 9/322)	(5/157, 11/323)	(7/190, 12/322),
(9/215, 14/323)	(11/215, 17/322)	(11/190, 19/323)	(17/265, 21/322),
(11/157, 23/323)	(14/190, 24/322)	(15/190, 26/323)	(19/215, 29/322),
(21/190, 36/322)	(22/190, 38/323).		

They are pairwise disjoint and lie in $(1/322, 1/5)$. Define

$$\mu(x) = \mathbf{1}_{\mathcal{C}}(x) + \sum_{i=1}^{18} \mathbf{1}_{\mathcal{E}_i}(x).$$

Proposition 3.4. *Let $N \ge 1$ and let $p > 3$ be a prime with $p^2 > 430N$. Then $\mu(N/p) \le v_p(D_N)$ and $v_p(D_N\, c_i(G_N)) \ge \mu(N/p)$ for $i = 1, 2$.*

Proof. Let $x = N/p$ with $\mu(x) > 0$. Then $1/322 < x < 2$, so $p < 322N$, $v_p(D_N) = 2$ and $\mu(x) \le 2$. If $x \notin \bigcup_i \mathcal{E}_i$, then $\mu(x) = \mathbf{1}_{\mathcal{C}}(x) \le [\,C = 1\,] \le M$ by Lemmas 3.3 and 3.2. If $x \in \mathcal{E}_1$, then $x \notin \mathcal{C}$ and $p > 265N$, so Lemma 3.2 gives $v_p(D_N\, c_i(G_N)) \ge 1 = \mu(x)$. If $x \in \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_{18}$, then $M \ge 1 + [\,C = 1\,] \ge \mu(x)$ by Table A.1, which lists the floors F_a for $a \in \{157, 190, 215, 240, 265, 322, 323\}$ on pieces with no jump inside. □

4. Density

Lemma 4.1. *Let $\Delta_N = \prod p^{\mu(N/p)}$ over primes $p > 3$ with $p^2 > 430N$. Then $\Delta_N \mid D_N$, $(D_N/\Delta_N)\, c(G_N) \in \mathbb{Z}^2$, and*

$$\lim_{N\to\infty} \frac{\log \Delta_N}{40N} = \delta := \frac{1}{40} \int_0^2 \mu(x)\, \frac{dx}{x^2} > 4.7670,$$
$$\lim_{N\to\infty} \frac{\log D_N}{40N} = c_D := \frac{645 + 132\log 4 + 107\log 3}{40} < 23.63856.$$

Proof. The first two claims follow from Proposition 3.4 and (3.1). If $\mu(N/p) > 0$, then $p > N/2$, so $p > 3$ and $p^2 > 430N$ for $N > 1720$. For each of the 488 intervals I of $\mathcal{C}$ and $\mathcal{E}_1, \dots, \mathcal{E}_{18}$, the prime number theorem gives

$$\sum_{N/p \in I} \log p = N \int_I x^{-2}\, dx + o(N).$$

Summing over I gives δ. Likewise $\log d_n = n + o(n)$ gives c_D, and $c_D < 23.63856$ by Table B.1.

The intervals of $\mathcal{C}$ with $r = 0$ contribute

$$\delta_0 = \frac{1}{40} \sum_{k=1}^{47} \Big(\frac{1}{a_k} - \frac{1}{b_k} \Big) = \frac{1}{40} \Big(\sum_{k=1}^{47} \frac{240}{k} - \sum_{\varrho_k \le 21} \frac{215}{k - j_k} - \sum_{\varrho_k \ge 22} \frac{265}{k + j_k + 1} \Big) > 1.6187,$$

with the two cases listed in Table A.2. Let $\ell = \sum_k (b_k - a_k) = 1139/22790$ and $\bar{y} = \ell^{-1} \sum_k \int_{a_k}^{b_k} y\, dy = 164041/1640160$. For $r \ge 1$, Jensen's inequality for $y \mapsto (r/5 + y)^{-2}$ gives

$$\sum_k \Big(\frac{1}{r/5 + a_k} - \frac{1}{r/5 + b_k} \Big) = \sum_k \int_{a_k}^{b_k} \frac{dy}{(r/5 + y)^2} \ge \frac{\ell}{(r/5 + \bar{y})^2}.$$

So the intervals with $1 \le r \le 9$ contribute at least $\frac{\ell}{40} \sum_r (r/5 + \bar{y})^{-2} > 0.0260$ (Table B.1). The intervals $\mathcal{E}_i$ contribute $\frac{202007398913}{64696932300} > 3.1223$. Thus $\delta > 1.6187 + 0.0260 + 3.1223 = 4.7670$. □

5. Analytic estimates

5.1. **The double integral.** Write $J_N^{(0)}$, $J_N^{(1)}$ for $J_4^{(0)}$, $J_4^{(1)}$ at the tuple (1.1).

Lemma 5.1. *Let $y = \eta/(\eta - 4)$. Then $R_4(x, y)\, dy = -f_N(x, \eta)\, d\eta$, where*

$$f_N(x, \eta) = \frac{\Psi(x, \eta)^N}{(x - \eta)(4 - \eta)}, \qquad \Psi(x, \eta) = \frac{x^{240}(1 - x)^{240}\, \eta^{190}(1 - \eta)^{190}}{(x - \eta)^{215}(4 - \eta)^{322}}.$$

In particular, with

$$\Phi(x, \tau) = \Psi(x, -\tau) = \frac{x^{240}(1 - x)^{240}\, \tau^{190}(1 + \tau)^{190}}{(x + \tau)^{215}(4 + \tau)^{322}},$$

we have

$$J_N^{(0)} = \int_0^1 \int_0^\infty \frac{\Phi(x, \tau)^N}{(x + \tau)(4 + \tau)}\, d\tau\, dx.$$

Proof. We have

$$y = \frac{-\eta}{4 - \eta}, \qquad 1 - y = \frac{4}{4 - \eta}, \qquad 1 + 3y = \frac{4(1 - \eta)}{4 - \eta},$$
$$x(1 - y) + 4y = \frac{4(x - \eta)}{4 - \eta}, \qquad dy = \frac{-4\, d\eta}{(4 - \eta)^2}.$$

The exponents are $j = h = 240N$, $k = 190N$, $l = 157N$, $j + q - m = 190N$, $j + k - m + 1 = 215N + 1$ and $k - l - q = -132N$. The powers of 4 cancel, since $-132 + 157 + 190 - 215 = 0$ and $4^{-1} \cdot 4 = 1$. Those of $4 - \eta$ add to $-(322N + 1)$, and $(-\eta)^{190N} = \eta^{190N}$. For $J_N^{(0)}$ put $\tau = -\eta$. Then τ runs over $[0, \infty)$ as y runs over $[0, 1)$, and $-f_N\, d\eta = f_N(x, -\tau)\, d\tau$. □

Lemma 5.2. *For $N \ge 1$, $10^{-9} e^{-40(18.88)N} < J_N^{(0)} < e^{-40(18.87333)N}$.*

Proof. Weighted AM–GM, $x+\tau \ge 215\,(x/25)^{25/215}(\tau/190)^{190/215}$, gives

$$\begin{aligned}\Phi &\le \Big(\frac{5}{43}\Big)^{25}\Big(\frac{38}{43}\Big)^{190} x^{215}(1-x)^{240}\,\frac{(1+\tau)^{190}}{(4+\tau)^{322}}\\ &\le \Phi_+ := \Big(\frac{5}{43}\Big)^{25}\Big(\frac{38}{43}\Big)^{190}\Big(\frac{43}{91}\Big)^{215}\Big(\frac{48}{91}\Big)^{240}\frac{(95/22)^{190}}{(161/22)^{322}},\end{aligned}$$

with maxima at $x = 43/91$ and $\tau = 73/22$. By Table B.1, $-\log\Phi_+ > 40(18.87333)$. Also

$$\int_0^1\!\!\int_0^\infty \frac{d\tau\,dx}{(x+\tau)(4+\tau)} = \int_0^1 \frac{\log(4/x)}{4-x}\,dx < \frac{1+\log 4}{3} < 1,$$

so $J_N^{(0)} < \Phi_+^N$. For the lower bound let $\mathcal{B} = [0.4711, 0.4713] \times [3.3999, 3.4001]$. Bounding each factor of Φ by its least value on $\mathcal{B}$ gives

$$\Phi \ge \Phi_- := \frac{0.4711^{240}\,0.5287^{240}\,3.3999^{190}\,4.3999^{190}}{3.8714^{215}\,7.4001^{322}} \qquad \text{on } \mathcal{B},$$

and $(x+\tau)(4+\tau) < 29$ there. By Table B.1, $-\log\Phi_- < 40(18.88)$. Hence $J_N^{(0)} \ge \operatorname{area}(\mathcal{B})\,\Phi_-^N/29 = 4\cdot 10^{-8}\,\Phi_-^N/29 > 10^{-9}e^{-40(18.88)N}$. □

5.2. **The residue integral.** Put $A = (3+4i)/5$, $B = (5-12i)/13$ and, for $0 \le t, u \le 1$,

$$X(t) = \frac{t}{t+A(1-t)}, \qquad Y(u) = \frac{u}{u+B(1-u)}, \qquad \omega = t(1-u)+u(1-t).$$

Lemma 5.3. *For $0 \le t, u \le 1$ and $x \in [0,1]$:*

(1) $|X'| \le 5/4$, $|Y'| \le 13/9$, $|X(1-X)| \le \frac{5}{4}t(1-t) \le \frac{5}{4}\omega$, $|Y(1-Y)| \le \frac{13}{9}u(1-u) \le \frac{13}{9}\omega$, $|Y| \le 1$ *and* $|4-Y| \ge 3$;
(2) $|X-Y| \ge 7\omega/\sqrt{65}$, *and* $|x-Y| \ge 2\omega_x/\sqrt{13}$ *with* $\omega_x = x(1-u)+u(1-x)$;
(3) $\operatorname{Im} X(t) < 0 < \operatorname{Im} Y(u)$ *for* $0 < t, u < 1$.

Proof. One has $|t+A(1-t)|^2 = 1-\frac{4}{5}t(1-t) \ge \frac{4}{5}$, $|u+B(1-u)|^2 = 1-\frac{16}{13}u(1-u) \ge \frac{9}{13}$, $X' = A/(t+A(1-t))^2$ and $1-X = A(1-t)/(t+A(1-t))$; similarly for Y. Also $|u+B(1-u)|^2 - u^2 = (1-u)(13-3u)/13 \ge 0$, so $|Y| \le 1$. Next $t(1-t) \le \omega$, since $\omega - t(1-t)$ equals $u(1-2t)+t^2 \ge 0$ for $t \le \frac{1}{2}$ and is at least $(1-t)^2$ for $t \ge \frac{1}{2}$; likewise $u(1-u) \le \omega$. For (2), $X - Y = \frac{tB(1-u)-uA(1-t)}{(t+A(1-t))(u+B(1-u))}$ and the denominator has modulus at most 1. With $\xi = t(1-u)$, $\zeta = u(1-t)$ and $\gamma = \operatorname{Re}(B\bar{A}) = -33/65$,

$$|tB(1-u) - uA(1-t)|^2 = \xi^2 + \zeta^2 - 2\gamma\xi\zeta = \tfrac{1-\gamma}{2}\,\omega^2 + \tfrac{1+\gamma}{2}(\xi-\zeta)^2 \ge \tfrac{49}{65}\,\omega^2.$$

For real x take $A = 1$, $t = x$ and $\gamma = \operatorname{Re} B = 5/13$. Part (3) follows from $\operatorname{Im} X = -\frac{4}{5}t(1-t)/|t+A(1-t)|^2$ and its analogue. □

Lemma 5.4. *For $N \ge 1$, $J_N^{(1)} = \pi^{-1}\operatorname{Im} Z_N$, where*

$$Z_N = \int_0^1\!\!\int_0^1 f_N\big(X(t), Y(u)\big)\,X'(t)\,Y'(u)\,dt\,du.$$

Proof. Residues are invariant under the change of variables of Lemma 5.1, and $y = x/(x-4)$ corresponds to $\eta = x$. Hence $J_N^{(1)} = -\int_0^1 \operatorname{Res}_{\eta=x} f_N(x,\eta)\,dx$. Fix $x \in (0,1)$. By Lemma 5.3, Y runs from 0 to 1 in the upper half-plane and $\bar{Y}$ in the lower one, both in the closed unit disc. So $\bar{Y}$ followed by Y reversed is a positively

oriented Jordan curve around x, and the other pole $\eta = 4$ lies outside. As f_N has real coefficients, the residue theorem gives

$$\operatorname*{Res}_{\eta=x} f_N = \frac{\overline{W(x)} - W(x)}{2\pi i} = -\frac{\operatorname{Im} W(x)}{\pi}, \qquad W(x) = \int_0^1 f_N(x, Y(u))\, Y'(u)\, du.$$

By Lemma 5.3, also with t replaced by x,

$$|f_N(x,Y)\,Y'| \ll_N \omega_x^{215N-1}, \qquad |f_N(X,Y)\,X'Y'| \ll_N \omega^{215N-1},$$

so both are bounded on $[0,1]^2$. For fixed $u \in (0,1)$ the function $w \mapsto f_N(w, Y(u))$ is holomorphic off $w = Y(u)$, and $\operatorname{Im} Y(u) > 0 \geq \operatorname{Im} X(t)$. Cauchy's theorem moves $\int_0^1 dx$ to the arc X, and Fubini's theorem gives $\int_0^1 W(x)\, dx = Z_N$. □

Lemma 5.5. *For $N \geq 1$, $|J_N^{(1)}| < e^{-40(18.8995)N}$.*

Proof. Put $r = t/(1-t) = e^R$ and $s = u/(1-u) = e^S$, so $X = r/(r+A)$ and $Y = s/(s+B)$. With

$$Q_1 = r^2 + \tfrac{6}{5}r + 1, \quad Q_2 = s^2 + \tfrac{10}{13}s + 1, \quad Q_3 = r^2 + \tfrac{66}{65}rs + s^2, \quad Q_4 = 9s^2 + \tfrac{120}{13}s + 16,$$

one has $|X(1-X)| = r/Q_1$, $|Y(1-Y)| = s/Q_2$, $|X-Y|^2 = Q_3/(Q_1Q_2)$ and $|4-Y|^2 = Q_4/Q_2$. So $\log|\Psi(X,Y)| = L$, where

$$L = 240\log r + 190\log s - \tfrac{265}{2}\log Q_1 + \tfrac{157}{2}\log Q_2 - \tfrac{215}{2}\log Q_3 - 161\log Q_4.$$

Let $\ell_\gamma(v) = \log(\cosh v + \gamma)$ and $\kappa = \log(4/3)$. Since $Q_1 = 2r(\cosh R + \frac{3}{5})$, $Q_2 = 2s(\cosh S + \frac{5}{13})$, $Q_3 = 2rs(\cosh(R-S) + \frac{33}{65})$ and $Q_4 = 24s(\cosh(S-\kappa) + \frac{5}{13})$,

$$\begin{aligned} L = {} & -\tfrac{323}{2}\log 2 - 161\log 24 - \tfrac{265}{2}\ell_{3/5}(R) + \tfrac{157}{2}\ell_{5/13}(S) \\ & - \tfrac{215}{2}\ell_{33/65}(R-S) - 161\,\ell_{5/13}(S-\kappa). \end{aligned}$$

Here $\ell_\gamma'' = (1+\gamma\cosh v)/(\cosh v + \gamma)^2 > 0$ and, for $0 < \gamma < 1$,

$$(\log \ell_\gamma'')' = -\sinh v\, \frac{2 + \gamma\cosh v - \gamma^2}{(1+\gamma\cosh v)(\cosh v + \gamma)}, \qquad |(\log \ell_\gamma'')'| \leq \frac{2|\sinh v|}{\cosh v + \gamma} < 2,$$

so $\ell_\gamma''(v-\kappa) \geq \frac{9}{16}\ell_\gamma''(v)$. Therefore

$$\begin{aligned} -d^2 L = {} & \tfrac{265}{2}\ell_{3/5}''(R)\, dR^2 + \tfrac{215}{2}\ell_{33/65}''(R-S)(dR - dS)^2 \\ & + \big(161\,\ell_{5/13}''(S-\kappa) - \tfrac{157}{2}\ell_{5/13}''(S)\big) dS^2 \end{aligned}$$

is positive definite, since $161 \cdot \frac{9}{16} > \frac{157}{2}$. So L is strictly concave and $L_{RS} > 0$. The partial derivatives are

$$\begin{aligned} L_R &= 240 - 265\,\frac{r(r+3/5)}{Q_1} - 215\,\frac{r(r+33s/65)}{Q_3}, \\ L_S &= 190 + 157\,\frac{s(s+5/13)}{Q_2} - 215\,\frac{s(s+33r/65)}{Q_3} - 322\,\frac{s(9s+60/13)}{Q_4}. \end{aligned}$$

Table B.2 gives $L_R, L_S > 0$ at $(r,s) = (\frac{29}{25}, \frac{139}{100})$ and $L_R, L_S < 0$ at $(\frac{59}{50}, \frac{71}{50})$. Consider the box $\frac{29}{25} \leq r \leq \frac{59}{50}$, $\frac{139}{100} \leq s \leq \frac{71}{50}$. Since $L_{RS} > 0$, we get $L_R > 0$ on its face $r = \frac{29}{25}$, $L_R < 0$ on $r = \frac{59}{50}$, $L_S > 0$ on $s = \frac{139}{100}$ and $L_S < 0$ on $s = \frac{71}{50}$. So the maximum of L on the box is an interior critical point, hence the global maximum. At $(r,s) = (\frac{117}{100}, \frac{7}{5})$, Table B.2 gives $-\frac{1}{4} < L_R < 0 < L_S < \frac{3}{10}$ and $L < -755.99$. Concavity gives

$$\sup L < -755.99 + \tfrac{1}{4}\log\tfrac{117}{116} + \tfrac{3}{10}\log\tfrac{71}{70} < -755.99 + \tfrac{1}{464} + \tfrac{3}{700} < -755.98.$$

By Lemma 5.3, $|X'Y'| < 2$, $|X - Y| \ge \frac{4}{5}\omega$ and $|4 - Y| \ge 3$. With $w = 1 - 2u$,

$$\int_0^1\int_0^1 \frac{dt\,du}{\omega} = \int_0^1 \frac{\log\frac{1-u}{u}}{1-2u}\,du = \frac{1}{2}\int_{-1}^1 \log\frac{1+w}{1-w}\,\frac{dw}{w} = 2\sum_{n\ge 0}\frac{1}{(2n+1)^2} = \frac{\pi^2}{4}.$$

By Lemma 5.4,

$$|J_N^{(1)}| \le \frac{1}{\pi}\cdot 2\cdot\frac{5}{4}\cdot\frac{1}{3}\cdot\frac{\pi^2}{4}\,e^{-40(18.8995)N} = \frac{5\pi}{24}\,e^{-40(18.8995)N} < e^{-40(18.8995)N}. \quad \square$$

6. Proof of Theorem 1

By Lemmas 5.2 and 5.5, $J_N^{(0)} > 10^{-9}e^{-40(18.88)N} > (\log 4)\,|J_N^{(1)}|$ for large N. Hence $G_N = J_N^{(0)} - (\log 4)J_N^{(1)} > 0$ and $G_N < 3e^{-40(18.87333)N}$. Put

$$\Lambda_N = \frac{D_N}{\Delta_N}\,G_N = \frac{D_N}{\Delta_N}\,c_1(G_N) - \frac{D_N}{\Delta_N}\,c_2(G_N)\,\theta.$$

By (3.1) and Lemma 4.1, $\Lambda_N \in \mathbb{Z} + \mathbb{Z}\theta$. Also $\Lambda_N > 0$ for large N, and

$$\limsup_{N\to\infty}\frac{\log\Lambda_N}{40N} \le c_D - \delta - 18.87333 < 23.63856 - 4.7670 - 18.87333 = -0.00177.$$

If $q_0\theta \in \mathbb{Z}$ for an integer $q_0 \ge 1$, then $q_0\Lambda_N$ is a positive integer for large N and $q_0\Lambda_N \to 0$, which is impossible. $\square$

Appendix A. Floor data

Table A.1 cuts each $\mathcal{E}_i$ at the jumps of the F_a. On each piece, $F_a = \lfloor ax_0\rfloor$ with x_0 the left end. The columns C, U_b, V_b follow from Lemma 3.2, and the last column is $1 + [\,C = 1\,]$. The first row uses (2.2) instead of C, U_b, V_b.

Table A.1. Floors on the pieces of $\mathcal{E}_1, \dots, \mathcal{E}_{18}$.

piece	F_{157}	F_{190}	F_{215}	F_{240}	F_{265}	F_{322}	F_{323}	C	U_{190}	U_{215}	V_{190}	V_{215}	$1+[C=1]$
$(1/322, 1/265)$	0	0	0	0	0	1	1	by (2.2)					1
$(1/215, 1/190)$	0	0	1	1	1	1	1	0	−1	1	−2	0	1
$[1/190, 2/323)$	0	1	1	1	1	1	1	0	1	1	0	0	1
$(2/215, 3/322)$	1	1	2	2	2	2	3	0	−1	1	−1	1	1
$(3/215, 4/265)$	2	2	3	3	3	4	4	0	−1	1	−1	1	1
$(4/190, 7/323)$	3	4	4	5	5	6	6	1	2	1	1	0	2
$(7/265, 9/323)$	4	5	5	6	7	8	8	0	1	−1	1	−1	1
$[9/323, 6/215)$	4	5	5	6	7	8	9	0	0	−2	1	−1	1
$[6/215, 9/322)$	4	5	6	6	7	8	9	−1	−1	0	0	1	1
$(5/157, 7/215)$	5	6	6	7	8	10	10	0	0	−2	1	−1	1
$[7/215, 8/240)$	5	6	7	7	8	10	10	−1	−1	0	0	1	1
$[8/240, 9/265)$	5	6	7	8	8	10	10	1	1	2	0	1	2
$[9/265, 11/323)$	5	6	7	8	9	10	10	0	1	1	0	0	1
$(7/190, 12/323)$	5	7	7	8	9	11	11	0	1	−1	1	−1	1
$[12/323, 8/215)$	5	7	7	8	9	11	12	0	0	−2	1	−1	1
$[8/215, 12/322)$	5	7	8	8	9	11	12	−1	−1	0	0	1	1
$(9/215, 8/190)$	6	7	9	10	11	13	13	0	−1	1	−2	0	1
$[8/190, 14/323)$	6	8	9	10	11	13	13	0	1	1	0	0	1
$(11/215, 10/190)$	8	9	11	12	13	16	16	0	−1	1	−1	1	1
$[10/190, 17/322)$	8	10	11	12	13	16	17	0	0	0	1	1	1
$(11/190, 14/240)$	9	11	12	13	15	18	18	−1	0	−1	1	0	1
$[14/240, 19/323)$	9	11	12	14	15	18	18	1	2	1	1	0	2

piece	F_{157}	F_{190}	F_{215}	F_{240}	F_{265}	F_{322}	F_{323}	C	U_{190}	U_{215}	V_{190}	V_{215}	$1+[C{=}1]$
$(17/265, 21/323)$	10	12	13	15	17	20	20	0	1	-1	1	-1	1
$[21/323, 14/215)$	10	12	13	15	17	20	21	0	0	-2	1	-1	1
$[14/215, 21/322)$	10	12	14	15	17	20	21	-1	-1	0	0	1	1
$(11/157, 17/240)$	11	13	15	16	18	22	22	-1	-1	0	0	1	1
$[17/240, 23/323)$	11	13	15	17	18	22	22	1	1	2	0	1	2
$(14/190, 24/323)$	11	14	15	17	19	23	23	0	1	-1	1	-1	1
$[24/323, 16/215)$	11	14	15	17	19	23	24	0	0	-2	1	-1	1
$[16/215, 24/322)$	11	14	16	17	19	23	24	-1	-1	0	0	1	1
$(15/190, 17/215)$	12	15	16	18	20	25	25	0	0	-2	1	-1	1
$[17/215, 19/240)$	12	15	17	18	20	25	25	-1	-1	0	0	1	1
$[19/240, 21/265)$	12	15	17	19	20	25	25	1	1	2	0	1	2
$[21/265, 26/323)$	12	15	17	19	21	25	25	0	1	1	0	0	1
$(19/215, 14/157)$	13	16	19	21	23	28	28	0	-1	1	-2	0	1
$[14/157, 17/190)$	14	16	19	21	23	28	28	0	-1	1	-1	1	1
$[17/190, 29/323)$	14	17	19	21	23	28	28	0	1	1	1	1	1
$[29/323, 29/322)$	14	17	19	21	23	28	29	0	0	0	1	1	1
$(21/190, 36/323)$	17	21	23	26	29	35	35	0	1	-1	1	-1	1
$[36/323, 24/215)$	17	21	23	26	29	35	36	0	0	-2	1	-1	1
$[24/215, 36/322)$	17	21	24	26	29	35	36	-1	-1	0	0	1	1
$(22/190, 25/215)$	18	22	24	27	30	37	37	0	0	-2	1	-1	1
$[25/215, 28/240)$	18	22	25	27	30	37	37	-1	-1	0	0	1	1
$[28/240, 31/265)$	18	22	25	28	30	37	37	1	1	2	0	1	2
$[31/265, 38/323)$	18	22	25	28	31	37	37	0	1	1	0	0	1

TABLE A.2. The two cases in the definition of b_k.

j_k	$\varrho_k \le 21$	$k - j_k$	$\varrho_k \ge 22$	$k + j_k + 1$
0	$1 \le k \le 4$	$1, \dots, 4$	$5 \le k \le 9$	$6, \dots, 10$
1	$10 \le k \le 13$	$9, \dots, 12$	$14 \le k \le 19$	$16, \dots, 21$
2	$20 \le k \le 23$	$18, \dots, 21$	$24 \le k \le 28$	$27, \dots, 31$
3	$29 \le k \le 33$	$26, \dots, 30$	$34 \le k \le 38$	$38, \dots, 42$
4	$39 \le k \le 42$	$35, \dots, 38$	$43 \le k \le 47$	$48, \dots, 52$

APPENDIX B. NUMERICAL VALUES

Values are truncated to the digits shown.

TABLE B.1. Constants.

quantity	value
$-\log \Phi_+/40$	$18.8733376741\dots$
$-\log \Phi_-/40$	$18.8775167011\dots$
c_D	$23.6385592638\dots$
δ_0	$1.6187800851\dots$
$\frac{\ell}{40} \sum_{r=1}^{9} (r/5 + \bar{y})^{-2}$	$0.0260767666\dots$
$\frac{1}{40} \sum_i \int_{\mathcal{E}_i} x^{-2}\, dx$	$3.1223644109\dots$

TABLE B.2. Values of L_R, L_S and L.

(r, s)	L_R	L_S	L
(29/25, 139/100)	$0.57635\ldots$	$0.62026\ldots$	
(59/50, 71/50)	$-0.52544\ldots$	$-1.02457\ldots$	
(117/100, 7/5)	$-0.22659\ldots$	$0.26833\ldots$	$-755.99795\ldots$

APPENDIX C. COMPUTER CHECKS

The two scripts in Table C.1 recheck only the statements listed there, in exact rational arithmetic. They are included as ancillary files. The proof does not use them. For $\log v$ the analytic script writes $v = 2^e q$ with $e \in \mathbb{Z}$, $1 \le q \le 2$, and uses 30 terms of $\log q = 2\sum_{n\ge 0} w^{2n+1}/(2n+1)$, $w = (q-1)/(q+1)$, with tail at most $2w^{61}/(61(1-w^2))$.

TABLE C.1. Scripts.

script	statements checked	time
`check_arith.py`	the intervals $\mathcal{E}_i$ and Tables A.1, A.2 as printed; Lemma 3.3; the tuples s_1, s_2, s_3, s' and the floor sums in Lemma 3.2; Proposition 3.4 for $1/265 \le x < 2$ at each jump of the F_a and between consecutive jumps, and $M = 0$, $\mu = 1$ on $\mathcal{E}_1$; the numbers in Lemma 4.1; 57/40 and 3.35 in Section 1; rows 4–6 of Table B.1	10 s
`check_arith.py --brute 1,2,3,4,5,6,7,8`	for $N \le 8$ and all primes p with $3 < p \le 323N$ and $p^2 > 430N$ (for larger p, $\mu(N/p) = 0$), the best bound from Lemma 3.1 over (2.1) with $b = 190,215$ and (2.2), all λ and all $s \in O(t)$, is at least $\mu(N/p)$	3 min
`check_analytic.py`	the exponents in Lemma 5.1; the constants in Lemma 5.2; the identities in Lemmas 5.3 and 5.5 at rational points; rows 1–3 of Table B.1 and Table B.2 as printed; $23.63856 - 4.7670 - 18.87333 = -0.00177$	1 s

USE OF AI TOOLS

Codex (OpenAI, GPT-5.6 Sol) searched for the parameters in (1.1) and wrote the Python code and the interval lists. Claude Code (Anthropic, Claude Fable 5) checked the code, the parameters and the tables. The author checked every step of the proof, every table and every reference, and takes full responsibility for the content of the paper.

REFERENCES

[1] F. Calegari, V. Dimitrov, and Y. Tang, *The linear independence of* 1, $\zeta(2)$, *and* $L(2, \chi_{-3})$, preprint (2024), arXiv:2408.15403v2.

[2] *NIST Digital Library of Mathematical Functions*, `https://dlmf.nist.gov/`, Release 1.2.8 of 2026-09-15, F. W. J. Olver et al., eds.

[3] M. Hata, *Rational approximations to the dilogarithm*, Trans. Amer. Math. Soc. **336** (1993), no. 1, 363–387. MR1147401

[4] T. Prellberg, *Three irrationality results for the dilogarithm: Rhin–Viola and Viola–Zudilin constructions at* $-1/4$*,* $1/5$*, and* $-1/3$, preprint (2026), arXiv:2609.06895v1.
[5] G. Rhin and C. Viola, *On a permutation group related to* $\zeta(2)$, Acta Arith. **77** (1996), no. 1, 23–56. MR1404975
[6] G. Rhin and C. Viola, *The permutation group method for the dilogarithm*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 3, 389–437. MR2185863
[7] G. Rhin and C. Viola, *Linear independence of* 1*,* Li_1 *and* Li_2, Mosc. J. Comb. Number Theory **8** (2019), no. 1, 81–96. MR3864310
[8] C. Viola and W. Zudilin, *Linear independence of dilogarithmic values*, J. Reine Angew. Math. **736** (2018), 193–223; preprint version: Max-Planck-Institut für Mathematik Preprint Series 2014 (26), `http://www.mpim-bonn.mpg.de/preblob/5515`. MR3769989

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