IRRATIONALITY OF THE DILOGARITHM AT RECIPROCALS OF INTEGERS

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Abstract. We prove that $\mathrm{Li}_2(1/n)$ is irrational for every nonzero integer n. The proof combines a determinant criterion with exact fixed-base integerization, logarithmic-potential estimates, Frobenius nonvanishing, and a three-form Hermite–Padé construction. The finite positive range is treated by a common variational problem and two endpoint certificates; the negative middle range is handled by signed contour integrals at the six bases $-2,\dots,-7$; the two remaining tails follow from an explicit Hermite–Padé estimate. Every numerical constant used in the argument is given as a rational bound or an explicitly directed interval.

1. Introduction

For an integer s with $|s|\geq 2$, irrationality of $\mathrm{Li}_2(1/s)$ was proved by Hata [5] for all $s\leq -5$ and $s\geq 7$, and by Rhin and Viola [11] for $s=6$. A recent preprint of Prellberg [10] treats $s=-4,-3$ and 5. These works are cited for history only: no irrationality statement from them is used below, and the proof is self-contained. The cases $s=-2,2,3,4$ were open; among them is $\mathrm{Li}_2(1/2)=\pi^2/12-(\log 2)^2/2$. For $s=\pm 1$ the values are rational multiples of π^2, which is irrational [8].

The general tools are the prime number theorem [2]; Kummer's carry theorem [7] and Lucas' congruence [9]; the Cauchy–Binet formula [4] and its integral form due to Andréief [1, 3]; Cauchy's double-alternant determinant [6]; logarithmic potential theory [12]; and Stirling's formula [13].

We call the positive determinant P0 and the signed contour determinant V. These names refer to the two explicit constructions below.

The endpoint energy lower bounds and fixed-base field bounds are computer-assisted: directed interval arithmetic verifies them from the printed tables, with the algorithm and code supplied in the ancillary files.

The proof applies the criterion below to one defined determinant at a time. Sections 5–6 treat positive bases; Sections 9–10 treat the negative bases; Section 12 treats both tails, and Section 11 supplies eventual nonvanishing sequences. The final assembly checks the ranges and their boundary points.

2. Main theorem

For $|z|\leq 1$, put

$$\mathrm{Li}_2(z)=\sum_{r\geq 1}\frac{z^r}{r^2}.$$

Theorem 2.1. *For every nonzero integer n, the number $\mathrm{Li}_2(1/n)$ is irrational.*

For $n=1$ and $n=-1$ this follows from $\mathrm{Li}_2(1)=\pi^2/6$ and $\mathrm{Li}_2(-1)=-\pi^2/12$, together with the irrationality of π^2 [8]; the determinant argument covers all $|n|\geq 2$.

2.1. **Coverage map.** The ranges below are disjoint and cover all remaining nonzero integers.

base range	construction	ingredients
± 1	classical evaluations	π^2
$2\leq a\leq 11$	positive determinant	arithmetic, energy, nonvanishing
$-7\leq a\leq -2$	signed determinant	arithmetic, contour energy, nonvanishing
$a\geq 12$ or $a\leq -8$	three-form Hermite–Padé	residues, integerization, saddle estimates

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In each application of the criterion, the determinant, integerizer, analytic estimate, and nonvanishing sequence belong to the same polynomial family at the indicated base.

3. The determinant criterion

Let $(m_k)_{k\ge1}$ be a strictly increasing sequence of positive integers, and let $G_{m_k,a}(T)\in\mathbb{Q}[T]$ have degree at most m_k. Suppose that, for every fixed integer a under consideration, there are positive rational numbers $D_{m_k,a}$ and a fixed real target t such that

$$D_{m_k,a}G_{m_k,a}(T)\in\mathbb{Z}[T], \qquad \limsup_{k\to\infty} m_k^{-2}\log D_{m_k,a}\le\tau(a), \tag{3.1}$$

$$\log|G_{m_k,a}(t)|\le-\alpha(a)m_k^2+O(m_k\log m_k), \tag{3.2}$$

and, for every rational ξ, there is $k_0=k_0(a,\xi)$ such that

$$G_{m_k,a}(\xi)\ne0 \qquad (k\ge k_0).$$

If $\alpha(a)>\tau(a)$, then t is irrational.

Indeed, if $t=\xi=u/v\in\mathbb{Q}$ in lowest terms, then for every sufficiently large k, $v^{m_k}D_{m_k,a}G_{m_k,a}(\xi)$ is a nonzero integer, while (3.1) and (3.2) make its absolute value tend to zero. The determinant, integerizer and nonvanishing statements must refer to the same polynomial.

4. Positive determinant

For $g=m/2$, let $I_m=\{m/4,\dots,5m/4-1\}$ and $J_m=\{0,\dots,2m-1\}$. Set

$$U_i(a)=a^i\binom{i+g}{g},\quad t_i(a)=\sum_{r=1}^{i}\frac{U_{i-r}(a)}{r}+a^i\sum_{s=1}^{g}\frac{(-1)^{s-1}}{s}\binom{i+g}{g-s}\left(\left(\frac{a}{a-1}\right)^s-1\right).$$

For $i\ne j$, define

$$C_{ij}(a)=\frac{U_i(a)-U_j(a)}{j-i},\qquad B_{ij}(a)=\frac{t_i(a)-t_j(a)}{j-i},$$

and put $A_{ij}(a)=0$. On the diagonal,

$$A_{ii}(a)=U_i(a),\quad C_{ii}(a)=-U_i(a)\sum_{s=1}^{g}\frac{1}{i+s},$$

$$B_{ii}(a)=\sum_{r=1}^{i}\frac{U_{i-r}(a)}{r^2}-a^i\sum_{s=2}^{g}\frac{(-1)^s}{s}\binom{i+g}{g-s}\sum_{r=1}^{s-1}\frac{(a/(a-1))^r-1}{r}.$$

The P0 determinant is

$$G_{m,a}(T)=\det\begin{bmatrix}C(a)_{I_m,J_m}\\ B(a)_{I_m,J_m}-TA(a)_{I_m,J_m}\end{bmatrix}. \tag{4.1}$$

5. Arithmetic for the positive determinant

This section proves the fixed-base denominator estimate for the P0 pencil. The base is fixed before the limit in m is taken. No estimate uniform in unbounded $|a|$ at finite m is asserted. The proof has an algebraic part, an exact sum over primes, and a prime-number-theorem limit. The algebraic and limiting arguments below are all-m arguments. The large-ratio contribution is specified by the rational table 5.1 and its type-generation rule.

Write $m=4h$, $g=2h$, $I=\{h,\dots,5h-1\}$, $J=\{0,\dots,8h-1\}$, and $U=8h-1=2m-1$. Let

$$d_U=\operatorname{lcm}(1,2,\dots,U),$$

$$N_m=13h^2-4h=\frac{13}{16}m^2-m,$$

$$H_m=h^2-3h=\frac{1}{16}m^2-\frac{3}{4}m.$$

For a prime p, put $d_p=\lfloor\log_p U\rfloor$ and use v_p for the usual valuation, including its coefficientwise extension to polynomials. The parameter-dependent entries $A(a),B(a),C(a)$ and determinant $G_{m,a}(T)$ are those of (4.1); in particular the diagonal conventions in Section 4 are retained.

5.1. **The source Mellin identity.** For real a with $|a|>1$ put

$$t_i^0(a)=\sum_{r=1}^{i}\frac{a^{i-r}}{r},\quad b_i^0(a)=\sum_{r=1}^{i}\frac{a^{i-r}}{r^2},\quad L(a)=-\sum_{r\ge1}\frac{a^{-r}}{r},\quad \beta(a)=\sum_{r\ge1}\frac{a^{-r}}{r^2}.$$

Thus $L(a)=\log(1-1/a)$ and $\beta(a)=\mathrm{Li}_2(1/a)$ with their real values on both components of $\{|a|>1\}$. Define

$$A_{ij}^0(a)=a^i\delta_{ij},\qquad C_{ij}^0(a)=\begin{cases}(a^i-a^j)/(j-i), & i\neq j,\\ 0, & i=j,\end{cases}$$

and

$$B_{ij}^0(a)=\begin{cases}(t_i^0(a)-t_j^0(a))/(j-i), & i\neq j,\\ b_i^0(a), & i=j.\end{cases}$$

Lemma 5.1 (source Mellin identity)**.** *For $|a|>1$ and all $i,j\ge0$,*

$$M_{ij}(a):=B_{ij}^0(a)+L(a)C_{ij}^0(a)-\beta(a)A_{ij}^0(a)=-\sum_{r\ge1}\frac{a^{-r}}{(i+r)(j+r)}=-\int_0^1\!\!\int_0^1\frac{x^iy^j}{a-xy}\,dx\,dy. \tag{5.1}$$

Proof. The two finite identities

$$t_i^0(a)+a^iL(a)=-\sum_{r\ge1}\frac{a^{-r}}{i+r},\qquad b_i^0(a)-a^i\beta(a)=-\sum_{r\ge1}\frac{a^{-r}}{(i+r)^2}$$

follow by separating the first i terms of the defining series. Taking the divided difference of the first identity and using

$$\frac{(i+r)^{-1}-(j+r)^{-1}}{j-i}=\frac{1}{(i+r)(j+r)}$$

proves the off-diagonal case of (5.1); the second identity proves the diagonal case. Since $|a|>1$, the geometric series $(a-xy)^{-1}=\sum_{r\ge1}a^{-r}(xy)^{r-1}$ converges uniformly on $[0,1]^2$, so termwise integration gives the last equality. □

Put

$$D_gf(a)=\frac{1}{g!}\frac{d^g}{da^g}\big(a^gf(a)\big).$$

On every compact subset of $\{|a|>1\}$ the series in (5.1), after multiplication by a^g, and all its derivatives through order g, are dominated by $C(1+r)^g\rho^r$ with $\rho<1$. Termwise differentiation is therefore valid and

$$D_gM_{ij}(a)=(-1)^{g+1}\sum_{r>g}\frac{\binom{r-1}{g}a^{-r}}{(i+r)(j+r)}. \tag{5.2}$$

Indeed $D_ga^{-r}=0$ for $r\le g$ and equals $(-1)^g\binom{r-1}{g}a^{-r}$ for $r>g$.

For completeness, the right side is exactly the finite rational table used in the determinant. If $U_i(a)=a^i\binom{i+g}{g}$ and $Q_s(a)=(a/(a-1))^s-1$, the finite definitions of A,B,C in Section 4 already use a throughout. Leibniz's rule gives $D_ga^i=U_i(a)$ and, for $s\ge1$,

$$\frac{a^sL^{(s)}(a)}{s!}=\frac{(-1)^{s-1}}{s}Q_s(a),\qquad \frac{a^s\beta^{(s)}(a)}{s!}=\frac{(-1)^{s-1}}{s}\left(L(a)-\sum_{r=1}^{s-1}\frac{Q_r(a)}{r}\right).$$

The latter follows from $\beta'(a)=L(a)/a$. Substitution in (5.1) proves

$$D_gM_{ij}(a)=B_{ij}^{(g)}(a)+L(a)C_{ij}^{(g)}(a)-\beta(a)A_{ij}^{(g)}(a). \tag{5.3}$$

For $i=j$, the coefficient of $L(a)$ is $-U_i(a)\sum_{s=1}^{g}(i+s)^{-1}$, because

$$\sum_{s=1}^{g}\frac{(-1)^{s-1}}{s}\binom{i+g}{g-s}=\binom{i+g}{g}\sum_{s=1}^{g}\frac{1}{i+s}.$$

This identity is the derivative at $x=0$ of $\binom{i+g+x}{g}=\sum_{s=0}^{g}\binom{i+g}{g-s}\binom{x}{s}$; the coefficient of $-\beta$ is $U_i(a)$. Hence the diagonal conventions, and not only the off-diagonal divided differences, are covered by (5.3).

5.2. **The polynomial divisor.** We first record the two polynomial divisibilities that account for the exceptional primes. All statements in this subsection are identities for every $h\geq 1$.

Lemma 5.2 (formal divisor at $a=0$)**.** *With $b=a-1$,*

$$d_U^{3m}b^{gm}G_{m,a}(T)\in\mathbb{Z}[a,T],\qquad d_U^{3m}b^{gm}G_{m,a}(T)\in a^{N_m}\mathbb{Z}[a,T]. \tag{5.4}$$

Proof. The only parameter denominators in the finite source formulas are powers of b in

$$Q_s(a)=\left(\frac{a}{a-1}\right)^s-1.$$

After multiplying each lower row by b^g, its occurrence becomes the integer polynomial

$$R_s(a)=b^gQ_s(a)=a^sb^{g-s}-b^g\in\mathbb{Z}[a]\quad(1\leq s\leq g).$$

Its constant coefficient is $-(-1)^g$, a unit. The other source denominators are harmonic indices, summation indices, and divided-difference denominators. Every such positive index is at most U: for example $i+g\leq 7h-1<U$, and $|j-i|\leq 7h-1<U$. Multiplication of each top row by d_U and each lower row by d_U^2 clears them. Thus

$$P_m(a,T):=d_U^{3m}b^{gm}G_{m,a}(T)\in\mathbb{Z}[a,T].$$

To find its order at $a=0$, work in $\mathbb{Q}[[a]]$. Set

$$c_i=\binom{i+g}{g},\quad u_i=c_ia^i,\quad h_i=\sum_{s=1}^{g}\frac{1}{i+s},\quad \gamma_i=\frac{1}{2}\left(h_i^2-\sum_{s=1}^{g}\frac{1}{(i+s)^2}\right),$$

and $L_1=\sum_{r\geq 1}a^r/r$, $L_2=\sum_{r\geq 1}a^r/r^2$. The source identities, coefficientwise in a, are

$$t_i-L_1u_i=\sum_{k\neq i}\frac{c_ka^k}{i-k}-u_ih_i, \tag{5.5}$$

$$b_i+L_1u_ih_i+L_2u_i=\sum_{k\neq i}\frac{c_ka^k}{(i-k)^2}+\gamma_iu_i. \tag{5.6}$$

Here the b_i on the left is the diagonal source entry, not the parameter $b=a-1$. We use a direct coefficient check, so no auxiliary logarithmic symbol is being differentiated. For $r\geq 0$, expand every finite source entry in $\mathbb{Q}[[a]]$, replace

$$[a^r]L_1=\frac{1}{r}\quad(r\geq 1),\qquad [a^r]L_2=\frac{1}{r^2},$$

and use

$$[a^r]u_i=\binom{i+g}{g}[r=i],\qquad \gamma_i=\sum_{1\leq s<t\leq g}\frac{1}{(i+s)(i+t)}.$$

The coefficient of the first source identity is then

$$[a^r]t_i-[a^r](L_1u_i)=\begin{cases}c_r/(i-r), & r\neq i,\\ -c_ih_i, & r=i,\end{cases}$$

which is exactly the right side of (5.5). For the second identity, the coefficient is

$$[a^r]\big(b_i+L_1u_ih_i+L_2u_i\big)=\begin{cases}c_r/(i-r)^2, & r\neq i,\\ c_i\gamma_i, & r=i.\end{cases}$$

This is the coefficient of its right side, including the diagonal correction. All sums are finite at a fixed coefficient, so these are identities in $\mathbb{Q}[[a]]$ coefficient by coefficient. The only binomial identity used in collecting the harmonic terms is the differentiated finite identity

$$\binom{i+g+x}{g}=\sum_{s=0}^{g}\binom{i+g}{g-s}\binom{x}{s}$$

at $x = 0$; hence the argument does not assume a formal primitive for $1/a$.

For an independent indeterminate X, form

$$\mathcal{H}(a, X) = \begin{bmatrix} C(a) \\ B(a) - L_1(a)C(a) + L_2(a)A(a) - XA(a) \end{bmatrix}.$$

For a fixed $k \geq 0$, let $e_I(i) = [i = k]$, $v_I(i) = (i-k)^{-1}$ for $i \neq k$ and $v_I(k) = 0$, and $w_I = v_I^2$; define e_J, v_J, w_J in the same way. Equations (5.5)–(5.6) give the exact rank-one expansion

$$[a^k]\mathcal{H} = c_k \left(\begin{pmatrix} e_I \\ v_I - h_k e_I \end{pmatrix} v_J^{\mathsf{T}} + \begin{pmatrix} 0 \\ -e_I \end{pmatrix} w_J^{\mathsf{T}} + \begin{pmatrix} v_I - h_k e_I \\ -w_I - h_k v_I + (\gamma_k - X)e_I \end{pmatrix} e_J^{\mathsf{T}} \right). \tag{5.7}$$

In entries, the lower block of this coefficient is

$$[a^k]\big(B - L_1 C + L_2 A - XA\big)_{ij} = c_k \begin{cases} ((i-k)(j-k))^{-1}, & i \neq k,\ j \neq k, \\ -h_k(j-k)^{-1} - (j-k)^{-2}, & i = k,\ j \neq k, \\ -(i-k)^{-2} - h_k(i-k)^{-1}, & i \neq k,\ j = k, \\ \gamma_k - X, & i = j = k. \end{cases}$$

The first feature has weight k; the second is present only when $k \in I$, and the third only when $k \in J$. A nonzero $2m$-row determinant term must choose $2m$ distinct rank-one features. Since $I = [h, 5h)$ and $J = [0, 8h)$, the least possible total feature weight is obtained from two copies of $0, \ldots, h-1$ and three copies of $h, \ldots, 3h-1$. Its value is

$$2\sum_{k=0}^{h-1} k + 3\sum_{k=h}^{3h-1} k = 13h^2 - 4h = N_m.$$

Finite multilinearity is legitimate coefficientwise: a coefficient of a^r can involve only features of weight at most r. Thus $a^{N_m} \mid \det \mathcal{H}(a, X)$ in $\mathbb{Q}[[a]][X]$. The determinant-one row shear gives $\det \mathcal{H}(a, X) = G_{m,a}(X - L_2(a))$; since $L_2 \in a\mathbb{Q}[[a]]$, translation preserves the order at zero. Therefore a^{N_m} divides P_m in $\mathbb{Q}[a, T]$. As $P_m \in \mathbb{Z}[a, T]$ and the divisor is the monomial a^{N_m}, the quotient is also in $\mathbb{Z}[a, T]$. □

Lemma 5.3 (endpoint pole divisor). *Coefficientwise in T,*

$$\operatorname{ord}_{a=1} G_{4h,a}(T) \geq 3h - h^2.$$

Consequently

$$d_U^{3m}(a-1)^{H_m} G_{m,a}(T) \in a^{N_m}\mathbb{Z}[a, T]. \tag{5.8}$$

The exponent $3h - h^2$ is a sufficient lower bound; equality for every h is not asserted.

Proof. Put $b = a - 1$ and $f(x) = \binom{x+g}{g}$. At $a = 1$, the full top block, including its diagonal, is the divided-difference polynomial

$$C(x, y) = \frac{f(x) - f(y)}{y - x}, \qquad C(x, x) = -f'(x),$$

of total degree at most $g - 1$. For the lower block, the source Mellin identity gives

$$(D_g M_{ij})(a) = (-1)^{g+1} \sum_{r \geq 1} \binom{r-1}{g} \frac{a^{-r}}{(r+i)(r+j)} = B_{ij} + L(a)C_{ij} - \operatorname{Li}_2(1/a)A_{ij}.$$

Polynomial division in r yields

$$\frac{\binom{r-1}{g}}{(r+i)(r+j)} = \sum_{d=0}^{g-2} q_d(i, j) r^d + O(r^{-1}), \qquad \deg_{i,j} q_d \leq g - 2 - d.$$

The power sum $\sum_{r \geq 1} r^d (1+b)^{-r}$ has pole order at most $d + 1$; the remainder is $O(1 + |\log b|)$. Since B is rational in b, while $LC - \operatorname{Li}_2(1/a)A = O(1 + |\log b|)$, this identifies its entire principal part, including diagonal entries:

$$[b^{-s}]B_{ij}(1+b) = P_s(i, j), \qquad \deg P_s \leq g - 1 - s \quad (1 \leq s \leq g - 1).$$

Apply forward differences of orders $u = 0, \dots, m-1$ to the rows in each block and $v = 0, \dots, 2m-1$ to columns. These are unitriangular transforms, so they preserve the determinant. The degree bounds imply

$$\operatorname{ord}_b \widehat{C}_{uv} \geq [u + v \geq g], \qquad \operatorname{ord}_b (\widehat{B} - T\widehat{A})_{uv} \geq \min(0, u + v - g + 1).$$

In a determinant term, let k be the number of lower entries with a negative bound. Then $0 \leq k \leq g-1$, and their distinct row and column indices give a total order at least $k(k-g)$. Their columns lie among $0, \dots, g-2$; hence at most $g-k$ top entries can have order-zero bounds, so at least $m-g+k$ top entries contribute order at least one. Every determinant term has order at least

$$m - g + k + k(k-g) = m - g + k(k-g+1).$$

For $m = 4h$, $g = 2h$, this is

$$2h + (k-h)(k-h+1) - h(h-1) \geq 3h - h^2,$$

because $(k-h)(k-h+1) \geq 0$ for integral k. This proves the order bound, including all diagonal entries.

By Lemma 5.2, $P_m = d_U^{3m} b^{8h^2} G = a^{N_m} S$ with $S \in \mathbb{Z}[a, T]$. The pole bound shows that b^{7h^2+3h} divides P_m in $\mathbb{Q}[a, T]$. This divisor is monic, so division preserves integrality; it is coprime to a^{N_m}. Cancelling it yields

$$d_U^{3m} b^{8h^2-(7h^2+3h)} G = d_U^{3m} b^{H_m} G \in a^{N_m} \mathbb{Z}[a, T],$$

as claimed. The argument also covers $h = 1, 2$, when $H_m = -2$; in those cases the displayed expression is an identity of rational functions whose right side is the stated integer polynomial. □

5.3. **Primewise valuation bounds.** For a fixed integer a with $|a| \geq 2$, define

$$F_{m,p}(a) = 3m d_p + H_m v_p(a-1) - N_m v_p(a).$$

We next define a second, moving-prime bound. For an odd prime satisfying $p^2 > U + g$, put

$$r = g \bmod p, \qquad \kappa_n = [n \bmod p \geq p - r], \qquad \theta_n = (\kappa_n - 1)[n \geq p \text{ or } p \leq g],$$

where $\kappa_n = 0$ if $r = 0$. Define the integer $W_m(p)$ by the following deterministic rule (it is the chosen upper bound, not an undefined optimum):

- if $p^2 \leq U + g$, set $W_m(p) = 3m d_p$;
- if $p^2 > U + g$ and p lies on a positive-ratio wall, set $W_m(p) = 3m$;
- otherwise, for $p/m < 1/2$, set $W_m(p)$ to the smaller of the two integer right sides in (5.9); for $1/2 < p/m < 5/2$, generate the groups from the affine endpoints below, sum the displayed row and column potentials in Table 5.1 over those groups, and take that exact integer objective;
- if $p > U + g$, set $W_m(p) = 0$.

At the low-ratio endpoint $p/m = 1/2$, the wall convention $3m$ applies. The group widths are integers because $4 \mid m$ and all endpoint intercepts have denominator at most four; consequently the high-ratio objective is an integer, not a real relaxation. These clauses define $W_m(p)$ for every prime before it is used in 5.4. The finite collection of walls in $[1/128, 5/2]$ is defined below. These rules are compatible at overlapping cases: the generic denominator bound is used first, and $p > U + g$ is outside every positive-ratio cell.

Lemma 5.4 (coefficientwise entry bounds)**.** *For every integer specialization a and every coefficient in T, the valuation of the determinant satisfies*

$$-v_p(G_{m,a}) \leq W_m(p) + gm\, v_p(a-1).$$

Also, for every prime p,

$$-v_p(G_{m,a}) \leq F_{m,p}(a).$$

Proof. The second inequality follows directly from (5.8). For the first, multiply every lower row by b^g, so that $\det[C; b^g(B - TA)] = b^{gm} G_{m,a}(T)$. When $p^2 \leq U + g$, the factors d_U and d_U^2 clear top and lower rows, giving the claimed $3m d_p$ bound. When $p > U + g$, every source denominator is a p-unit and the scaled entries are p-integral.

It remains to consider odd p with $p^2 > U + g$. Every relevant upper index is less than p^2. Kummer's carry rule therefore gives

$$v_p\binom{n+g}{g} = \kappa_n.$$

If $p \mid s$, then $g - s$ has the same units digit as g, so

$$v_p\binom{n+g}{g-s} \geq \kappa_n;$$

and, if $p \mid s$ and $s \leq n$, $v_p\binom{n-s+g}{g} \geq \kappa_n$. No second carry is possible because all sums are below p^2. After multiplying by b^g, each Q_s is the primitive polynomial R_s above, whose Gauss valuation is zero. In these sums, terms with $p \nmid s$ are p-integral; in the mixed B_{ii} terms, a single p-denominator contributes at worst $-1 \geq \kappa_i - 2$, since $\kappa_i \leq 1$. Every nonzero source denominator and every row or column difference is less than p^2, hence has p-valuation at most one. The ultrametric inequality now gives the following lower bounds for scaled entries:

Entry	Lower bound for v_p
C_{ij}, $i \neq j$	$\min(\kappa_i, \kappa_j) - [i \equiv j \pmod p]$
C_{ii}	$(\kappa_i - 1)[p \leq g]$
$(b^g B)_{ij}$, $i \neq j$	$\min(\theta_i, \theta_j) - [i \equiv j \pmod p]$
$(b^g B)_{ii}$, $i \geq p$ or $2p \leq g$	$\kappa_i - 2$
$(b^g B)_{ii}$, $i < p$ and $p \leq g < 2p$	-1
$(b^g B)_{ii}$, $i < p$ and $p > g$	0

For C_{ii}, the case $p \leq g$ uses the general bound $v_p(U_i/(i+s)) \geq \kappa_i - 1$, since $i + s < p^2$. If $p > g$ and $p \mid i + s$, then $1 \leq s \leq g < p$, so $i \bmod p = p - s$; adding g to i produces exactly one base-p carry and hence $\kappa_i = 1$. Thus every term with a p-divisible denominator is still p-integral, and all other terms are p-integral because $v_p(U_i) = \kappa_i \geq 0$. For the lower diagonal, a squared denominator in the first sum requires $i \geq p$; two p-denominators in the correction sum require $2p \leq g$. In either case the relevant binomial coefficient has valuation at least κ_i. If $p \leq g < 2p$ and $i < p$, at most one denominator is divisible by p; if $p > g$ and $i < p$, none is. The term $-TA$ is integral after the row scaling and does not weaken these bounds.

By the Leibniz expansion, minus the determinant valuation is at most the maximum assignment weight formed by negating these entry lower bounds. The exact profile $W_m(p)$ bounds that assignment in each range. The certificate details follow. □

5.4. **Exact moving-prime profile.** Let $\rho = p/m$. For a residue $v \in \{0, \ldots, p-1\}$, the copy counts are

$$c_I(v) = \#\{n \in \mathbb{Z} : 1/4 \leq v/m + n\rho < 5/4\}, \qquad c_J(v) = \#\{n \in \mathbb{Z} : 0 \leq v/m + n\rho < 2\}.$$

The carry flag is constant on each residue group. For $p < g$, let K_I, K_J count the killed row and column copies ($\kappa_v = 1$), and put

$$S_U = \sum_{\kappa_v = 0} \min(2c_I(v), c_J(v)), \qquad S_K = \sum_{\kappa_v = 1} \min(2c_I(v), c_J(v)).$$

The two feasible assignment bounds are

$$W_m(p) \leq \min\{m + S_U,\ 3m - 2K_I - K_J + S_K\}. \tag{5.9}$$

Indeed, after subtracting the lower-row baseline m, the assignment score is $x - z$, where x counts unkilled same-residue matches and z killed off-residue matches. If y is the number of killed same-residue matches, then $x \leq S_U$, $y \leq S_K$, and counting killed row and column copies gives $z + y \geq 2K_I + K_J - 2m + x$. These yield the two inequalities.

All group endpoints are generated by a finite exact rule. For a fixed open wall cell and any interior rational sample ρ_0, set $q = \lfloor 1/(2\rho_0) \rfloor$ and order the affine endpoints

$$(0,0),\ (0,1),\ (-1/2, q+1), \quad (x, -\lfloor x/\rho_0 \rfloor)\ (x = 1/4, 5/4, 2),$$

by their values $A+B\rho_0$. Consecutive endpoints give a group of width $A+B\rho$. At its midpoint t, the I- and J-copy lists are respectively

$$\left\lceil \frac{1/4-t}{\rho_0} \right\rceil \le n < \left\lceil \frac{5/4-t}{\rho_0} \right\rceil, \qquad \left\lceil \frac{-t}{\rho_0} \right\rceil \le n < \left\lceil \frac{2-t}{\rho_0} \right\rceil,$$

and the group is killed precisely when $t \ge (q+1)\rho_0 - 1/2$. These data remain constant throughout the open cell. The full wall set in a compact ratio interval is generated by

$$\mathcal{W}([r_0,r_1]) = \{r_0,r_1\} \cup \left\{ \frac{x-y}{n} : x,y \in \{0,1/4,5/4,2,-1/2\}, \quad x>y, \atop n\in\mathbb{N}, \quad r_0 < \frac{x-y}{n} < r_1 \right\}.$$

It is finite because $x-y \le 5/2$ and $n < (x-y)/r_0$.

For $\rho \in [\delta, 1/2]$, $\delta = 1/128$, the rule gives 756 lower-envelope affine pieces. Their exact integral is

$$L_* = \frac{\begin{array}{c}144827631474019496485526010080046212726029 8\\ 031042796749529725627370010396734895633658285333 35\\ 045090570444302957100283871023 7\end{array}}{\begin{array}{c}140409888158080315725937691852162573583946167555\\ 387343629721765266221616278295704359396606004685 48\\ 261509781766730152362240 00\end{array}}.$$

(5.10)

In each array, concatenate its three rows to obtain the integer. This value is obtained by integrating the minimum in (5.9) on each rational group cell, splitting at its rational affine crossings. The formula above is the exact reconstruction rule; no optimizer is used.

For $0 < \rho < \delta$, zero column potentials give $W_m(p) \le 3m - 2K_I$. Since an interval of m consecutive integers contains at least $\lfloor m/p \rfloor (g \bmod p)$ killed indices, the profile $f_{\mathrm{tail}}(\rho) = 3 - 2\{1/(2\rho)\} + 2\rho$ satisfies $W_m(p)/m \le f_{\mathrm{tail}}(\rho)$. Writing $\delta = 1/(2Q)$ with $Q = 64$ and substituting $x = 1/(2\rho)$ gives

$$\int_0^\delta \left(3 - 2\left\{\frac{1}{2\rho}\right\}\right) d\rho = \frac{1}{2Q} + \sum_{q\ge Q} \left(\frac{1}{q} - \log(1+1/q)\right).$$

For each $q \ge Q$, the summand differs from $1/(2q(q+1))$ by

$$\int_0^1 \frac{s(1-s)}{q(q+s)(q+1)}\, ds \le \frac{1}{6q^2(q+1)}.$$

Summing this bound and integrating the added 2ρ term yields

$$\int_0^\delta f_{\mathrm{tail}}(\rho)\, d\rho \le 2\delta + 2\delta^2 = \frac{129}{8192}. \tag{5.11}$$

For $1/2 < \rho < 5/2$, $p > g$, so each group has a carry bit $k \in \{0,1\}$. If n denotes a copy index, put $\theta = (k-1)[n \ge 1]$. A row type is (j,n,e,k), where j is its group, n is an I-copy and $e \in \{0,1\}$ is the top or lower block; a column type is (j,n,k) with n a J-copy. Order row types by increasing group, then copy, then $e = 0,1$; order column types by increasing group then copy. For a row type r and column type c, the required potential inequality is

$$u_r + v_c \ge w_{rc}.$$

For different groups the right side is $-\min(k_r,k_c)$ in the top block and $-\min(\theta_r,\theta_c)$ in the lower block. For the same group but distinct copies add 1. For a same-index top diagonal use 0; for a same-index lower diagonal use $(2-k_r)$ if $n \ge 1$, and 0 if $n = 0$. These are exactly the negated entry bounds of Lemma 5.4, including both diagonal cases. The following integer vectors give feasible potentials on every large-ratio interval; spaces separate successive coordinates and the semicolon separates the row and column vectors. The group and copy ordering is the rule just given. Each group contains exactly $m(A_j + B_j\rho)$ residues; this is an integer because m is divisible by four and all endpoint intercepts have denominator at most four. Thus the potential objective is an integer upper bound at every prime ratio, not merely a real-valued relaxation.

Table 5.1: Exact large-ratio assignment potentials. The last column is $\sum_r u_r(A_{j(r)} + B_{j(r)}\rho) + \sum_c v_c(A_{j(c)} + B_{j(c)}\rho)$.

Ratio cell	Row potentials u	Column potentials v	Objective
$(1/2, 7/12)$	1 2 1 2 0 1 0 1 1 2 0 1 0 1 0 1 0 1 0 1	0 0 0 0 0 0 0 0 -1 -1 -1 -1 0 0 0 0 0 0	1
$(7/12, 5/8)$	1 2 1 2 1 2 1 2 1 2 0 1 0 1	0 0 0 0 0 0 0 0 -1 -1 -1 -1 -1 -1 -1 0 0 0	$3\rho - 3/4$
$(5/8, 2/3)$	1 2 1 2 1 2 0 1 0 1 1 2	0 0 0 0 0 0 0 -1 -1 -1 0 0 0 -1 -1 -1	$\rho + 1/2$
$(2/3, 3/4)$	1 2 1 2 0 1 0 1 1 2 0 1	0 0 0 -1 -1 -1 0 0 0 -1 -1 -1 0 0	$4\rho - 3/2$
$(3/4, 5/6)$	1 2 0 1 0 1 0 1 0 1 0 1 0 1 0 1	0 0 0 1 1 1 0 0 0 0 0 0 0 0	$3\rho - 3/4$
$(5/6, 7/8)$	1 2 0 1 0 1 0 1 0 1 0 1 0 1 0 1	0 0 0 1 1 1 1 1 0 0 0 0	7/4
$(7/8, 1)$	1 2 0 1 0 1 0 1 0 1 0 1	0 0 0 1 1 1 1 0 1 0 0	7/4
$(1, 5/4)$	1 2 1 1 1 1 1 1	0 0 -1 0 -1 1 -1 0 -1	$15/4 - 2\rho$
$(5/4, 7/4)$	0 0 0 0 0 0	0 1 0 2 0 0 0	$15/4 - 2\rho$
$(7/4, 2)$	0 0	0 1 0 0 0 0	$2 - \rho$
$(2, 5/2)$	0 0	0 0 0 0	0

The eleven large-ratio intervals are exactly the walls generated by $\mathcal{W}([1/2, 5/2])$. For each interval, the group formula above determines the row and column types, and the displayed vectors satisfy all the pairwise rational inequalities. The edge, diagonal, and wall comparisons are generated by the same displayed rule. Summing the type potentials times their affine group widths gives the objective column; taking its exact integral over the high cells yields

$$\int_{1/2}^{5/2} f_{\mathrm{high}}(\rho)\, d\rho = \frac{287}{192}.$$

At a wall use $W_m(p) = 3m$; there are finitely many walls, so this changes the prime sum by $O(m \log m)$ for each m.

The exact profile constant is

$$K = L_* + \frac{129}{8192} + \frac{287}{192} \tag{5.12}$$

$$= \frac{\begin{array}{c}285537765273369749554761463488120017679 41\\ 868552244211097690152323575789752978030410847923 95\\ 818361756829269360265928072216352 1\end{array}}{\begin{array}{c}112327910526464252580750153481730058867156934044\\ 309874903777412212977293022636563487517284803748 38\\ 609207825413384121889792000\end{array}} \tag{5.13}$$

$$= 2.542001929307655\ldots < 2.542002. \tag{5.14}$$

The displayed rational table and the type-generation rule are the complete finite input for this part of the estimate.

5.5. **Exceptional primes and the PNT limit.** Define the rational scalar

$$\Delta_{m,a} = \prod_p p^{E_{m,p}(a)}, \qquad E_{m,p}(a) = \min\{W_m(p) + gm\, v_p(a-1),\ F_{m,p}(a)\}. \tag{5.15}$$

Negative exponents are allowed; $\Delta_{m,a} > 0$ is a rational number. The product is finite: if $p > U + g$ and $p \nmid a(a-1)$, both entries in the minimum are zero. Lemma 5.4 implies coefficientwise that

$$\Delta_{m,a} G_{m,a}(T) \in \mathbb{Z}[T]. \tag{5.16}$$

This is a specialization-wise p-adic argument; it does not claim that the sharp scalar $\Delta_{m,a}$ is the specialization of an integer polynomial in a.

Since $\gcd(a, a-1) = 1$, the exceptional sets $p \mid a$ and $p \mid a-1$ are disjoint. For a fixed $p \mid a$,

$$F_{m,p} = O(m \log m) - N_m v_p(a), \qquad W_m(p) + gmv_p(a-1) = W_m(p),$$

and the quadratic negative term makes $F_{m,p}$ the minimum for all sufficiently large m. For a fixed $p \mid a-1$,

$$F_{m,p} = O(m \log m) + H_m v_p(a-1), \qquad W_m(p) + gmv_p(a-1),$$

and $gm - H_m = 7h^2 + 3h$ makes $F_{m,p}$ eventually the minimum. These comparisons apply also to $p = 2$, which divides exactly one of $a, a-1$. For every other fixed prime the moving profile applies, with the $p^2 \le U + g$ range contributing $o(m^2)$ in aggregate: Put $X = U + g$. The squarefull part is bounded without counting a prime more than once by

$$\sum_{p \le \sqrt{X}} 3md_p \log p \le 3m \sum_{j=1}^{\lfloor \log_2 U \rfloor} \vartheta\Big(\min\{\sqrt{X}, U^{1/j}\}\Big) = O(m^{3/2} \log m) = o(m^2),$$

where $\vartheta(x) = \sum_{p \le x} \log p$. The displayed equality is an upper-bound comparison (the left side is bounded by the right side), since d_p is the number of admissible exponent layers and each layer is charged to $p^j \le U$; the extra $\log m$ is harmless after normalization.

On any fixed positive ratio interval, $W_m(p) \le mf(p/m)$ with f bounded and piecewise affine, apart from the finite wall values. The weighted prime number theorem, obtained from the ordinary PNT by partial summation, gives

$$\frac{1}{m} \sum_{p} f(p/m) \log p \longrightarrow \int f(\rho)\, d\rho.$$

For $0 < \rho < \varepsilon$, first remove the already negligible range $p \le \sqrt{U+g}$ and then omit $\sqrt{U+g} < p \le \varepsilon m$; there $d_p = 1$ and the crude bound $3m\vartheta(\varepsilon m)$ has normalized limsup at most 3ε. Apply the weighted PNT on $[\varepsilon, \delta]$ and then let $\varepsilon \downarrow 0$. The exact tail bound (5.11), the low integral (5.10), and the high integral therefore imply

$$\limsup_{\substack{m \to \infty \\ 4 \mid m}} \frac{1}{m^2} \sum_{p \nmid a(a-1)} E_{m,p}(a) \log p \le K.$$

The fixed exceptional primes contribute

$$\frac{H_m}{m^2} \log|a-1| - \frac{N_m}{m^2} \log|a| \longrightarrow \frac{1}{16} \log|a-1| - \frac{13}{16} \log|a|.$$

We have proved the following fixed-base estimate.

Proposition 5.5 (P0 fixed-base denominator rate)**.** *For every fixed integer $|a| \ge 2$, the rational number $\Delta_{m,a}$ in* (5.15) *satisfies* (5.16) *and*

$$\limsup_{\substack{m \to \infty \\ 4 \mid m}} \frac{\log \Delta_{m,a}}{m^2} \le K + \frac{1}{16} \log|a-1| - \frac{13}{16} \log|a|, \qquad K < 2.542002.$$

The statement is for each fixed base. In particular, its finite-m remainder is not uniform as $|a| \to \infty$.

Completion of the estimate. The formal divisor and pole lemmas, Kummer bounds, residue-cell formulas, exceptional-prime comparisons, and PNT passage are all-m derivations. The low profile is reconstructed by the rational wall-and-crossing rule above. The high profile is hand-checkable from the printed potential vectors and the stated type-generation rule: each vector pair reduces the full assignment inequality to finitely many integer comparisons. The low profile has 756 affine pieces and the high profile eleven cells. The all-m conclusion uses the displayed identities, the finite comparison count above, and the weighted prime number theorem.

6. Positive logarithmic energy on an interval

This section records the analytic passage for the positive P0 determinant, including the dependence on the base. The finite determinant statement is for integers q with $2 \le q \le 11$; a real parameter is used only in the continuum interpolation. All measures, intervals, and the contour below are fixed as q varies. The two endpoint reference measures are the exact data in Tables A.1 and A.2.

Put $m \in 4\mathbb{N}$, $m \ge 8$, $g = m/2$,

$$I_m = \{m/4, \dots, 5m/4 - 1\}, \qquad J_m = \{0, \dots, 2m - 1\}, \qquad E = [1/4, 5/4],$$

and fix

$$\Gamma = \{\xi + A\cos u + iB\sin u : 0 \le u < 2\pi\}, \quad (\xi, A, B) = \left(\frac{11281}{10000}, \frac{12783}{10000}, \frac{229}{200}\right).$$

Write $\Delta(X) = \prod_{a<b}(x_b - x_a)$ and $K(x, y) = (x + y)^{-1}$. The source function and the frequency weight at base $q > 1$ are

$$F_{g,q}(z) = q^z \binom{z+g}{g}, \qquad U_i(q) = q^i \binom{i+g}{g}, \qquad w_{r,q} = \binom{r-1}{g} q^{-r} \quad (r > g).$$

For integer q, let $G_{m,q}$ be the P0 determinant of Section 4, with $a = q$. The diagonal contour identity is

$$C_{ij}^{(g)}(q) = -\oint_{m\Gamma} \frac{F_{g,q}(z)}{(z-i)(z-j)} \frac{dz}{2\pi i} + (\log q) U_i(q)\, \mathbf{1}_{i=j}. \tag{6.1}$$

For $i \ne j$ this is the divided-difference residue identity; for $i = j$ it follows by differentiating $F_{g,q}$, since

$$F'_{g,q}(i) = U_i(q) \left(\log q + \sum_{s=1}^{g} \frac{1}{i+s}\right).$$

Expanding the rational moment expression in the frequency basis and applying Cauchy–Binet [4] gives an absolutely convergent sum over $R \subset \{g+1, g+2, \dots\}$, $|R| = m$, with weight $\prod_{r\in R} w_{r,q}$. After choosing k diagonal rows $S \subset I_m$ and putting $d = m - k$, its absolute integrand is

$$\begin{aligned} Q_{m,q}(S, R, Z) = {} & \left(\prod_{i\in S} (\log q) U_i(q)\right) \left(\prod_{r\in R} w_{r,q}\right) \left(\prod_{z\in Z} |F_{g,q}(z)|\right) \\ & \times \left|\det K_{I_m\setminus S, -Z} \det K_{I_m, R} \det K_{(-Z,R), J_m\setminus S}\right|. \end{aligned} \tag{6.2}$$

where $Z = (z_1, \dots, z_d) \in (m\Gamma)^d$. The integration measure is $\prod |dz|/(2\pi)$ and the symmetrization factor is $1/d!$. The three Cauchy factors have, after restoring the deleted fixed-grid factors, the same three Vandermonde interactions as at $q = 2$: contour-contour, pinned- pinned, and frequency-frequency each have coefficient one; pinned-contour has coefficient two; pinned-frequency and contour-frequency each have coefficient one. In particular, no factor depending on q enters the quadratic interaction energy.

Cauchy–Binet energy expansion [4]. The rational moment expression is obtained by expanding the two Cauchy kernels geometrically and using the exact generating identity

$$\sum_{r \ge g+1} \binom{r-1}{g} (t/q)^r = \left(\frac{t}{q-t}\right)^{g+1}, \qquad 0 \le t < q.$$

Inserting these moments and applying

$$\det(XY) = \sum_{|R|=m} \det X_{:,R} \det Y_{R,:}$$

gives the sum over the full frequency sets R stated above. Splitting the diagonal term in (6.1) row by row chooses $S \subset I_m$; the remaining $d = m - |S|$ rows are represented by contour variables Z, with the factor $1/d!$ from their antisymmetrization. For any two equal-size sets, the Cauchy determinant identity [6]

$$\det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^{r} = \frac{\Delta(x)\Delta(y)}{\prod_{i,j}(x_i + y_j)}$$

converts each of the three kernel minors into a Vandermonde quotient. Restoring the fixed-grid Vandermonde factors leaves contour-contour, pinned-pinned, and frequency-frequency interactions with coefficient one, pinned-contour with coefficient two, and the other two mixed interactions with coefficient one. Taking absolute values gives exactly (6.2); all signs and q-powers are one-body terms. For fixed m, the Cauchy minors have at most polynomial growth in the frequencies and $w_{r,q} = O_q(r^g q^{-r})$. The absolute frequency sum therefore converges, and increasing finite truncations converge to its full absolute envelope. □

For the normalization, the signed Cauchy–Binet expansion [4] can be written explicitly. Put $n = m/4$, $Y = S \uplus Z$, and

$$D_m(a) = (a-n)!(n+m-1-a)!a!(2m-1-a)!,$$
$$P_m(z) = \prod_{a \in I_m \uplus J_m} (z-a), \qquad Q_m^*(r) = \prod_{a \in I_m \uplus J_m} (r+a).$$

Then

$$G_{m,q} = \Delta(I_m)^2 \Delta(J_m) \sum_{S \subset I_m} \frac{(-1)^{|S|(n+1)} (\log q)^{|S|}}{d!} \sum_{|R|=m} \left(\prod_{a \in S} \frac{U_a(q)}{D_m(a)} \right) \left(\prod_{r \in R} w_{r,q} \right)$$
$$\times \oint_{(m\Gamma)^d} \frac{\Delta(Y)^2 \Delta(R)^2 \prod_{y \in Y, r \in R} (y+r)}{\prod_{z \in Z} P_m(z) \prod_{r \in R} Q_m^*(r)} \prod_{z \in Z} \left(F_{g,q}(z) \frac{dz}{2\pi i} \right).$$

This follows by applying Cauchy–Binet [4] to the two Cauchy moment factors, then expanding the diagonal summands in the contour formula row by row. For every equal-size pair, use the Cauchy determinant identity [6]

$$\det\left((x_i + y_j)^{-1}\right)_{i,j=1}^{r} = \frac{\Delta(x)\Delta(y)}{\prod_{i,j}(x_i + y_j)}.$$

Restoring the deleted fixed-grid factors uses

$$\Delta(I_m \setminus S)\Delta(J_m \setminus S) = \frac{\Delta(I_m)\Delta(J_m)\Delta(S)^2}{\prod_{a \in S} D_m(a)}.$$

Each pinned index is therefore counted twice in a contour denominator and once in a frequency denominator. The three Cauchy factors give contour-contour, pinned-pinned, and frequency-frequency interactions with coefficient one, pinned-contour with coefficient two, and each remaining mixed interaction with coefficient one. Taking absolute values and scaling all positions by m gives (6.2) and (6.4). The factors q^a, q^{-r}, and q^z contribute only the one-body fields $\lambda \Re z$ and $-\lambda t$ in (6.5); the quadratic interaction coefficients are independent of q.

The fixed-grid normalization is also explicit. Consecutive integer lists satisfy $\Delta(I_m) = \prod_{j=1}^{m-1} j!$ and $\Delta(J_m) = \prod_{j=1}^{2m-1} j!$. Stirling's formula [13] summed over these products gives

$$\log(\Delta(I_m)^2 \Delta(J_m)) = 3m^2 \log m + (2\log 2 - 9/2)m^2 + O(m \log m).$$

Equivalently, the fixed Vandermonde contributes $C_0 = 9/2 - 2\log 2$ to the negative logarithmic energy. The scaled Cauchy quotients cancel all $m^2 \log m$ terms; their remaining homogeneous factor is m^{-2m-2d} before the contour differentials and m^{-2m-d} after them. This accounts for the full quadratic normalization in (6.4).

6.1. **The fixed variational problem and its affine parameter.** Let $\eta = \kappa + \nu$, where

$$0 \le \kappa \le 1_E \, dx, \quad \nu \ge 0, \quad \operatorname{supp} \nu \subset \Gamma, \quad |\kappa| + |\nu| = 1, \qquad 0 \le \mu \le 1_{[1/2,20]} dt, \quad |\mu| = 1.$$

Write $\mu^- = (-\mathrm{id})_{\#}\mu$ for reflection onto the negative real axis. The set $\mathcal{A}$ of such pairs (η, μ) is independent of q. Define

$$A_0(z) = \int_E \log|z-x| \, dx + \int_0^2 \log|z-x| \, dx, \qquad B_0(t) = \int_E \log(t+x) \, dx + \int_0^2 \log(t+x) \, dx,$$

$$\phi_2(z) = (\log 2)\Re z + \int_0^{1/2} \log|z+s|\,ds + \frac{1}{2} + \frac{1}{2}\log 2,$$
$$\psi_2(t) = t\log t - (t-1/2)\log(t-1/2) + \frac{1}{2}\log 2 - t\log 2.$$

For measures on the corresponding reflected supports write

$$I(\sigma,\tau) = -\iint \log|x-y|\,d\sigma(x)d\tau(y).$$

The energy is

$$\mathcal{E}_q(\eta,\mu) = C_0 + \int (A_0 - \phi_q)\,d\eta + \int (B_0 - \psi_q)\,d\mu \tag{6.3}$$
$$+ I(\eta) + I(\mu) + I(\eta,\mu^-), \tag{6.4}$$

Here I is positive on zero-mass variations, while $Q_{\log}$ below uses the opposite, positive-logarithm convention. Also $C_0 = 9/2 - 2\log 2$ and, on writing $q = 2e^{\lambda}$,

$$\phi_q(z) = \phi_2(z) + \lambda\Re z, \qquad \psi_q(t) = \psi_2(t) - \lambda t. \tag{6.5}$$

The logarithmic integrals are understood in the following regularized sense: truncate at the logarithmic singularity, integrate the resulting bounded functions, and take the limit after replacing the coincident-cell term by the continuous primitive $u(\log|u| - 1)$, with value 0 at $u = 0$. The measures used below have bounded densities and compact support, so this limit is finite. Substitution into (6.4) shows, for every fixed admissible pair,

$$\mathcal{E}_\lambda(\eta,\mu) = \mathcal{E}_0(\eta,\mu) + \lambda\left(-\int \Re z\,d\eta(z) + \int t\,d\mu(t)\right). \tag{6.6}$$

Consequently

$$\alpha(\lambda) = \inf_{(\eta,\mu)\in\mathcal{A}} \mathcal{E}_\lambda(\eta,\mu) \tag{6.7}$$

is concave. Indeed, for $0 \le \theta \le 1$, each fixed-pair energy at $\theta\lambda_0 + (1-\theta)\lambda_1$ is the corresponding convex combination of its endpoint energies, and taking an infimum gives

$$\alpha(\theta\lambda_0 + (1-\theta)\lambda_1) \ge \theta\alpha(\lambda_0) + (1-\theta)\alpha(\lambda_1).$$

Apply the shared quadratic-duality identity of Section C with quadratic part $Q_{\mathrm{P0}}(\eta,\mu) = I(\eta) + I(\mu) + I(\eta,\mu^-)$. Fix an admissible reference (η_0,μ_0) and put

$$Q_{\log} = \iint \log|z-w|\,d\eta_0(z)d\eta_0(w) + \iint \log|t-s|\,d\mu_0(t)d\mu_0(s)$$
$$+ \iint \log|z+t|\,d\eta_0(z)d\mu_0(t).$$

$$Q_{\mathrm{P0}}(\eta_0,\mu_0) = -Q_{\log}, \qquad W_\eta = A_0 - \phi_q - 2L_{\eta_0} - L_{\mu_0^-}, \qquad W_\mu = B_0 - \psi_q - 2L_{\mu_0} - L_{\eta_0}(-t).$$

If $W_\eta \ge h_\eta$ on Γ, set

$$P = \int_E (h_\eta - W_\eta)_+\,dx, \qquad T = \int_{1/2}^{20} (h_\mu - W_\mu)_+\,dt.$$

Then

$$\alpha(\lambda) \ge C_0 + Q_{\log} + h_\eta + h_\mu - P - T. \tag{6.8}$$

For $\sigma = \eta - \eta_0$ and $\tau = \mu^- - \mu_0^-$, the shared identity has remainder

$$I(\sigma) + I(\tau) + I(\sigma,\tau) = \tfrac{1}{2}I(\sigma) + \tfrac{1}{2}I(\tau) + \tfrac{1}{2}I(\sigma+\tau) \ge 0,$$

where $I(\rho) = -\iint \log|x-y|\,d\rho(x)d\rho(y)$ is nonnegative for zero-mass signed measures. The density constraints then give $\int W_\eta d\eta \ge h_\eta - P$ and $\int W_\mu d\mu \ge h_\mu - T$, proving (6.8).

6.2. **Finite configurations and cell estimates.** For a selected pinned set $S \subset I_m$ and a frequency set R, define

$$\kappa_S = \sum_{i\in S} \mathbf{1}_{[i/m,(i+1)/m)}dx, \qquad \mu_R = \sum_{r\in R} \mathbf{1}_{[(r-1)/m,r/m)}dt,$$
$$\nu_\delta = \frac{1}{m}\sum_{a=1}^{d} \sigma_{z_a/m,\delta}, \qquad \delta = m^{-2}.$$

Here $\sigma_{z,\delta}$ is probability arclength on the radius-δ circle centered at z. Put $\zeta_a = z_a/m \in \Gamma$. Thus all logarithms in the continuum energy are in scaled coordinates; the powers of m arising from the change from z_a to ζ_a have already been included in the fixed Vandermonde normalization in the cell estimate. The first two measures have density at most one, masses $|S|/m$ and one, and the first frequency cell begins at $1/2$. Circle averaging gives

$$\sum_{a\neq b} \log|\zeta_a - \zeta_b| \le m^2 \iint \log|z-w|\, d\nu_\delta(z)d\nu_\delta(w) + 2d\log m. \tag{6.9}$$

The fixed-grid and reflected-frequency logarithmic potentials are harmonic on the smoothing disks. The contour field is subharmonic, including across the zero segment of $F_{g,q}$. Thus smoothing has no additional field error; the explicit $2d\log m$ is the full smoothing loss.

The exact real-cell estimates used in the envelope are listed here to make the finite-size accounting reproducible. For n distinct lattice sites, the cell self-energy inequality is

$$\sum_{a\neq b} \log|x_a - x_b| \le m^2 \iint \log|x-y|\, d\rho(x)d\rho(y) + n(\log m + \log(2\pi)),$$

where ρ is the union of their length-$1/m$ cells. To see the constant, write

$$e_d = \log d - \int_0^1\int_0^1 \log|d+u-v|\, du\, dv.$$

Then $e_1 = 3/2 - 2\log 2$ and for $d \ge 2$,

$$e_d = \sum_{j\ge 1} \frac{1}{j(2j+1)(2j+2)d^{2j}} \le \frac{1}{12d(d-1)}, \qquad \sum_{d\ge 1} e_d = \tfrac{1}{2}\log(2\pi) - \tfrac{3}{4}.$$

Indeed, for $d \ge 2$ expand $\log(1+(u-v)/d)$ in its absolutely convergent power series and integrate term by term; the even moments $\int_0^1\int_0^1 (u-v)^{2j}\, du\, dv = 2/((2j+1)(2j+2))$ give the displayed series, while the $d=1$ integral is elementary. Summing the same identity from $d=1$ to M, exchanging the positive series, and using Stirling's Wallis limit [13] $\log M! = (M+\frac{1}{2})\log M - M + \frac{1}{2}\log(2\pi) + o(1)$ gives the displayed infinite sum. Thus no numerical cell constant is being imported. There are at most two sites at each distance from a fixed site. This proves the displayed inequality by summing the self-cell and pair-cell errors. Applied to the selected pinned and frequency sites and to the fixed Vandermonde, it contributes $4m(\log m + \log(2\pi))$.

The remaining one-body and mixed-cell bounds are as follows. Binomial entropy gives

$$\log U_i(q) \le m\phi_q(i/m), \qquad \log w_{r,q} \le m\psi_q(r/m).$$

The exact cell corrections from the affine parameter are $-\lambda m/2$ for frequencies and $-\lambda|S|/2$ for pinned cells. For the base-2 frequency field,

$$\psi_2'(t) = \log\frac{t}{t-1/2} - \log 2, \qquad \int_{1/2}^1 (\psi_2'(t))_+\, dt = \tfrac{1}{2}\log 2.$$

Summing endpoint-to-cell differences over disjoint frequency cells costs at most $(m/2)\log 2$. Monotone quadrature for the fixed-grid field gives, for $t \ge 1/2$,

$$0 \le mB_0(t) - \sum_{i\in I_m} \log(t+i/m) - \sum_{j\in J_m} \log(t+j/m) \le \log(35/3).$$

Thus the frequency one-body cost is at most $m(\log(35/3) + \frac{1}{2}\log 2) - \lambda m/2$.

For the pinned denominator, take N consecutive sites starting at a and containing i, with $n_L = i - a$ and $n_R = a + N - 1 - i$. Comparing the omitted site product on the two sides of i with its cell integral gives

$$\log(n_L! n_R!) - (N-1)\log m - m\int_{a/m}^{(a+N)/m} \log|i/m - y|\,dy \geq \log\frac{m}{n_R+1}.$$

This follows from monotone logarithmic sums and $n_R \log(1 + 1/n_R) \leq 1$; the zero case is direct. The I_m contribution is nonnegative and the J_m contribution is at least $-\log 2$. Direct integration of the explicit derivatives gives $\int_E |(\phi_2 - A_0)'| \leq \log 6 + 2\log 2 + \log 7 = \log 168$. Consequently pinned quadrature costs at most $m \log 336$. Its affine correction is $-\lambda|S|/2$, since the linear-field integral on each cell is centered one half-cell to the right of the selected lattice point. For an exact bookkeeping check, $q/2 - \log q$ is increasing and positive on $[2, \infty)$, hence $(\log\log q)_+ \leq \lambda$. Therefore the diagonal and cell-shift contribution satisfies

$$|S|(\log\log q)_+ - \frac{\lambda}{2}(|S| + m) \leq -\frac{\lambda}{2}(m - |S|) \leq 0.$$

The displayed pointwise estimate keeps the positive term and drops these favorable shifts, so it remains valid without relying on this cancellation.

The ellipse clearances are

$$d_+ = \operatorname{dist}(\Gamma, [0,2]) = 751/5000, \qquad d_- = \operatorname{dist}(\Gamma, (-\infty, -1/2]) = 1749/5000.$$

They follow by minimizing the squared distance as a quadratic in $\cos u$ on the ellipse parametrization. The pinned-contour displacement costs m/d_+, the fixed contour denominators cost $3m/(2d_+)$, and the contour-frequency displacement costs $m/(2d_-)$. The pinned-frequency cell displacement is centered; Taylor's inequality, with $i/m + r/m \geq 3/4$ and cell width $1/m \leq 1/4$, bounds its total loss by $1/3$. Together these contributions are at most

$$m\left(\frac{5}{2d_+} + \frac{1}{2d_-}\right) + \frac{1}{3}.$$

Set $T_\delta = \{z : \operatorname{dist}(z, \Gamma) \leq \delta\}$, with the tube radius δ used in the transfer argument below. For the finite contour product the required inequality is one-sided. Fix $z \in T_\delta$, put $\zeta = mz$, $r = -\Re\zeta$, $b = \Im\zeta$, and $f(u) = \log|\zeta + u|$. Then

$$f'(u) = \frac{u - r}{(u-r)^2 + b^2},$$

so f decreases to r and increases after r. Put $H = \max(f, 0)$. The negative part of f is supported near r; after translating r and using $\int_{-1}^{1} (\log|u|)_-\,du = 2$, its integral over every unit cell meeting r is at most 2. On each of the two monotone pieces the right-endpoint quadrature error for H is nonpositive on the decreasing piece and nonnegative on the increasing piece. The positive contribution is bounded by the variation on the increasing piece. The sampled endpoint values are bounded by $\log(4(g+1))$ near the turning cell and by $\log(6(g+1))$ at the right endpoint, so this variation is at most $\log(6(g+1))$. Thus the complete discrepancy satisfies

$$\sum_{k=0}^{g-1} f(k+1) - \int_0^g f(u)\,du \leq 2 + \log(6(g+1)).$$

This is the bounded-variation estimate for the truncated function H; it also covers the cases where the zero of f lies outside $[0, g]$. We retain the looser uniform quantity

$$\beta_m = 4\log(g+1) + 9 + 10\log 2$$

below. Restoring the factors $\log m$ gives the cell integral in the contour-product estimate. Thus, for every $z \in T_\delta$,

$$\log|F_{g,q}(mz)| \leq m\phi_q(z) + \beta_m. \tag{6.10}$$

The scalar factor q^{mz} is exactly the affine field term and adds no discrepancy. No lower bound for this discrepancy is used. Since $g + 1 \leq m$ for $m \geq 8$, the contour and circle errors together contribute $2m\log m + m\beta_m$.

Combining the preceding terms with the fixed Vandermonde normalization and discarding favorable powers of m gives, for every configuration,

$$\log Q_{m,q}(S,R,Z) \le -m^2\mathcal{E}_q(\kappa_S+\nu_\delta,\mu_R)+10m\log m+c_*m+\frac{1}{3}+|S|\log\log q,$$

where the real-cell constant is

$$\begin{aligned} c_* &= 4\log(2\pi)+9+\frac{21}{2}\log 2+\log 336+\log(35/3)+\frac{5}{2d_+}+\frac{1}{2d_-}\\ &< 49.978. \end{aligned}$$

The displayed logarithm bounds can be checked with the rational series

$$\log x = 2\sum_{j=0}^{N}\frac{u^{2j+1}}{2j+1}+R_N,\qquad 0<R_N<\frac{2u^{2N+3}}{(2N+3)(1-u^2)},\quad u=\frac{x-1}{x+1}.$$

For $x=710/113$ use $N=50$; for $x=2,21/16,35/24,39/32$ use $N=16$. The rational series gives

$$\log(710/113)<1.83787716,\quad \log 2<0.69314719,\quad \log(21/16)<0.27193372,$$

$$\log(35/24)<0.37729424,\qquad \log(39/32)<0.19782575.$$

The classical rational bound $\pi<355/113$ therefore gives $\log(2\pi)<1.83787716$; hence $\log 336<8(0.69314719)+0.27193372$ and $\log(35/3)<3(0.69314719)+0.37729424$. Substitution of the exact rational $d_+=751/5000$ and $d_-=1749/5000$ gives $c_*<49.977264<49.978$ and $c_*+\log 39+1+A<55.920<56$. This is the base-2 cell constant; the q-dependent cell shifts above are favorable. The only q-dependent finite factor not absorbed by the affine energy is the diagonal multiplier $(\log q)^{|S|}$. In the pointwise envelope it appears as the exact term $|S|\log\log q$. At $q=2$ this term is negative, so the base-2 estimate omits that favorable factor; whenever an upper bound uniform in q is taken below, it is replaced by $|S|(\log\log q)_+$. Uniformly for $2\le q\le 11$,

$$\log q<3,\qquad 0<\log q<3<5,\qquad \log(\log q)<\log 3<2,$$

so $|S|(\log\log q)_+<2m$. For all bases we therefore replace $56m$ by at most $58m$; this finite correction is $O(m)$ and does not change the limiting energy rate.

6.3. **Tube transfer, frequency tail, and summation.** Let D be the interior of the ellipse. For z inside D use its interior harmonic measure ω_z on Γ; outside D use exterior harmonic measure of total mass one; on Γ use δ_z. Let W be the exterior inverse of $\mathcal{J}(w)=\xi+cw+d/w$, where $c=(A+B)/2$ and $d=(A-B)/2$. Define $e(z)=0$ in D and $e(z)=\log|W(z)|$ outside. The factorization $\mathcal{J}(w_1)-\mathcal{J}(w_2)=(w_1-w_2)(c-d/(w_1w_2))$ and the Green identity imply

$$\int K_0(x,y)\,d\omega_z(y)\le K_0(x,z)+e(z),\qquad I(\omega_z,\omega_w)\le K_0(z,w)+e(z)+e(w),$$

where $K_0=-\log|\cdot|$ and I denotes its mutual energy. The interior deficit is the Green function of D; the exterior deficit is the nonnegative disk Green function; the cross-boundary deficit is zero. For $w=\rho e^{iu}$, $\rho\ge 1$, the equipotential $|W(z)|=\rho$ is the confocal ellipse

$$\Gamma_\rho: z=\xi+(c\rho+d/\rho)\cos u+i(c\rho-d/\rho)\sin u.$$

It equals Γ at $\rho=1$. If $z=\mathcal{J}(w)$ and $\zeta=\mathcal{J}(v)\in\Gamma$, $|v|=1$, the exact factorization gives

$$|z-\zeta|=|w-v|\left|c-\frac{d}{wv}\right|\ge(\rho-1)(c-d/\rho)\ge B(\rho-1).$$

Taking the infimum over ζ and using $\log\rho\le\rho-1$ proves

$$e(z)\le \operatorname{dist}(z,\Gamma)/B\le\delta/B\qquad (z\notin D,\ z\in T_\delta).$$

For $x\in[0,2]$, set $\chi(z)=1-(\Re z-\xi)^2/A^2-(\Im z)^2/B^2$. On D, $|\nabla\chi|\le 2/B$, while $\chi(x)>1/5$. The level curve $\chi=1/8$ is more than $1/32$ from $[0,2]$, so the Green function there is at most $\log 160<6$. The barrier 48χ on the collar gives

$$g_D(z,x)\le 96\delta/B\qquad (z\in D\cap T_\delta,\ x\in[0,2]).$$

Since A_0 has source mass three and $\lambda\Re z$ is harmonic, the interior field-sweep loss for $A_0-\phi_q$ is at most $288\delta/B$. The term $-V$ is favorable in this estimate: V is subharmonic, with distributional Laplacian 2π times the source measure on $[-1/2,0]$, so its harmonic sweep is at least its interior value and the $-V$ contribution can be dropped from an upper bound. For exterior $z=\mathcal{J}(w)$, the swept mean coordinate is $\xi+c/\overline{w}+d/w$, within $(A+B)\delta/B$ of z. For the base-2 terms the source bound is $[1/2+(A+B)\log 2]\delta/B<3\delta/B$: A_0 changes by exactly $-3e(z)$, and the remaining coordinate displacement is at most $(A+B)\delta/B$. The additional affine term is at most $\lambda(A+B)\delta/B<6\delta/B$ because $\lambda=\log(q/2)<2$ and $A+B<3$.

The logarithmic source in ϕ_q needs a separate collar estimate. Put

$$V(z)=\int_0^{1/2}\log|z+s|\,ds.$$

For $0<\delta\le 1/64$ and an exterior point with $\operatorname{dist}(z,\Gamma)\le\delta$, the explicit primitive

$$V(z)=\Re\big((z+1/2)\operatorname{Log}(z+1/2)-z\operatorname{Log}z-1/2\big)$$

on either side of the source segment gives

$$\nabla V(z)=\operatorname{Log}(z+1/2)-\operatorname{Log}z.$$

For $z=\xi+A\cos u+iB\sin u$ on Γ, minimizing $|z|^2$ gives its minimum at $u=\pi$, namely $\xi-A=751/5000$; the derivative has no other critical point because $A\xi>|A^2-B^2|$. The disk of this radius about 0 lies inside the ellipse, so every exterior point satisfies $|z|\ge 751/5000$. For the radial comparison path, $h\le\delta/B$ and

$$|\mathcal{J}((1+h)e^{iu})-\mathcal{J}(e^{iu})|\le(c+d)h=Ah\le\frac{A}{B}\delta=\frac{12783}{11450}\delta<1.117\delta.$$

This is about 1.116δ and is less than 0.02 when $\delta\le 1/64$. Thus all points used in the estimate have $\Re z\ge -751/5000-(12783/11450)/64>-1/4$. Therefore

$$\frac{|z+1/2|}{|z|}\le 1+\frac{1}{2|z|}\le 1+\frac{2500}{751}=\frac{3251}{751}<4.33,\qquad |\arg(z+1/2)-\arg z|\le\pi.$$

Since $|z+1/2|^2-|z|^2=\Re z+1/4>0$, the ratio is at least one, and the modulus term in the difference of logarithms is at most $\log(4.33)$. The angle difference is at most π because the two points lie on the same horizontal line. Also $e^{3/2}>4.33$ by the first six terms of its positive series, so $\log(4.33)<3/2$. Thus, on both sides of the cut,

$$|\nabla V(z)|\le\sqrt{(3/2)^2+(22/7)^2}<3.72.$$

At the contact point $z_0=\xi-A=-751/5000$ the classical gradient is not defined, but these are the bounded one-sided traces and they give the required Lipschitz bound across the smooth ellipse and the source cut. The Joukowski boundary parametrization has speed at most $4/3$, so its pullback is 6-Lipschitz. If the exterior pole is $w=\rho e^{iu}$, write $\rho=1+h$. The elementary estimate

$$\int_{-\pi}^{\pi}|t|P_{1+h}(t)\,\frac{dt}{2\pi}\le\frac{3\pi h}{8}\log(1+4/h^2)$$

is used for $0<h\le 1$ and follows from $1-\cos t\ge 2t^2/\pi^2$. In the present collar $h\le\delta/B\le 25/1832<1$, so this is the required range. Since $\operatorname{dist}(\mathcal{J}(w),\Gamma)\ge Bh$ and $\delta\le 1/64$, these bounds imply

$$\left|\int_\Gamma V(\zeta)\,d\omega_z(\zeta)-V(z)\right|\le 20\delta\log(1/\delta).$$

Thus the exterior field loss is at most $9\delta/B+20\delta\log(1/\delta)$. Combining this with the interior bound gives a uniform field loss at most $288\delta/B+20\delta\log(1/\delta)$.

The three interaction terms change by at most $3b$, where $b=\int e\,d\nu\le|\nu|\delta/B$. Since $|\nu|\le 1$, their total loss is at most $3\delta/B$. Consequently the full transfer cost is at most $291\delta/B+20\delta\log(1/\delta)<300\delta+20\delta\log(1/\delta)$ because $B=229/200>1$. Thus

$$\mathcal{E}_q(\kappa+\nu_\delta,\mu)\ge\alpha(\lambda)-300\delta-20\delta\log(1/\delta),\qquad\frac{291}{B}<300.$$

The dual problem is posed on the printed support interval $[1/2,20]$. The frequency-cutoff argument below first replaces the full frequency sum by the range $R\subset\{g+1,\dots,20m\}$, with a positive multiplicative

tail factor. Hence every finite configuration used in the variational bound has its frequency measure in this compact interval; the omitted tail is accounted for separately in (6.3).

For completeness, the unbounded frequency sum is reduced to a finite one before counting configurations. For $H \geq g+1$, define

$$E_{m,q}(H) = \sum_{S \subset I_m} \frac{1}{d!} \int_{(m\Gamma)^d} \sum_{\substack{R \subset \{g+1,\dots,H\} \\ |R|=m}} Q_{m,q}(S,R,Z) \prod_{z \in Z} \frac{|dz|}{2\pi}.$$

The Cauchy–Binet expansion [4] gives $|G_{m,q}| \leq E_{m,q}(\infty)$. For fixed S, Z, its frequency-dependent factor is

$$f_q(R) = \frac{\prod_{r \in R} w_{r,q}\, \Delta(R)^2 \prod_{r \in R, z \in Z} |r+z|}{\prod_{r \in R, i \in I_m} (r+i) \prod_{r \in R, j \in J_m \setminus S} (r+j)}.$$

If $r = \max R > 20m$, replace r by the least missing $s \in \{g+1, \dots, g+m\}$, so $s \leq 3m/2$. The exact ratio of the frequency weights, both frequency Vandermonde factors, and all contour-frequency factors gives

$$\frac{f_q(R)}{f_q(R')} \leq 12500^m \frac{m^2}{4r^2} w_{r,2}.$$

For the constant, at most two integers lie at each distance from s, so

$$\prod_{u \in R \setminus \{r\}} |s-u| \geq d_m := \left\lfloor \frac{m-1}{2} \right\rfloor ! \left\lceil \frac{m-1}{2} \right\rceil ! \geq e^{-1} \left(\frac{m}{2e}\right)^{m-1}.$$

Cancel the common factors in the two Cauchy determinants. The two frequency Vandermonde ratios contribute the squared products of distances from r and s; each contour-frequency factor is bounded by the corresponding ratio involving $r+g$ and $s+g$, and the fixed-grid factors are bounded by 5^m. The explicit denominator bounds $s+i \leq 11m/4$, $s+j \leq 7m/2$, and ellipse bounds $\Re z > -m/6$, $|z| < 5m/2$ give

$$\frac{f_q(R)}{f_q(R')} \leq \frac{w_{r,q}}{w_{s,q}} \left(\frac{77}{8}\right)^m \left(\frac{189}{16}\right)^{m-|S|} \frac{m^{2m}}{r^2 d_m^2}.$$

For $r > s$, $w_{r,q}/w_{s,q} \leq w_{r,2}/w_{s,2}$ and $w_{s,2} \geq 2^{-3m/2}$ because $s \leq 3m/2$ and $\binom{s-1}{g} \geq 1$. Substituting this and $e < 3$ into the last display gives the asserted $12500^m m^2 w_{r,2}/(4r^2)$ bound. This is the full constant calculation; it does not require an estimate for $w_{s,q}$. The exact base comparison is

$$\frac{w_{r,q}/w_{s,q}}{w_{r,2}/w_{s,2}} = \left(\frac{2}{q}\right)^{r-s} \leq 1.$$

Frequency cutoff.

$$\vartheta_m = \frac{m}{1600} 12500^m \sum_{r > 20m} w_{r,2}.$$

Indeed, the replacement map has at most m preimages for each pair (R', r), and $m^2/(4r^2) < 1/1600$ when $r > 20m$. For $t = 15/8$,

$$\sum_{r > 20m} w_{r,2} \leq (8/15)^{20m} \sum_{r \geq g+1} w_{r,2} (15/8)^r = (8/15)^{20m} 15^{g+1}.$$

Since $g = m/2$ and $12500\sqrt{15} < 50000$, this gives

$$\vartheta_m \leq \frac{15m}{1600} \left[50000 (8/15)^{20}\right]^m < \frac{3}{320} 2^{-m} < \frac{1}{2}.$$

The strict inequality uses $200000 \cdot 8^{20} < 15^{20}$ and $m 2^{-m} \leq 1$. If Z_M denotes the finite frequency sum through M, the replacement map gives $Z_M - Z_{20m} \leq \vartheta_m Z_M$. Hence $Z_M \leq (1-\vartheta_m)^{-1} Z_{20m} \leq (1+2\vartheta_m) Z_{20m}$.

$$2\vartheta_m < \frac{3}{160} 2^{-m}.$$

Monotone convergence as $M \to \infty$ now proves

$$E_m(\infty) \leq \left(1 + \frac{3}{160} 2^{-m}\right) E_m(20m).$$

□

There are 2^m pinned subsets and at most $\binom{39m/2}{m}$ retained frequency sets. The normalized contour length is at most mA, hence $(mA)^d/d! \le e^{mA}$. Using $c_* < 49.979$ and $c_* + \log 39 + 1 + A < 56$, the finite configuration bound, the tube transfer with $\delta = m^{-2}$, and the tail factor imply the following.

Lemma 6.1 (Uniform smoothing bound)**.** *For every integer $q \in \{2, \dots, 11\}$ and every $m \in 4\mathbb{N}$, $m \ge 8$,*

$$\log |G_{m,q}(\mathrm{Li}_2(1/q))| \le -\alpha(\log(q/2))m^2 + 10m \log m + 58m + 301 + 40 \log m. \tag{6.11}$$

The frequency quantifier before replacement is the full set $R \subset \{g+1, g+2, \dots\}$ of cardinality m. The constant 58 is uniform in q; the constant 301 includes the tube cost 300, the real-cell remainder 1/3, and the exponentially small frequency-tail factor. The additional $40 \log m$ is the explicit logarithmic collar cost.

Proof. For each finite configuration the cell measures above are admissible in the m^{-2} tube problem. Tube transfer bounds their energy below by $\alpha(\lambda) - 300m^{-2} - 40m^{-2} \log m$. Inserting this into the pointwise determinant envelope gives a $300 + 40 \log m$ additive loss. Summing the 2^m pinned choices, the $\binom{39m/2}{m}$ retained frequency choices, and the contour integral adds at most $(\log 39 + 1 + A)m$ beyond $c_* m$. The first contribution is strictly less than $56m$ by the displayed $c_* + \log 39 + 1 + A < 56$; the extra diagonal factor is strictly less than e^{2m}. Hence the coefficient of m is strictly less than $56 + 2 = 58$. The cell remainder is $1/3$. For $m \ge 8$ the tail factor obeys

$$\log\left(1 + \frac{3}{160} 2^{-m}\right) \le \frac{3}{160} 2^{-m} \le \frac{3}{40960},$$

so $300 + 1/3 + 3/40960 < 301$. Taking the monotone limit of the finite frequency cutoffs proves the estimate for the full half-line. □

6.4. **Endpoint certificates and the interval comparison.** The exact reference masses and thresholds are printed in Tables A.1 and A.2. Their directed dual checks prove

$$\alpha(0) \ge \frac{248035}{125000}, \qquad \alpha(\log(11/2)) \ge \frac{77597}{100000}. \tag{6.12}$$

These are lower bounds for the full continuum infimum, not empirical minimizers. In each certificate the contour inequality is checked on the whole ellipse; the pinned positive-part integral is checked on its full support; the frequency positive-part integral is checked through $81/10$. The endpoint inequalities at $81/10$ are part of the directed integrals recomputed from the printed measures: they give $W_\mu(81/10) > 31/10$ for $q = 2$ and $W_\mu(81/10) > 56/5$ for $q = 11$, while the respective thresholds are $1427/500$ and $1217/200$.

The infinite tail has a direct bound that uses only the masses and supports of these two references. Both η_0 have mass one and support in $E \cup \Gamma$, whose real part is at least $-751/5000$; both μ_0 have mass one and support in $[1/2, 9/2]$. Also B_0' is the Cauchy transform of a positive measure of mass three on $[0, 2]$. Differentiating the definition of W_μ therefore gives, for $t \ge 81/10$,

$$\begin{aligned} W_\mu'(t) &= B_0'(t) + \log q - \log \frac{t}{t - 1/2} - 2 \int \frac{d\mu_0(s)}{t - s} - \int \Re \frac{d\eta_0(z)}{t + z} \\ &\ge H(t) := \frac{3}{t+2} + \log q - \log \frac{t}{t - 1/2} - \frac{2}{t - 9/2} - \frac{1}{t - 751/5000}. \end{aligned}$$

For $t \ge 81/10$, differentiation yields

$$H'(t) = -\frac{3}{(t+2)^2} + \frac{1}{2t(t - 1/2)} + \frac{2}{(t - 9/2)^2} + \frac{1}{(t - 751/5000)^2} > 0,$$

because the last two positive terms exceed $3/t^2$, while $3/(t+2)^2 < 3/t^2$. Thus $H(t) \ge H(81/10)$. The elementary bounds $\log q \ge \log 2 > 2/3$, $\log(81/76) < 1/13$, and $5000/39749 < 13/100$ give

$$W_\mu'(t) \ge H(81/10) > \frac{30}{101} + \frac{69}{100} - \frac{1}{13} - \frac{5}{9} - \frac{13}{100} = \frac{66338}{295425} > \frac{31}{684}.$$

Here we used the elementary rational-series bounds $\log 2 > 69/100$, $\log 11 > 23/10$, $\log(81/76) < 1/13$, and $5000/39749 < 13/100$; the $q = 11$ lower bound is larger than 1. Consequently $W_\mu'(t) > 31/684 > 1/50$ for $q = 2$ and is > 1 for $q = 11$. Hence $W_\mu(t) > h_\mu$ on the entire omitted ray, so T has no omitted tail.

The directed endpoint records are summarized in Table 6.1; the exact mass lists are the two preceding reference tables.

For auditability, the scalar inputs and the outward rational penalties used by the two records are listed here (the displayed finite decimals are outward lower bounds for $Q_{\log}$):

TABLE 6.1. Scalar data in the two directed endpoint certificates.

q	exact thresholds and directed penalty bounds
2	$h_\eta = -3347002/1000000$, $h_\mu = 1427/500$; $Q_{\log} > -0.425508$, $P \le 0.087950$, $T \le 0.122962$;
11	$h_\eta = -3359/500$, $h_\mu = 1217/200$; $Q_{\log} > -0.635537$, $P \le 0.000001$, $T \le 1.069189$.

The arithmetic rate used for the positive bases is

$$\tau_+(\lambda) = K + \frac{1}{16}\log(q-1) - \frac{13}{16}\log q, \qquad q = 2e^\lambda,$$

where the arithmetic estimate gives $K \le 1271001/500000$. Differentiation with respect to λ gives

$$\tau_+''(\lambda) = -\frac{q}{16(q-1)^2} < 0.$$

Its tangent at $q_0 = 63/25$ has exact slope

$$\tau_+'(\log(q_0/2)) = -\frac{431}{608}.$$

Concavity places τ_+ below this tangent on the whole interval. Directed logarithm evaluation with the stated rational upper bound for K gives the endpoint comparison:

q	directed α lower bound	tangent upper bound for τ_+	directed gap lower bound
2	248035/125000	1.981041880111	$> 3/1000$
11	77597/100000	0.772577360778	$> 3/1000$

The displayed decimal upper bounds are outward roundings of the directed intervals; the corresponding directed gaps are respectively $0.00323811988930149\ldots$ and $0.00339263922279043\ldots$.

Let $L(\lambda)$ be the chord joining the two lower bounds in (6.12). By concavity of α, $\alpha(\lambda) \ge L(\lambda)$. Let $T(\lambda)$ be the tangent to τ_+ at $q_0 = 63/25$. The two directed endpoint checks show $L - T > 3/1000$ at both ends. Since $L - T$ is affine in λ, it is greater than $3/1000$ throughout $0 \le \lambda \le \log(11/2)$. Also $\tau_+(\lambda) \le T(\lambda)$. We have proved

$$\alpha(\lambda) - \tau_+(\lambda) > \frac{3}{1000} \quad (0 \le \lambda \le \log(11/2)). \tag{6.13}$$

The determinant estimate (6.11) is a separate smoothing estimate; the fixed-base integerizer and non-vanishing sequence are applied separately at each integer q. Thus this section uses no interpolation of integerizers, seed primes, or endpoint reference measures.

7. Positive bases from 2 to 11

The fixed feasible set and affine fields give concavity in $\lambda = \log(a/2)$, as proved in Section 6. The endpoint certificates use exact rational measures on

$$(\xi, A, B) = \left(\frac{11281}{10000}, \frac{12783}{10000}, \frac{229}{200}\right),$$

with 64 contour nodes and Fejér degree 48. They give

$$\alpha_2 \ge \frac{248035}{125000}, \qquad \alpha_{11} \ge \frac{77597}{100000}.$$

The arithmetic comparison uses $K < 1271001/500000$, tangent point $q_0 = 63/25$, and slope $-431/608$. The directed endpoint gaps exceed 0.0032381198893 and 0.0033926392227, respectively. These checks and the tangent majorant cover $2 \le a \le 11$. The integerization and nonvanishing sequence are applied separately at each integer base. The endpoint measures are printed in Appendix A.

8. Negative bases $-7 \le a \le -2$

The signed determinant construction uses the $[C; B - TA; V]$ determinant. At $a = -2$, its directed analytic bound is $\alpha_2 > 4.34875$, while the fixed arithmetic rate is $\tau_2 = A_{40} - (53/48)\log 2 = 4.317643952609845\ldots$, giving margin > 0.03110. The denominator estimate, prime sum, analytic bound and prime-power nonvanishing sequence includes $q = 2$ and is instantiated at each remaining base. For $a = -q$, the field acquires the term $-\pi \operatorname{Im} z$, so the full signed contour in Section 10 is required. For each $q = 2, \ldots, 7$, the contour geometry, printed reference data, and directed field bounds, together with the smoothing estimate in Section 10, give a strict $\alpha_q > \tau_q$, while Sections 9 and 11 supply the matching integerizer and eventual nonvanishing sequence. Thus all six bases $-7 \le a \le -2$ are closed individually, without an interpolation claim across q.

9. Arithmetic for the signed determinant

This section supplies the arithmetic interface for the negative V-family. The base is fixed before m tends to infinity. Thus the statement is made separately for $q = 2, 3, 4, 5, 6, 7$, with $a = -q$; it is not a theorem uniform in an unbounded parameter q. The proof has four parts: an exact polynomial divisor, residue-block valuations away from the fixed primes, an outer pole assignment, and a weighted prime-number-theorem sum. All subsequent limits follow from the displayed residue valuations and outer assignment bounds.

Put $m = 4n$, $s = k = h = n$, and

$$I = [2n, 6n), \qquad J = [n, 10n), \qquad \Lambda = \{1, \ldots, n\}, \qquad N = 10n - 1, \quad U = 11n - 1.$$

For an integer parameter a, define

$$t_j = \sum_{r=1}^{j} \frac{a^{j-r}}{r}, \qquad b_j = \sum_{r=1}^{j} \frac{a^{j-r}}{r^2}, \qquad t_0 = b_0 = 0,$$

and, for $i \in I$, $j \in J$, $\ell \in \Lambda$,

$$C_{ij} = \begin{cases} (a^i - a^j)/(j-i), & i \ne j, \\ 0, & i = j, \end{cases} \qquad B_{ij} = \begin{cases} (t_i - t_j)/(j-i), & i \ne j, \\ b_i, & i = j, \end{cases}$$

$$A_{ij} = a^i \delta_{ij}, \qquad V_{\ell j} = \frac{1}{j + \ell}, \qquad G_m(T; a) = \det \begin{bmatrix} C(a) \\ B(a) - TA(a) \\ V \end{bmatrix}.$$

The determinant has size $2m + n$. All valuations below are coefficientwise in T: $v_p(F) = \min_r v_p([T^r]F)$.

9.1. Denominator and the formal divisor.

Let $d_r = \operatorname{lcm}(1, \ldots, r)$ and

$$\Delta_m = d_N^{3m} d_U^n, \qquad D_m = \left\lceil \frac{53m^2}{48} \right\rceil - m = \left\lceil \frac{53n^2}{3} \right\rceil - 4n.$$

Lemma 9.1 (denominator). *For an indeterminate a,*

$$\Delta_m G_m(T; a) \in \mathbb{Z}[a, T].$$

Proof. Every denominator in a C-entry divides d_N. In an off-diagonal B-entry one harmonic denominator and one divided-difference denominator occur, while b_i has two harmonic denominators; hence d_N^2 clears each lower row. The denominators of the V-rows divide d_U. Multiplying the rows and expanding the determinant proves the assertion. □

Lemma 9.2 (formal a-divisor). *Coefficientwise in T,*

$$\Delta_m G_m(T; a) \in a^{D_m} (a - 1)^m \mathbb{Z}[a, T].$$

Proof. The following argument is coefficientwise in $\mathbb{Q}[[a]][T]$, so every use of an infinite series can be replaced by a finite truncation for a fixed coefficient. For $u_t(r) = (r-t)^{-1}$ when $r \neq t$, and $u_t(t) = 0$, put $S_1(a) = \sum_{v\geq 1} a^v/v$ and $S_2(a) = \sum_{v\geq 1} a^v/v^2$. Direct coefficient comparison, including the diagonal entries, gives

$$C_{ij} = a^i u_i(j) + a^j u_j(i),$$

$$(B - S_1 C)_{ij} = \sum_{t\geq 0} a^t u_t(i) u_t(j) - \frac{a^i + a^j}{(i-j)^2} \quad (i \neq j),$$

and

$$(B - TA)_{ii} = \sum_{t\geq 0} a^t u_t(i)^2 - a^i(T + S_2(a)).$$

The last formula is the diagonal term and is the reason no off-diagonal limiting convention is needed.

Expand first along the n rows of V. In each remaining $2m$-row minor, use the preceding rank-one expansion. A nonzero term selects $2m = 8n$ distinct features. If b, x, y, z count, respectively, the series features, the upper boundary features, the lower boundary features, and the column boundary features, then $b + x + y + z = 8n$. Distinctness gives the lower bound

$$\Psi \geq \binom{b}{2} + \binom{x}{2} + x\max(2n, b) + 2ny + \binom{y}{2} + nz + \binom{z}{2}.$$

Here the b series indices are distinct; an upper boundary index cannot be one of them because the two corresponding right vectors coincide. The x and y indices are distinct members of $I = [2n, 6n)$, while the z column indices are distinct members of $J = [n, 10n)$. These are the only features used in the count, so a collision can only delete a term and cannot lower the displayed lower bound. Writing $B_0 = b + x$, the right side is at least

$$Q(B_0, y) = B_0^2 + B_0 y + y^2 - 9nB_0 - 7ny + 40n^2 - 4n = \frac{53n^2}{3} - 4n + r^2 + rs + s^2$$

for suitable real r, s. Since the order is integral, every coefficient of every $2m$-minor has order at least D_m. The V-minors are rational constants independent of a, T, so expansion along V gives $a^{D_m} \mid G_m(T; a)$ in $\mathbb{Q}[a, T]$. Combining this with the denominator lemma and monic division gives $a^{D_m} \mid \Delta_m G_m$ in $\mathbb{Z}[a, T]$.

Finally every C-entry is divisible by $a-1$ in $\mathbb{Q}[a]$, including the zero diagonal convention. There are m such rows, hence $(a-1)^m \mid G_m$. The two monic factors a^{D_m} and $(a-1)^m$ are coprime, so their product divides the integral polynomial $\Delta_m G_m$. This proves the lemma for every $n \geq 1$, not only for the finite verification values. □

For $a = -q$, the formal divisor implies the exact coefficientwise bound

$$v_p([T^r]G_m(T; -q)) \geq v_p(q)D_m + v_p(q+1)m - 3m\lfloor \log_p N \rfloor - n\lfloor \log_p U \rfloor.$$

The two credited prime sets are disjoint because $\gcd(q, q+1) = 1$. Notice that this argument includes composite q and powers p^e; no square-free assumption is being made. For reference, the finite exceptional sets in the five middle cases are

q	$\mathcal{P}_q = \{p : p \mid q(q+1)\}$	q	$\mathcal{P}_q$
3	$\{2, 3\}$	4	$\{2, 5\}$
5	$\{2, 3, 5\}$	6	$\{2, 3, 7\}$
7	$\{2, 7\}$		

The $q = 2$ row is $\{2, 3\}$, and is treated by the same rule.

9.2. **The p-unit residue blocks.** Fix an odd prime $p \nmid q(q+1)$ with $p^2 > \max(N, U)$, and write $i = u + kp$, $j = u + \ell p$, $0 \le u < p$. Since $a = -q$ and $a^p \equiv a \pmod p$, splitting the defining sums at multiples of p gives

$$pt_{u+kp} = \sum_{\substack{1 \le r \le u+kp \\ p \nmid r}} \frac{pa^{u+kp-r}}{r} + \sum_{s=1}^{k} \frac{a^{u+p(k-s)}}{s} \equiv a^u t_k \pmod p,$$

$$p^2 b_{u+kp} = \sum_{\substack{1 \le r \le u+kp \\ p \nmid r}} \frac{p^2 a^{u+kp-r}}{r^2} + \sum_{s=1}^{k} \frac{a^{u+p(k-s)}}{s^2} \equiv a^u b_k \pmod p, \qquad p^2 T A_{ij} \equiv 0 \pmod p.$$

Here $k, \ell < p$ follows from $i \le N$, $j \le U$, and $p^2 > \max(N, U)$, so each displayed denominator s is a p-unit. Thus the scaled entries satisfy

$$pC_{u+kp,u+\ell p} \equiv a^u C_{k\ell} \pmod p, \qquad p^2 B_{u+kp,u+\ell p} \equiv a^u B_{k\ell} \pmod p,$$

whereas

$$pV_{r,u+\ell p} \equiv 0 \quad \text{unless } r + u = pw, \qquad pV_{r,u+\ell p} = \frac{1}{\ell + w} \quad (r + u = pw).$$

The denominators in the last expression are units because $p^2 > U$. Cross-residue C and B entries vanish after the displayed row scalings. The reduced residue-u block has $2|K_u| + |W_u|$ rows and $|L_u|$ columns. Therefore its rank over $\mathbb{F}_p$ is at most

$$\min(2|K_u| + |W_u|, |L_u|)$$

where

$$K_u = \{k : 1/2 \le u/m + kp/m < 3/2\}, \quad L_u = \{\ell : 1/4 \le u/m + \ell p/m < 5/2\},$$
$$W_u = \{w \ge 1 : 0 < wp/m - u/m \le 1/4\}.$$

Let rk_u denote the actual rank of this reduced block. After permuting rows and columns by residue class, the globally scaled square matrix of order $2m+n$ is block diagonal with precisely these blocks. Hence its rank modulo p equals $\sum_u \mathrm{rk}_u$, and its determinant has valuation at least

$$(2m+n) - \sum_u \mathrm{rk}_u \,.$$

The total row-scaling factor is p^{3m+n}: the C, B, and V rows were scaled by p, p^2, and p, respectively. Undoing this factor gives

$$v_p(G_m(T; -q)) \ge -\left(m + \sum_u \mathrm{rk}_u\right).$$

These valuations are coefficientwise in T. Indeed, the Gauss valuation on $\mathbb{Q}_p(T)$ is discrete with residue field $\mathbb{F}_p(T)$; Smith normal form over its valuation ring gives $v_p(\det M) \ge (2m+n) - \mathrm{rank}(M \bmod p)$ for every integral square matrix M of order $2m+n$. Thus only the actual block ranks rk_u are credited, not a count of non-divisible rows. Summing the block ranks gives the exponent used below. The only rank saving deliberately credited in the valuation sum is the profile $K = \{0\}, L = \{0, 1\}, W = \varnothing$. After dividing its rows by the unit a^u, its quotient matrix is

$$\begin{pmatrix} 0 & 1-a \\ 0 & -1 \end{pmatrix},$$

which has rank one over every residue field. This is a universal identity; no accidental rank deficiency of another profile is credited.

For $\rho = p/m$, define

$$K(x, \rho) = \{k \ge 0 : 1/2 \le x + k\rho < 3/2\}, \quad L(x, \rho) = \{\ell \ge 0 : 1/4 \le x + \ell\rho < 5/2\},$$
$$W(x, \rho) = \{w \ge 1 : 0 < w\rho - x \le 1/4\}, \qquad 0 \le x < \rho,$$

and define

$$\begin{aligned}\mathfrak{r}(x,\rho) &= \min(2|K(x,\rho)| + |W(x,\rho)|, |L(x,\rho)|) \\ &\quad - \mathbf{1}_{\{K(x,\rho)=\{0\},\, L(x,\rho)=\{0,1\},\, W(x,\rho)=\varnothing\}}, \\ \mathfrak{c}(\rho) &= 1 + \int_0^\rho \mathfrak{r}(x,\rho)\, dx.\end{aligned}$$

with the rank-one profile subtracted on its exact active cells. The endpoint conventions are left-closed/right-open for K, L, and $w\rho - 1/4 \le x < w\rho$ for W. Every wall has the form $x = c + b\rho$ with $4c, b \in \mathbb{Z}$. Since $m = 4n$, the function is constant on each lattice interval $[u/m, (u+1)/m)$, and hence

$$\sum_{u=0}^{p-1} \mathfrak{r}(u/m, p/m) = m \int_0^{p/m} \mathfrak{r}(x, p/m)\, dx.$$

The residue-block lemma is therefore

$$v_p(G_m(T; -q)) \ge -m\mathfrak{c}(p/m).$$

The proof uses only the p-unit conditions and the index geometry, so it is independent of the value of the fixed q.

For the arrangement with cutoff $R = 40$, all active walls on $1/40 < \rho < 5/4$ are enumerated by the finite affine rule above. Define the finite critical set explicitly by

$$\mathcal{W}_{40} = \{1/40, 5/4\} \cup \left\{ \frac{c - c'}{b' - b} : \; x = c + b\rho,\; x = c' + b'\rho \text{ are listed walls, } \frac{1}{40} < \frac{c - c'}{b' - b} < \frac{5}{4} \right\}.$$

At each open interval of this set, choose the least rational midpoint and generate the x-cells and profiles by the endpoint inequalities above; the profile is constant on the resulting cells. Their exact ordering gives 412 ρ-slabs, 2472 x-cells and 582 profiles; the one intentionally deficient profile integrates to the saving $5/48$. More explicitly, retain on $1/40 \le \rho \le 5/4$ and $0 \le x \le \rho$ the affine walls

$$\begin{gathered} x = 0, \quad x = \rho, \quad x = \tfrac{1}{2} - k\rho, \quad x = \tfrac{3}{2} - k\rho, \\ x = \tfrac{1}{4} - \ell\rho, \quad x = \tfrac{5}{2} - \ell\rho, \quad x = w\rho - \tfrac{1}{4}, \quad x = w\rho. \end{gathered}$$

for all nonnegative k, ℓ and positive w whose line meets the closed strip. Only finitely many such lines occur; sort their pairwise rational intersections and evaluate the exact midpoint profile on each cell. This finite rule gives the 412 ρ-slabs, 2472 x-cells and 582 profiles without floating-point geometry. The corresponding exact finite comparisons use only the displayed residue blocks and the stated rank formula.

9.3. **Outer poles and precedence.** For $p > 5n$, use row potentials $\lambda_B(i), \lambda_V(\ell)$ and column potentials $\chi(j)$ as follows:

range	$(\lambda_B, \lambda_V, \chi)$	cost	pole
$5n < p < 6n$	$(1 + [i \ge p], 1, [j \ge p + 2n])$	$19n - 2p$	C, B, V
$6n \le p < 8n$	$([i + p < 10n], 1, [j \ge p])$	$19n - 2p$	C, B, V
$8n \le p < 10n$	$(0, 1, [j \ge p])$	$11n - p$	C, B, V
$10n \le p \le U$	$(0, [p - \ell \in J], 0)$	$11n - p$	V

Here brackets are indicator functions. A C-entry has at most a simple pole at $j = i + p$; a B-entry has the simple harmonic pole and, when the residue agrees, the second pole (including the diagonal double pole); and a V-entry has a simple pole at $j + \ell = p$ or $2p$. The potentials in the table dominate these orders entry by entry. In particular, the two families of V-poles are explicitly included, and no pole is assigned to two ranges. This proves

$$v_p(G_m(T; -q)) \ge -E_p^{\mathrm{out}}, \qquad E_p^{\mathrm{out}} = \begin{cases} 19n - 2p, & 5n < p < 8n, \\ 11n - p, & 8n \le p \le U. \end{cases}$$

The C-row potential in the last row is 0; its absence there records that no C-pole occurs once $p > 10n$. The argument uses only p-unit cancellation and pole locations; it is unchanged when -2 is replaced by any fixed $-q$ in the p-unit branch.

For completeness, fix the finite set $\mathcal{P}_q = \{p : p \mid q(q+1)\}$. Let $\mathcal{W}_{40}$ be the exact set of affine ρ-walls in this arrangement, with $1/40$ and $5/4$ included as closed endpoints. Define the baseline exponent

$$e_{\mathrm{nat}}(p) = 3m \left\lfloor \log_p N \right\rfloor + n \left\lfloor \log_p U \right\rfloor,$$

which is the coefficientwise exponent supplied by $\Delta_m = d_N^{3m} d_U^n$, and assign the exponent to each prime by the following ordered rule:

$$E_{p,q} = \begin{cases} e_{\mathrm{nat}}(p) - v_p(q) D_m - v_p(q+1)m, & p \in \mathcal{P}_q, \\ e_{\mathrm{nat}}(p), & p \notin \mathcal{P}_q,\ p \le m/40, \\ e_{\mathrm{nat}}(p), & p \notin \mathcal{P}_q,\ p^2 \le \max(N,U), \\ e_{\mathrm{nat}}(p), & p \notin \mathcal{P}_q,\ p/m \in \mathcal{W}_{40}, \\ m + \sum_u \mathfrak{r}(u/m, p/m), & \substack{p \notin \mathcal{P}_q,\ 1/40 < p/m < 5/4, \\ p/m \notin \mathcal{W}_{40}}, \\ E_p^{\mathrm{out}}, & 5n < p \le U,\ p \notin \mathcal{P}_q, \\ 0, & p > U. \end{cases}$$

The first matching line always takes precedence; the cases are therefore an ordered partition even when a small-prime condition overlaps a wall. The outer line begins after the inner range. The coefficientwise bounds above imply

$$D_{m,q}^* := \prod_p p^{E_{p,q}}, \qquad D_{m,q}^* G_m(T; -q) \in \mathbb{Z}[T].$$

The exponent in the first line can be negative; this is allowed because the formal divisor supplies the corresponding coefficient valuation. No primitive content or gcd equality is asserted.

9.4. **Prime summation and the fixed-base rate.** Let $\vartheta(y) = \sum_{p \le y} \log p$ and

$$\varepsilon_{\mathrm{PNT}}(X) = \sup_{y \ge X} \left| \frac{\vartheta(y)}{y} - 1 \right|.$$

For every affine profile f on a rational ρ-cell, Stieltjes summation gives

$$m \sum_{t_0 m < p < t_1 m} f(p/m) \log p = m^2 \int_{t_0}^{t_1} f(\rho)\, d\rho + O\big(\varepsilon_{\mathrm{PNT}}(m/40) m^2 + m \log(3m)\big).$$

The baseline strip contributes $13/160$, the deliberately credited profile contributes $-5/48$, and the outer potentials contribute $45/32$. The exact rational value for this arrangement (including all intervening cells) is

$$\begin{aligned} A_{40} &= N_{40}/D_{40} = 5.082993964478118\ldots, \\ N_{40} &= 1322633638034219254503386228478097423171245514 97, \\ D_{40} &= 2602075956173237187743865003560151668425466 8800. \end{aligned}$$

Higher prime powers contribute $O(m^{3/2} \log(3m))$. The exact sum over the fixed exceptional set is

$$\sum_{p \in \mathcal{P}_q} E_{p,q} \log p = \sum_{p \in \mathcal{P}_q} e_{\mathrm{nat}}(p) \log p - D_m \log q - m \log(q+1).$$

Only the first term on the right is the exceptional-prime cost; since $\mathcal{P}_q$ is fixed and $N, U = O(m)$, it is $O_q(m \log(3m))$. The $(q+1)^m$ formal factor contributes the separate linear term $-m \log(q+1)$. As $D_m = 53m^2/48 - m + O(1)$, the other formal term contributes $-\frac{53}{48} m^2 \log q + O_q(m)$. Thus

$$\log D_{m,q}^* \le \left(A_{40} - \frac{53}{48} \log q \right) m^2 + r_{q,m},$$

where

$$r_{q,m} = O\Big(\varepsilon_{\mathrm{PNT}}(m/40) m^2 + m^{3/2} \log(3m) + m \log(3m) + C_q^{\mathrm{exc}} m \log(3m) + B_q m \Big),$$

$C_q^{\mathrm{exc}} = 13|\mathcal{P}_q|/4$, and B_q is fixed after q is fixed. Since $\varepsilon_{\mathrm{PNT}}(X) \to 0$, this is $o_q(m^2)$. The quadratic rates are

q	$A_{40} - (53/48)\log q$
3	3.869942895741
4	3.552293940742
5	3.305906269499
6	3.104592883873
7	2.934384841563

For $q = 2$, the same formal rate is $4.317643952609845\ldots$; its analytic passage is the $q = 2$ instance of Section 10.

Proposition 9.3 (fixed-base arithmetic transfer)**.** *For each fixed $q \in \{2,3,4,5,6,7\}$, the determinant has a rational coefficientwise integerizer satisfying*

$$\limsup_{m\to\infty,\ 4|m} m^{-2}\log D^*_{m,q} \le A_{40} - \frac{53}{48}\log q.$$

For $q = 2,3,4,5,6,7$, the proof is the formal divisor, residue/outer bounds, wall precedence and PNT argument above. The finite comparisons instantiate the displayed formulas; they do not replace the all-m argument.

10. Signed logarithmic energy and smoothing

For each fixed $q \in \{2,3,4,5,6,7\}$, directed interval checks on the printed reference and the smoothing estimate below give an all-m determinant bound. The interval checks certify the energy rate; the smoothing lemma supplies the estimate for every $m = 4n \ge 64$.

Throughout, q is a fixed integer with $2 \le q \le 7$, $a = -q$, $m = 4n$. The three local shape parameters are set to $s = k = h = n$, so that $L = m + h = 5m/4$. The three masses are $(M_x, M_v, M_z) = (1, 5/4, 1)$. Write

$$I(\sigma,\eta) = \iint \log|u-w|\,d\sigma(u)d\eta(w), \qquad D_q(\sigma,\eta) = \iint \log|q+uv|\,d\sigma(u)d\eta(v),$$

and

$$\mathcal{Q}_q(\mu) = I(\mu_x,\mu_x) + I(\mu_v,\mu_v) + I(\mu_z,\mu_z) + I(\mu_v,\mu_z) - D_q(\mu_x,\mu_v). \tag{10.1}$$

The supports used for the continuous problem are

$$\mathcal{K}_x = \mathcal{K}_v = [0,1], \qquad \mathcal{K}_z(q) = [-q,-r]\cup[r,1]\cup\{|z| = r\}, \qquad r = 3/5.$$

For a reference triple $\nu = (\nu_x,\nu_v,\nu_z)$ of these masses, put

$$U_i(w) = \int \log|w-t|\,d\nu_i(t),$$
$$D_v(w) = \int \log|q+wt|\,d\nu_v(t),$$
$$D_x(w) = \int \log|q+tw|\,d\nu_x(t).$$

and define the three fields

$$\begin{aligned} \widehat{F}_x(w) &= 2U_x(w) - D_v(w),\\ \widehat{F}_v(w) &= 2U_v(w) + U_z(w) - D_x(w),\\ \widehat{F}_z(w) &= 2U_z(w) + U_v(w), \end{aligned} \tag{10.2}$$

For the full logarithmic energy we add the one-body terms separately:

$$\begin{aligned} F_x(w) &= \widehat{F}_x(w) + \tfrac{3}{4}\log|w|,\\ F_v(w) &= \widehat{F}_v(w) + \tfrac{1}{4}\log|w|,\\ F_z(w) &= \widehat{F}_z(w) - \tfrac{5}{4}\log|w|. \end{aligned} \tag{10.3}$$

The constant field from the determinant prefactor is

$$C_0(q) = \tfrac{3}{2}\log q. \tag{10.4}$$

We first identify the determinant at the dilogarithm value. Put

$$M_{ij}(q) = \int_0^1\int_0^1 \frac{x^i y^j}{q+xy}\,dx\,dy, \qquad \beta_q = \mathrm{Li}_2(-1/q), \qquad \lambda_q = \log(1+1/q).$$

For $j \ge 0$ set the two moment sequences

$$m_j = \int_0^1 \frac{u^j}{q+u}\,du, \qquad n_j = \int_0^1 \frac{u^j \log(1/u)}{q+u}\,du.$$

Polynomial division gives, for $j \ge 1$,

$$m_j = -qm_{j-1} + \frac{1}{j}, \qquad n_j = -qn_{j-1} + \frac{1}{j^2},$$

while $m_0 = \lambda_q$ and $n_0 = -\beta_q$. Hence, with the polynomials t_j, b_j of Section 9,

$$m_j = (-q)^j\lambda_q + t_j, \qquad n_j = b_j - (-q)^j\beta_q.$$

The substitution $u = xy$ yields

$$M_{ij}(q) = \frac{m_i - m_j}{j-i} \quad (i \ne j), \qquad M_{ii}(q) = n_i,$$

and therefore

$$M(q) = B(-q) + \lambda_q C(-q) - \beta_q A(-q). \tag{10.5}$$

Adding λ_q times each C row to the corresponding row of $B(-q) - \beta_q A(-q)$ preserves the determinant. Thus the determinant of Section 9, evaluated at $T = \beta_q$, is exactly the determinant with middle block $M(q)$.

10.1. **The path and the signed energy.** Let $\gamma_+(t) = re^{it}$ for $0 \le t \le \pi$ and $\gamma_-(t) = re^{it}$ for $\pi \le t \le 2\pi$. Orient the two real segments increasingly and set

$$\Gamma_q = [-q,-r] + [r,1] + (c_q - 1)\gamma_+ + c_q\gamma_-, \qquad c_q = \frac{\log q + i\pi}{2\pi i}. \tag{10.6}$$

For every integer d,

$$\int_{\Gamma_q} z^{d-1}\,dz = \begin{cases} (1-(-q)^d)/d, & d \ne 0, \\ 0, & d = 0. \end{cases} \tag{10.7}$$

For $d \ne 0$, integrate z^{d-1} on each piece using the primitive z^d/d; the two circular half-arcs contribute exactly the terms needed to cancel the intermediate endpoints $\pm r$. For $d = 0$, the real pieces give $-\log q$, while the circular pieces give $i\pi(2c_q - 1) = \log q$. This also fixes the branch convention at the zero Laurent moment.

The same Andreief identity [1, 3] and Cauchy–Vandermonde expansions [6] as for the determinant in Section 9, now using (10.7), give

$$G_m(\mathrm{Li}_2(-1/q); -q) = \frac{q^{3m^2/2-m}}{(m!)^2 L!}\int_{[0,1]^m}\int_{[0,1]^L}\int_{\Gamma_q^m} \Phi_q(x,v,z)\,dz\,dv\,dx, \tag{10.8}$$

where

$$\Phi_q = \frac{\Delta(x)^2\Delta(v)^2\Delta(z)^2\prod_{d=1}^m\prod_{b=1}^L(z_d - v_b)\prod_{i=1}^m x_i^{3m/4}\prod_{b=1}^L v_b^{m/4}\prod_{d=1}^m z_d^{-5m/4}}{\prod_{i=1}^m\prod_{b=1}^L(q + x_i v_b)}.$$

Proof of the integral representation. By (10.7), $C_{ij} = (-q)^i\int_{\Gamma_q} z^{j-i-1}\,dz$. Put $f_i(v) = \int_0^1 x^i/(q+xv)\,dx$. Then the lower rational block and the V block are respectively

$$M_{ij} = \int_0^1 f_i(v)v^j\,dv, \qquad V_{\ell j} = \int_0^1 v^{\ell-1}v^j\,dv.$$

For integrable families, Andreief's identity [1, 3] is

$$\det\left(\int f_i(t)g_j(t)\,dt\right)_{i,j=1}^{N} = \frac{1}{N!}\int \det(f_i(t_b))_{i,b=1}^{N}\det(g_j(t_b))_{b,j=1}^{N}\prod_{b=1}^{N}dt_b.$$

Apply it successively to the L lower/V rows, the m contour rows, and the m functions f_i. These three applications account respectively for $1/L!$, $1/m!$, and $1/m!$:

$$G_m = \frac{1}{(m!)^2 L!}\int D_z D_{zv} D_x D_{xv}\,dx\,dv\,dz.$$

Here $e_\Delta = m(m-1)/2$ and, with $\mathbf{w} = (z_1,\dots,z_m,v_1,\dots,v_L)$, the four determinants are

$$D_z = \det\big((-q)^i z_d^{-i-1}\big)_{i\in I,d} = (-1)^{e_\Delta}(-q)^{\sum_{i\in I} i}\Delta(z)\prod_d z_d^{-3m/2},$$

$$D_{zv} = \det(\mathbf{w}_b^j)_{b,j\in J} = (-1)^{mL}\Delta(z)\Delta(v)\prod_d z_d^{m/4}\prod_b v_b^{m/4}\prod_{d,b}(z_d - v_b),$$

$$D_x = \det(x_{i'}^i)_{i\in I,i'} = \Delta(x)\prod_{i'} x_{i'}^{m/2},$$

$$D_{xv} = \det\begin{pmatrix}((q+x_{i'}v_b)^{-1})_{i',b}\\ (v_b^{\ell-1})_{\ell,b}\end{pmatrix} = \frac{(-1)^{e_\Delta}q^{e_\Delta}\Delta(x)\Delta(v)\prod_{i'}x_{i'}^{m/4}}{\prod_{i',b}(q+x_{i'}v_b)}.$$

The last equality is the Cauchy–Vandermonde identity [6]. For completeness, for $t_{i'} = q/x_{i'}$ it follows from

$$\det\begin{pmatrix}((t_{i'}+v_b)^{-1})_{i',b}\\ (v_b^{\ell-1})_{\ell,b}\end{pmatrix} = \frac{\Delta(t)\Delta(v)}{\prod_{i',b}(t_{i'}+v_b)},$$

after multiplying the first m rows by $x_{i'}^{-1}$ and using $\Delta(t) = (-1)^{e_\Delta}q^{e_\Delta}\Delta(x)\prod_{i'}x_{i'}^{1-m}$. Here is a proof of the identity, for every $m,h\ge 0$ with $L = m+h$. Treat the t_i and v_b as independent indeterminates. The case $m=0$ is the Vandermonde determinant. Assume the formula for $m-1$ and all $h\ge 0$, and regard both sides as rational functions of v_L. Expanding the determinant along its last column, its residue at $v_L = -t_i$ is $(-1)^{i+L}$ times the complementary minor. By induction this residue is

$$(-1)^{i+L}\frac{\Delta(\widehat{t_i})\Delta(v_1,\dots,v_{L-1})}{\prod_{j\ne i}\prod_{b<L}(t_j+v_b)}.$$

The proposed quotient has the same residue: substitute $v_L = -t_i$, cancel the factors $t_j - t_i$ and $t_i + v_b$, and use $\Delta(t) = (-1)^{i-1}\Delta(\widehat{t_i})\prod_{j\ne i}(t_j - t_i)$. Hence the difference has no finite poles. After cancellation of the denominator $\prod_i(t_i+v_L)$, the difference is a polynomial in v_L of degree at most $h-1$. It vanishes at each of the $L-1$ values $v_L = v_b$ $(b<L)$, since both sides have the factor $\Delta(v)$. For $m\ge 1$, $L-1 = m+h-1\ge h$, so these roots force the polynomial to be zero. When $h=0$, the difference tends to zero at infinity instead. This proves the Cauchy–Vandermonde identity.

Multiplying the four displayed factors gives precisely Φ_q: the $m/2$ from D_x and the $m/4$ from D_{xv} combine to the displayed $3m/4$ power of each $x_{i'}$. The sign is $(-1)^{\sum_{i\in I} i+2e_\Delta+mL} = 1$: here $\sum_{i\in I} i = 2n(8n-1)$ and $mL = 20n^2$ are even. The q-power is

$$\sum_{i\in I} i + e_\Delta = m(2n) + m(m-1) = \frac{3m^2}{2} - m, \qquad (m = 4n,\ L = 5n).$$

The factors $q + x_i v_b$ are at least q, while $|z|\ge r$ on Γ_q; all three integrations therefore converge absolutely. The $m!$, $L!$, and $m!$ factors are exactly the three Andreief symmetrizations, proving (10.8). □

For triples of positive measures of the stated masses, define

$$\mathcal{E}_q(\mu) = C_0(q) + \mathcal{Q}_q(\mu) + \tfrac{3}{4}\int \log|x|\,d\mu_x + \tfrac{1}{4}\int \log|v|\,d\mu_v - \tfrac{5}{4}\int \log|z|\,d\mu_z. \tag{10.9}$$

Lemma 10.1 (signed-energy concavity for fixed q)**.** *Let μ, ν be triples of positive measures of masses $(1, 5/4, 1)$, supported in compact subsets of the indicated supports or their sufficiently small tubular neighborhoods. Assume that the pairings in $\mathcal{Q}_q$ are finite real numbers. No absolute logarithmic moment assumption is imposed on a competitor's x- or v-component. Then*

$$\mathcal{Q}_q(\mu) \le -\mathcal{Q}_q(\nu) + \int \widehat{F}_x \, d\mu_x + \int \widehat{F}_v \, d\mu_v + \int \widehat{F}_z \, d\mu_z,$$

where the hatted fields are formed from ν. Consequently the same inequality for the full energy is obtained by adding the displayed one-body terms to the hatted fields, i.e. by using F_x, F_v, F_z in the field bounds.

Proof. By the shared quadratic-duality identity in Section C, it suffices to prove that the zero-mass quadratic remainder $\mathcal{Q}_q(\mu - \nu)$ is nonpositive. First suppose that all v-supports avoid zero. For a signed measure σ of mass zero, write $\|\sigma\|^2 = -I(\sigma, \sigma) \ge 0$; this is the zero-mass logarithmic-energy positivity lemma. Put $R_q(v) = -q/v$. The exact identities

$$\log|q + xv| = \log|x - R_q(v)| + \log|v|,$$
$$\log|R_q(v) - R_q(w)| = \log q + \log|v - w| - \log|v| - \log|w|$$

Integrating against zero-mass measures cancels all one-variable terms and gives

$$D_q(\sigma_x, \sigma_v) = I(\sigma_x, (R_q)_*\sigma_v),$$
$$I((R_q)_*\sigma_v, (R_q)_*\sigma_v) = I(\sigma_v, \sigma_v).$$

Let $\sigma_i = \mu_i - \nu_i$, and set $\sigma'_x = \sigma_x$, $\sigma'_v = (R_q)_*\sigma_v$, $\sigma'_z = -(R_q)_*\sigma_z$. Then

$$\mathcal{Q}_q(\sigma) = -\|\sigma'_x\|^2 - \|\sigma'_v\|^2 - \|\sigma'_z\|^2 + \langle\sigma'_x, \sigma'_v\rangle + \langle\sigma'_v, \sigma'_z\rangle, \qquad \langle\alpha, \beta\rangle = -I(\alpha, \beta).$$

The right side equals

$$-\tfrac{1}{2}\big(\|\sigma'_x - \sigma'_v\|^2 + \|\sigma'_v - \sigma'_z\|^2 + \|\sigma'_x\|^2 + \|\sigma'_z\|^2\big) \le 0.$$

The shared identity gives the asserted quadratic inequality. For a competitor μ_v whose support meets zero, fix a smooth probability measure ρ supported in $[3/4, 7/8]$. For $0 < \delta < 1/4$, let $\lambda_\delta = \mu_v|_{\{|v|<\delta\}}$, let $e_\delta = \lambda_\delta(\mathbb{C})$, and set

$$\mu_{v,\delta} = \mu_v - \lambda_\delta + e_\delta\rho.$$

Finite logarithmic self-energy excludes an atom at zero, so $e_\delta \to 0$ and $\mu_{v,\delta} \Rightarrow \mu_v$. Absolute integrability of the logarithmic pairings in $\mathcal{Q}_q$ implies that the integrals over the deleted sets tend to zero. The replacement terms involving $e_\delta\rho$ tend to zero as well: ρ is separated from $\{|v| < \delta\}$, and its logarithmic potential is bounded on the compact supports. The kernel $\log|q + uv|$ is bounded and continuous on the stated tubes because $q \ge 2$ and $|u|, |v| \le 1 + o(1)$. Hence $\mathcal{Q}_q(\mu_x, \mu_{v,\delta}, \mu_z) \to \mathcal{Q}_q(\mu)$. The reference fields $\widehat{F}_i$ are continuous and bounded on these compact sets, so their integrals also converge. Applying the already proved inequality to $\mu_{v,\delta}$ and taking the limit proves the quadratic inequality without requiring $\int |\log|v|| \, d\mu_v < \infty$.

The one-body integrals in $\mathcal{E}_q$ are then interpreted as extended real integrals. On the compact x- and v-supports, $\log|w|$ is bounded above and can have integral $-\infty$; on the z-support, $|z| \ge r - o(1)$, so the z-term is finite. Adding these linear terms to the quadratic inequality is therefore valid in the extended reals. In particular, when the full energy is $-\infty$ the asserted upper bound is automatic. This argument uses no implication from finite self-energy to a finite absolute logarithmic moment. □

Each printed reference has finitely many positive-length rational interval rows, compact z support in $[-q, -3/5]$, and Poisson centres with $|c| < 3/5$; the x and v chains start at the positive endpoints listed in the data section and end at 1. Hence all six reference supports are compact and avoid zero. Interval densities are bounded by the finite maximum of their rational mass/length ratios, while a Poisson row of mass w has density

$$\frac{w}{2\pi}\frac{r^2 - |c|^2}{|re^{i\theta} - c|^2} \le \frac{w(r + |c|)}{2\pi r(r - |c|)}, \qquad r = 3/5.$$

Thus the printed references have finite logarithmic moments directly from their rational supports, without any general self-energy implication.

Combining the lemma with the one-body terms gives the explicit full-energy upper bound used below:

$$\mathcal{E}_q(\mu) \le C_0(q) - \mathcal{Q}_q(\nu) + \int F_x \, d\mu_x + \int F_v \, d\mu_v + \int F_z \, d\mu_z. \tag{10.10}$$

Thus, if the full fields are bounded above by $K_{x,q}, K_{v,q}, K_{z,q}$ on their respective admissible supports, then

$$\mathcal{E}_q(\mu) \le -\big(\mathcal{Q}_q(\nu) - C_0(q) - K_{x,q} - \tfrac{5}{4}K_{v,q} - K_{z,q}\big). \tag{10.11}$$

The inversion is valid uniformly for the fixed bases considered here: on $|v| \le 1 + 3/4096$, $|R_q'(v)| = q/|v|^2 > 1$ for $q \ge 2$. Thus a bounded arclength density pushes forward to a bounded arclength density on each inverted segment or circle. No $q = 2$ numerical identity is used in the concavity argument.

10.2. **Directed fixed-base certificates.** For a reference triple ν_q, suppose its energy lower endpoint is $Q_{q,-}$ and its support-wide field upper bounds are $F_x \le K_{x,q}$, $F_v \le K_{v,q}$, $F_z \le K_{z,q}$. Lemma 10.1 and (10.9) give the quadratic rate

$$\alpha_q = Q_{q,-} - \tfrac{3}{2}\log q - K_{x,q} - \tfrac{5}{4}K_{v,q} - K_{z,q}. \tag{10.12}$$

The following are the directed interval-bound results. The field thresholds are exact terminating decimals. The displayed α values are strict lower bounds rounded down, and each τ is an outward-rounded upper bound.

TABLE 10.1. Fixed-base directed certificates for $a = -q$.

q	rows	K_x	K_v	K_z	α_q	τ_q	margin
2	41/48/62/149	-4.621	-4.475	-0.085	> 4.34875	< 4.31765[1]	> 0.03110
3	40/48/62/143	-5.076	-4.632	0.388	> 3.96707	< 3.86995	> 0.09712
4	40/48/62/141	-5.408	-4.727	0.779	> 3.68211	< 3.55230	> 0.12981
5	39/48/62/137	-5.670	-4.791	1.108002	> 3.45240	< 3.30591	> 0.14649
6	41/47/61/135	-5.886	-4.839	1.391	> 3.26092	< 3.10460	> 0.15632
7	39/47/62/131	-6.071	-4.876	1.639	> 3.09651	< 2.93439	> 0.16212

The energy lower endpoints used in (10.12) are printed separately so that the energy and field slacks can be checked independently:

TABLE 10.2. Directed lower endpoints for the reference energies.

q	$Q_{q,-}$
2	> -4.911272
3	> -4.863007
4	> -4.776198
5	> -4.684191
6	> -4.595191
7	> -4.511619

For each $q = 2, \dots, 7$, put $\delta = 1/250$. The boxes $0 \le x, v \le \delta$ are discharged by the endpoint bounds (10.13). On $[\delta, 1]$ and on the two real z intervals, including the complete negative interval $[-q, -3/5]$, the subdivision recursively bisects each rational midpoint. A cell is accepted exactly when its directed upper field interval is finite and strictly below the displayed threshold; otherwise both children are visited, in left-first order, with maximum depth 30. Circle cells use the same rule in the parameter $t \in [0, 1]$, and the endpoints $t = 0, 1$ are checked directly. This is the complete partition rule. The six references contain 1733 rows in all, and their rows are printed in Section B. The smoothing lemma below supplies the complex-tube transfer and the all-m estimate.

[1] The exact rate is $\tau_2 = A_{40} - (53/48)\log 2 = 4.317643952609845\ldots$; the displayed decimal is a directed upper bound.

10.3. **Reference densities and directed endpoint disks.** All constants below are computed from the signed decimal tables in Section B. For each interval row $[a,b]$ of mass w, put $H = w/(b-a)$. Let $H_{x,q}$, $H_{v,q}$ and $H_{z,q}$ be the largest such value in the x, v and negative-real z chains. For a Poisson row (c,w) write

$$\rho_c = 2^{-120}\left\lceil 2^{120}|c|\right\rceil, \qquad P_q = \sum_{(c,w)} \frac{w(r+\rho_c)}{r(r-\rho_c)}.$$

The ceiling is evaluated by integer square root on the printed rational coordinates, and the construction checks $\rho_c < r$ row by row. Since the Poisson density on $|z| = r$ is at most $w(r+\rho_c)/(2\pi r(r-\rho_c))$ per unit arclength, and a circle meets a disk of radius t in arclength at most $2\pi t$, the reference measures satisfy $\nu_i(B(w,t)) \le A_{i,q}t$, where

$$A_{x,q} = 2H_{x,q}, \qquad A_{v,q} = 2H_{v,q}, \qquad A_{z,q} = 2H_{z,q} + P_q.$$

Define

$$B_{i,q} = 6A_{i,q} + 2M_i, \qquad \Sigma_q = 2B_{x,q} + \frac{7}{2}B_{v,q} + \frac{13}{4}B_{z,q}, \qquad \mathcal{C}_q = 10 + 3\Sigma_q.$$

In the table, Σ_q is rounded upward to 10^{-3} and this rounded rational value is used in $\mathcal{C}_q$. The path variation is

$$\mathcal{T}_q = q + 1 - 2r + r\sqrt{\pi^2 + \log^2 q}.$$

The $E_{x,q}$ and $E_{v,q}$ columns are rounded upward to 10^{-4}; the $\mathcal{T}_q$ column is rounded upward to 10^{-6}.

For $\delta = 1/250$, put $P(t) = t(\log t - 1)$ and define the exact real integrals

$$J_{x,q} = \int \log(t+\delta)\, d\nu_x(t), \quad J_{v,q} = \int \log(t+\delta)\, d\nu_v(t), \quad J_{z,q} = \int \log(|z|+\delta)\, d\nu_z(z).$$

An interval contributes $w(P(b+\delta) - P(a+\delta))/(b-a)$; for a negative z interval use endpoints $-a, -b$ in increasing order. The Poisson rows together contribute their total mass times $\log(r+\delta)$. For $|w_0| \le \delta$, the triangle inequality and the masses give

$$F_x(w_0) \le E_{x,q} := 2J_{x,q} - \frac{5}{4}\log(q-\delta) + \frac{3}{4}\log\delta,$$
$$F_v(w_0) \le E_{v,q} := 2J_{v,q} + J_{z,q} - \log(q-\delta) + \frac{1}{4}\log\delta. \tag{10.13}$$

The endpoint bounds are directed upper endpoints rounded upward at 10^{-4}. The same table gives Σ_q rounded upward at 10^{-3} and $\mathcal{T}_q$ rounded upward at 10^{-6}; throughout the proof set $\mathcal{C}_q = 10 + 3\Sigma_q$ using the printed Σ_q.

TABLE 10.3. Endpoint and all-m transfer data recomputed from the printed tables.

q	$E_{x,q}$	$E_{v,q}$	Σ_q (upper)	$\mathcal{T}_q$ (upper)
2	−6.2821	−4.4935	14092.327	3.730292
3	−6.7656	−4.6587	13219.499	4.796889
4	−7.1126	−4.7555	12702.122	5.860319
5	−7.3835	−4.8208	12351.983	6.917916
6	−7.6061	−4.8688	12095.881	7.969979
7	−7.7947	−4.9057	11898.583	9.017257

The endpoint entries satisfy $E_{x,q} < K_{x,q}$ and $E_{v,q} < K_{v,q}$, where the field bounds are in Table 10.1. The displayed Σ_q values give the explicit coefficients $\mathcal{C}_q = 10 + 3\Sigma_q$; no unprinted density or endpoint constant is used below.

10.4. **The all-m smoothing estimate.**

Lemma 10.2 (logarithmic-potential modulus)**.** *If a compactly supported positive measure η of mass M satisfies $\eta(B(w,t)) \le At$ for every disk, then for $d = |w - w'| \le 1/3$,*

$$|U_\eta(w) - U_\eta(w')| \le (6A + 2M)d\big(1 + \log(1/d)\big).$$

Proof. Let $N = \{z : |w - z| \le 2d\}$. On N both distances are at most $3d$. Approximating the closed disk from outside gives $\eta(N) \le 2Ad$. For either centre $w_* = w, w'$, layer cake bounds the integral on N by

$$\int_N \log \frac{1}{|w_* - z|}\, d\eta(z) \le \int_0^\infty \min(2Ad, Ae^{-s})\, ds = 2Ad\big(1 + \log(1/(2d))\big).$$

Off N, the logarithmic difference is at most $2d/|w - z|$, while another layer-cake integration gives $\int_{|w-z|>2d} |w - z|^{-1} d\eta(z) \le M + A\log(1/(2d))$. Adding these bounds gives at most $6Ad\log(1/d) + 2Md$, which is bounded by the displayed expression. □

For $q = 2, \dots, 7$, let α_q be the strict lower endpoints in Table 10.1. In particular, the printed table verifies $\alpha_q > \tau_q$ in each of the six cases. Then for every $m = 4n \ge 64$,

$$\log |G_m(\mathrm{Li}_2(-1/q); -q)| \le -\alpha_q m^2 + \mathcal{C}_q m \log m, \qquad \mathcal{C}_q = 10 + 3\Sigma_q. \tag{10.14}$$

Proof. Fix q and a nonzero integrand point in (10.8). Put $\epsilon = m^{-2}$. Replace each atom u of its empirical measures by the uniform probability measure on $u + 2i\epsilon + \epsilon e^{i\vartheta}$. Every point of this circle is within 3ϵ of u. The common translation preserves pairwise centres; subharmonicity of $\log|z|$ shows that every off-diagonal positive logarithmic pair energy does not decrease. Each diagonal circle has self-energy $\log\epsilon$, and there are $m + 5m/4 + m = 13m/4$ atoms. Thus, writing μ^ϵ for the smoothed empirical triple and $\mathcal{Q}_{q,\mathrm{disc}}$ for the three self-energies with their diagonals removed,

$$\mathcal{Q}_{q,\mathrm{disc}}(\mu) \le \mathcal{Q}_q(\mu^\epsilon) + \frac{13}{4m}\log(1/\epsilon) + C_{D,q}\epsilon, \quad C_{D,q} = 8L_{D,q}, \quad L_{D,q} = \frac{1 + 3/4096}{q - (1 + 3/4096)^2}.$$

Indeed, on the two smearing disks $|u|, |v| \le 1 + 3\epsilon$ and $|uv - xv_0| \le 6\epsilon + 9\epsilon^2$. The denominator stays at distance at least $q - (1 + 3/4096)^2$ from zero. The mean-value bound for $\log|q + uv|$, multiplied by the cross mass $5/4$, is at most

$$\left(\frac{15}{2} + \frac{45}{4}\epsilon\right) L_{D,q}\epsilon \le C_{D,q}\epsilon,$$

since $\epsilon \le 1/4096$.

The smoothed measures are tame and their v-support avoids zero, so Lemma 10.1 applies. The modulus lemma and the constants in Table 10.3 bound the changes in all reference potentials by

$$\Sigma_q(3\epsilon)\big(1 + \log(1/(3\epsilon))\big).$$

For the two denominator potentials in the dual fields, each derivative in the field variable is at most $L_{D,q}$ on the tube. Their total mass coefficient is $5/2$, so their transfer error is at most $C_{D,q}\epsilon$.

For the positive one-body terms, if a real centre $t \ge \delta/2$, the circle mean of $\log|t + 2i\epsilon + \epsilon e^{i\vartheta}|$ is $\log|t + 2i\epsilon| \le \log t + 8\epsilon^2/\delta^2$. The combined weighted loss for x and v is therefore at most $(17/2)\epsilon^2/\delta^2$. If $t < \delta/2$, the whole smearing circle lies in $|w| \le \delta$, so (10.13) gives the endpoint bound directly. For the negative z one-body term, $|z| \ge r$, $|z^\epsilon - z| \le 3\epsilon$, and $3\epsilon < r$ give

$$|\log|z^\epsilon| - \log|z|| \le \frac{3\epsilon}{r - 3\epsilon}.$$

Consequently,

$$-\frac{5}{4}\log|z^\epsilon| \le -\frac{5}{4}\log|z| + \frac{15\epsilon}{4(r - 3\epsilon)}.$$

The centres lie on the certified supports; combining the last three estimates with the endpoint bounds and the directed field bounds yields

$$\begin{aligned}\frac{\log|\Phi_q|}{m^2} \le\ & -\mathcal{Q}_q(\nu_q) + K_{x,q} + \frac{5}{4}K_{v,q} + K_{z,q} + \frac{13}{4m}\log(1/\epsilon) \\ & + 3\Sigma_q\epsilon\big(1 + \log(1/(3\epsilon))\big) + \left(2C_{D,q} + \frac{15}{4(r - 3/4096)}\right)\epsilon + \frac{17}{2}\frac{\epsilon^2}{\delta^2}.\end{aligned}$$

Here the coefficient $2C_{D,q}$ accounts separately for the denominator in the smoothed energy and in the transferred dual fields. For all $q \geq 2$, the rational comparison $100(1+3/4096) < 101(2-(1+3/4096)^2)$ gives $L_{D,q} < 101/100$, hence $C_{D,q} < 8.08$. Also $15/(4(r-3/4096)) < 7$, and

$$\frac{17}{2\delta^2 m^2} \leq \frac{531250}{4096} < 130 \qquad (m \geq 64).$$

The total constant after multiplying by m^2 is therefore less than $16.2 + 7 + 130 < 200$.

The path length is exactly $\mathcal{T}_q$. Since $q \leq 7$ and $e > 8/3$, we have $\log q < 2$. Also $\pi < 22/7$, so $\sqrt{\pi^2 + \log^2 q} < 4$, whence $\mathcal{T}_q/q < 21/10 < e$. The prefactor in (10.8) contributes $(3m^2/2 - m)\log q$, and the path contributes $m \log \mathcal{T}_q$; their linear part is $m \log(\mathcal{T}_q/q) \leq m$. Add these terms and drop the favorable $-\log((m!)^2 L!)$. With $\epsilon = m^{-2}$, the diagonal term is $(13/2)m \log m$. Also

$$3\Sigma_q\big(1 + \log(m^2/3)\big) \leq 3\Sigma_q m \log m, \quad 200 \leq m \log m, \quad m \leq \tfrac{1}{4} m \log m$$

for $m \geq 64$. The resulting coefficient is at most $13/2 + 3\Sigma_q + 1 + 1/4 < 10 + 3\Sigma_q$. Finally the definition of α_q in (10.12) gives (10.14). If the determinant integrand vanishes, the bound holds trivially; otherwise all empirical measures above are defined and the argument applies. This proves the claim for every $m = 4n \geq 64$, not just for the finite certificate values. □

Thus the directed margins and the all-m analytic estimate are separate inputs: the former supplies $\alpha_q > \tau_q$, while (10.14) supplies the full quantifier with an explicit finite coefficient for each fixed $q = 2, \ldots, 7$.

11. Seed determinants and eventual nonvanishing

This section proves the nonvanishing input for the determinants used in each of the two constructions. The P0 family has a base-dependent auxiliary prime; the signed construction uses one explicitly listed prime for each base $a = -r$, $2 \leq r \leq 7$. The prime-power variable below is always separate from the base parameter. Finite seed calculations are printed, but nonvanishing at all sufficiently high powers follows from the residue-block identities proved here, not from extrapolating a finite table.

11.1. **The P0 seed and its base-dependent prime.** Write $G^{(0)}_{m,a}(T)$ for the P0 determinant of (4.1). At $m = 4$ its row and column sets are $I_4 = \{1,2,3,4\}$ and $J_4 = \{0,1,\ldots,7\}$. Exact elimination in this the 8×8 determinant gives

$$G^{(0)}_{4,a}(0) = \frac{a^9(a-1)^2 H(a)}{D}, \qquad D = 2090188800000 = 2^{17}3^6 5^5 7. \tag{11.1}$$

For an exact coefficient record, write $H(a)=\sum_{j=0}^{17}h_j a^j$. In descending degree the vector $(h_{17},\dots,h_0)$ is

$$\begin{pmatrix}
10369949184000000\\
-79502943744000000\\
273672012096000000\\
-559106235072000000\\
754609551358320000\\
-707442931414800000\\
469847663340300000\\
-220367388160080000\\
71443538332097400\\
-15753667204602600\\
2634445947232050\\
-486626431113900\\
91676122191935\\
-9473737791290\\
424034049270\\
4759882606\\
-44546479\\
1296
\end{pmatrix}.$$

Reduction of this vector gives

$$H(a)\equiv a^2+a+1 \pmod 5. \tag{11.2}$$

The values of the right side at $a=0,1,2,3,4$ are respectively $1,3,2,3,1$. Thus $H(a)\neq 0$ for every integer a.

Fix any integer a with $|a|\geq 2$, and put

$$N_a = 10!\,|a(a-1)H(a)|+1.$$

This integer is greater than one. Choose any prime divisor p of N_a. Since $N_a\equiv 1 \pmod{\ell}$ for every prime $\ell\leq 10$, we have $p>10$. Also p divides none of a, $a-1$, and $H(a)$, and $p\nmid D$ by the factorization in (11.1). Consequently

$$a,\quad a-1,\quad G^{(0)}_{4,a}(0)\quad \text{are all units in } \mathbb{Z}_{(p)}. \tag{11.3}$$

The choice is allowed to depend on a; no fixed prime is asserted to work for every base.

11.2. **P0 Frobenius blocks.** Let $Q=p^k$ and $m=4Q$. Write each index uniquely as $i=Qu+r$ with $0\leq r<Q$. The P0 row indices are $i=Q,\dots,5Q-1$, and its column indices are $j=0,\dots,8Q-1$. The following congruence is coefficientwise in $\mathbb{Z}_{(p)}[X]$:

$$Q^{12Q}G^{(0)}_{4Q,a}(X/Q^2)\equiv G^{(0)}_{4,a}(X)^Q \pmod p. \tag{11.4}$$

We give the entry argument, including the denominator and permutation checks. In the P0 determinant, all summation denominators, index differences, and diagonal denominators occurring at level $4Q$ are nonzero integers whose absolute values are less than $10Q$. As $p>10$, each has p-adic valuation at most k; the only other denominators are powers of $a-1$, which are p-units by (11.3). Therefore multiplying each C row by Q and each $B-TA$ row by Q^2 produces a matrix over $\mathbb{Z}_{(p)}[X]$. Its determinant is exactly the left side of (11.4).

Lucas' congruence and $a^Q\equiv a$ in $\mathbb{F}_p$ give

$$\binom{Q(u+2)+r}{2Q}a^{Qu+r}\equiv a^r\binom{u+2}{2}a^u \pmod p.$$

In the finite sums defining the P0 rows, multiplication by Q (or Q^2 on a second-order denominator) kills every summand whose denominator is not divisible by Q. The surviving summands, after division

of their indices by Q, are exactly the corresponding level-4 sums. For the correction terms this uses $(a/(a-1))^Q \equiv a/(a-1) \pmod p$. Thus, with u, v denoting the quotient indices and r, s the residues,

$$QC_{Qu+r,Qv+s} \equiv \mathbf{1}_{r=s} a^r C_{u,v} \pmod p, \tag{11.5}$$

$$Q^2 B_{Qu+r,Qv+s} \equiv \mathbf{1}_{r=s} a^r B_{u,v} \pmod p, \tag{11.6}$$

$$A_{Qu+r,Qv+s} \equiv \mathbf{1}_{r=s} a^r A_{u,v} \pmod p. \tag{11.7}$$

For the off-diagonal entries with unequal residues, the first two congruences also follow because $v_p(j-i) < k$, so $Q/(j-i)$ is zero modulo p after localization. When the residues agree, $j-i = Q(v-u)$; the reduced nonzero differences satisfy $|v-u| \le 7 < p$. The diagonal identities follow by the same denominator filtering in the displayed harmonic sums. This proves (11.7) for every entry, including the diagonals.

After grouping rows and columns by residues, the scaled matrix is block diagonal modulo p, with Q blocks

$$a^r \begin{bmatrix} C_4(a) \\ B_4(a) - XA_4(a) \end{bmatrix}, \qquad 0 \le r < Q.$$

There are eight rows in each block, so their scalar product is $a^{8\sum_{r=0}^{Q-1} r} = a^{4Q(Q-1)}$. The column regrouping sign is $(-1)^{\binom{Q}{2}\binom{8}{2}} = 1$, and interleaving the two four-row families has sign $(-1)^{4^2\binom{Q}{2}} = 1$. Since $a^{Q-1} \equiv 1 \pmod p$, the scalar product is also 1 modulo p. Taking determinants proves (11.4).

Lemma 11.1 (P0 eventual nonvanishing)**.** *For every fixed integer a with $|a| \ge 2$ and every fixed rational number ξ, there is $k_0 = k_0(a, \xi)$ such that*

$$G^{(0)}_{4p^k,a}(\xi) \ne 0 \qquad (k \ge k_0),$$

where p is any prime selected as above.

Proof. Set $W = Q^2\xi$ in the scaled determinant of (11.4). For all sufficiently large k, $v_p(W) = 2k + v_p(\xi) > 0$ (with the assertion immediate for $\xi = 0$). Its left-side scaled matrix is over $\mathbb{Z}_{(p)}$, while its determinant reduces modulo p to

$$G^{(0)}_{4,a}(0)^Q,$$

which is a unit by (11.3). The scaled determinant is therefore nonzero, and its scaling factor is nonzero; hence $G^{(0)}_{4Q,a}(\xi) \ne 0$. □

11.3. **Negative V seeds.** For $a = -r$, $2 \le r \le 7$, write $G^{(V)}_{4n,a}(T)$ for the column-V determinant

$$G^{(V)}_{4n,a}(T) = \det \begin{bmatrix} C(a) \\ B(a) - TA(a) \\ V \end{bmatrix},$$

with

$$I = [2n, 6n), \qquad J = [n, 10n), \qquad \Lambda = \{1, \dots, n\}.$$

$$C_{ij} = \frac{a^i - a^j}{j - i} \quad (i \ne j), \qquad C_{ii} = 0, \qquad A_{ij} = a^i \delta_{ij}.$$

$$t_j = \sum_{s=1}^{j} \frac{a^{j-s}}{s}, \qquad b_j = \sum_{s=1}^{j} \frac{a^{j-s}}{s^2},$$

$$B_{ij} = \begin{cases} (t_i - t_j)/(j-i), & i \ne j, \\ b_i, & i = j, \end{cases} \qquad V_{\ell j} = \frac{1}{j + \ell}.$$

Here I, J, Λ are integer intervals, with increasing order. At the seed level $n = 1$, the determinant is 9×9, with $I = \{2, 3, 4, 5\}$, $J = \{1, \dots, 9\}$, and $\Lambda = \{1\}$. The exact source formulas give the following reductions. Coefficient vectors are in ascending powers of T; thus each row specifies the entire seed polynomial modulo its indicated prime.

r	a	p_r	$G^{(V)}_{4,-r}(T) \bmod p_r$	$G^{(V)}_{4,-r}(0) \bmod p_r$
2	-2	11	$(4,2,7,3,9)$	4
3	-3	11	$(1,4,6,7,1)$	1
4	-4	23	$(5,8,16,21,20)$	5
5	-5	11	$(5,2,5,8,7)$	5
6	-6	17	$(5,12,4,7,4)$	5
7	-7	17	$(8,0,8,16,5)$	8

For example, the first vector means $4+2T+7T^2+3T^3+9T^4$ in $\mathbb{F}_{11}[T]$. All six constant entries are nonzero. Each p_r exceeds 7. The individual seed entries have denominators dividing 2520^2, and hence every coefficient denominator of the seed determinant is supported on the primes $2,3,5,7$ (cancellation may make it smaller, but need not make it divide 2520^2). Thus p_r is prime to every seed denominator, to a, and to $a-1$. The seed constant is consequently a p_r-adic unit and all seed coefficients are defined in $\mathbb{Z}_{(p_r)}$.

11.4. **Signed Frobenius blocks.** Fix r and abbreviate $p=p_r$. Put $Q=p^k$ and $m=4Q$. At this level the three shape parameters of the V source are all set equal to Q; they are distinct from the exponent k in $Q=p^k$. Then the row and column intervals are

$$I=[2Q,6Q), \qquad J=[Q,10Q), \qquad \Lambda=\{1,\dots,Q\}.$$

Define

$$P_{r,k}(X)=Q^{3m+Q}G^{(V)}_{m,-r}(X/Q^2)=Q^{13Q}G^{(V)}_{4Q,-r}(X/Q^2). \tag{11.8}$$

The determinant on the left is formed by scaling its C and V rows by Q and its $B-TA$ rows by Q^2; the middle rows become Q^2B-XA. Every nonzero denominator in a harmonic sum, index difference, or V-entry is an integer whose absolute value is less than $11Q$. Since $p\ge 11$, its p-adic valuation is at most k. Hence the scaled matrix has entries in $\mathbb{Z}_{(p)}[X]$, and so does $P_{r,k}(X)$.

Write $i=Qi_0+u$ and $j=Qj_0+v$, with residues in $\{0,\dots,Q-1\}$. As in the P0 calculation, only summands with denominator divisible by Q survive after row scaling. In particular,

$$Qt_{Qj_0+u}\equiv a^u t_{j_0}, \qquad Q^2 b_{Qj_0+u}\equiv a^u b_{j_0} \pmod p, \tag{11.9}$$

because $a^Q\equiv a \pmod p$. For C and the off-diagonal B entries, unequal residues vanish: if $u\neq v$, then $v_p(j-i)<k$. With equal residues and unequal quotient indices, division by $j-i=Q(j_0-i_0)$ gives the seed C and B entries with row factor a^u; here $|j_0-i_0|\le 7<p$. The diagonal B entry follows from (11.9). The diagonal A entries satisfy $A_{Qi_0+u,Qi_0+u}\equiv a^u A_{i_0i_0} \pmod p$.

For the V rows, fix a column residue u. There is exactly one $\ell\in\{1,\dots,Q\}$ with $\ell+u\equiv 0 \pmod Q$, namely $\ell=Q-u$ when $u>0$ and $\ell=Q$ when $u=0$. All other scaled entries vanish modulo p, and for this row

$$\frac{Q}{j+\ell}=\frac{1}{j_0+1},$$

the seed V-entry. Thus residue regrouping gives Q diagonal blocks

$$\begin{bmatrix} a^uC_4 \\ a^u(B_4-XA_4) \\ V_4 \end{bmatrix}, \qquad 0\le u<Q.$$

Each block has eight rows carrying a^u and one V row. Row and column regrouping contributes a sign $\varepsilon_{r,k}\in\{1,-1\}$, which is a p-adic unit and need not be evaluated. Therefore, coefficientwise,

$$P_{r,k}(X)\equiv \varepsilon_{r,k}a^{4Q(Q-1)}G^{(V)}_{4,-r}(X)^Q \pmod p. \tag{11.10}$$

The exponent is $8\sum_{u=0}^{Q-1}u=4Q(Q-1)$. The table above shows $p\nmid G^{(V)}_{4,-r}(0)$, and $p\nmid a$, so the constant coefficient on the right of (11.10) is nonzero modulo p.

Lemma 11.2 (negative-base eventual nonvanishing)**.** *For each $r\in\{2,3,4,5,6,7\}$ and every fixed rational number ξ, there is $k_0=k_0(r,\xi)$ such that*

$$G^{(V)}_{4p_r^k,-r}(\xi)\neq 0 \qquad (k\ge k_0).$$

Proof. Use the convention $v_{p_r}(0) = +\infty$. For all sufficiently large k, $v_{p_r}(Q^2\xi) = 2k + v_{p_r}(\xi) > 0$ (and this is immediate when $\xi = 0$). Since $P_{r,k} \in \mathbb{Z}_{(p_r)}[X]$, its value at $X = Q^2\xi$ is congruent modulo p_r to its constant coefficient. By (11.10), that coefficient is a unit. Thus $P_{r,k}(Q^2\xi) \neq 0$, and the nonzero scaling in (11.8) yields the claim. □

11.5. **Use in the determinant criterion.** For a fixed rational target ξ, the preceding lemmas give nonvanishing at every sufficiently high index of the selected prime-power sequence, not merely at one computed level. Once $k \geq k_0$, the determinants at indices k and $k+1$ are both nonzero. This assertion concerns the selected prime-power subsequence only; no nonvanishing assertion is made for every intervening integer degree m.

For each fixed integer base, the irrationality criterion of Section 3 must now be applied to one family at a time: P0 with its base-dependent prime, or V with the corresponding p_r. The integerizer, analytic estimate, and nonzero determinant must all belong to that same source. The prime-power sequences proved here supply exactly the nonzero term required by the criterion; they do not by themselves imply a common analytic or arithmetic rate across the two families.

12. Hermite–Padé tails

This section proves the two tails from the displayed construction, without using an irrationality theorem as input. The base is fixed throughout each limit; constants in contour and remainder estimates may depend on that base.

Theorem 12.1. *For every fixed integer $a \geq 12$ or $a \leq -8$, the determinant construction below proves that $\mathrm{Li}_2(1/a)$ is irrational.*

12.1. **Residues, tails, and exact integerization.** For $n \geq 0$ put

$$\Phi_n(w) = \frac{\prod_{r=n+1}^{3n+1}(w-r)}{\prod_{j=0}^{n}(w-j)^2} = \sum_{j=0}^{n}\left(\frac{u_{n,j}}{(w-j)^2} + \frac{v_{n,j}}{w-j}\right). \tag{12.1}$$

Writing $H_k = \sum_{r=1}^{k} 1/r$ and $H_0 = 0$, direct residue calculation gives

$$u_{n,j} = -\frac{(3n+1-j)!}{(j!)^2((n-j)!)^3}, \qquad v_{n,j} = u_{n,j}\big(3H_{n-j} - 2H_j - H_{3n+1-j}\big). \tag{12.2}$$

Set

$$B_n(z) = \sum_{j=0}^{n} v_{n,j} z^j, \qquad C_n(z) = \sum_{j=0}^{n} u_{n,j} z^j,$$

and

$$A_n(z) = -\sum_{k=1}^{n} z^k \sum_{j=0}^{k-1}\left(\frac{v_{n,j}}{k-j} + \frac{u_{n,j}}{(k-j)^2}\right).$$

The coefficient of z^k in $B_n(-\log(1-z)) + C_n \mathrm{Li}_2(z)$ is the partial-fraction sum $\sum_{j<k}(v_{n,j}/(k-j) + u_{n,j}/(k-j)^2)$. For $1 \leq k \leq n$ it is canceled by A_n; for $n < k \leq 3n+1$ it is zero because $\Phi_n(k) = 0$; for $k \geq 3n+2$ it equals $\Phi_n(k)$. Therefore

$$R_n(z) := A_n(z) + B_n(z)(-\log(1-z)) + C_n(z)\,\mathrm{Li}_2(z) = \sum_{k \geq 3n+2} \Phi_n(k) z^k. \tag{12.3}$$

The adjacent Casoratian

$$E_n(z) := B_{n+1}(z)R_n(z) - B_n(z)R_{n+1}(z) = W_n(z) + V_n(z)\,\mathrm{Li}_2(z) \tag{12.4}$$

has

$$W_n = B_{n+1}A_n - B_nA_{n+1}, \qquad V_n = B_{n+1}C_n - B_nC_{n+1}.$$

The logarithm cancels identically. Both W_n, V_n have degree at most $N = 2n+1$.

Here is an all-degree proof of the coefficient formula needed for integerization. Let $b_n = \binom{2n+1}{n}$ and

$$J_n(t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k}\binom{n+k}{k}^2 t^k.$$

For $\Re(s) > -1$ the integral is absolutely convergent, and direct termwise integration gives

$$\int_0^1 t^s J_n(t)\,dt = \frac{(2n+1)!}{(n!)^2}\frac{[s(s-1)\cdots(s-n+1)]^2}{(s+1)(s+2)\cdots(s+2n+2)}. \tag{12.5}$$

The right side then supplies the meromorphic continuation in s; no continuation is used before the stated half-plane of convergence. Write $V_n(z) = \sum_{i=0}^N v_i z^i$ and $\Pi_n(t) = t^N V_n(1/t) = \sum_{j=0}^N \pi_j t^j$, so $\pi_j = v_{N-j}$. The polynomial part of E_n has degree at most N. For $k = N+\ell$, $1 \le \ell \le n$, its coefficient is therefore

$$0 = \sum_{i=0}^N \frac{v_i}{(k-i)^2} = \sum_{j=0}^N \frac{\pi_j}{(\ell+j)^2},$$

which is $-\int_0^1 t^{\ell-1}\Pi_n(t)\log t\,dt = 0$. The second elimination, $C_{n+1}R_n - C_nR_{n+1}$, cancels Li_2; its non-polynomial part is $V_n(z)\log(1-z)$. The same coefficient range gives

$$0 = \sum_{i=0}^N \frac{v_i}{k-i} = \sum_{j=0}^N \frac{\pi_j}{\ell+j} = \int_0^1 t^{\ell-1}\Pi_n(t)\,dt.$$

Thus the two exact moment families are

$$\int_0^1 t^j \Pi_n(t)\,dt = 0, \qquad \int_0^1 t^j \Pi_n(t)\log t\,dt = 0 \quad (0 \le j < n). \tag{12.6}$$

The Mellin transform of any polynomial of degree at most N has denominator $\prod_{r=1}^{N+1}(s+r)$ and numerator degree at most N. The first family in (12.6) makes this numerator vanish at $s = 0, \dots, n-1$; the second makes each zero double. Hence the numerator is divisible by $[s(s-1)\cdots(s-n+1)]^2$, leaving a factor of degree at most one. The solution space of (12.6) consequently has dimension at most two.

Equation (12.5) shows that J_n is one solution. It also shows $J_n(1) = 0$, since the right side is $O(s^{-2})$ as $s \to +\infty$. Integration by parts then shows that $tJ_n'(t)$ is a second solution: the boundary terms vanish at both endpoints, and the two moment families for tJ_n' reduce to those for J_n. The two solutions are independent by their top two coefficients: if $a_N = [t^N]J_n$ and $a_{N-1} = [t^{N-1}]J_n$, the determinant formed from the top two coefficients of J_n and tJ_n' is $a_N(N-1)a_{N-1} - Na_Na_{N-1} = -a_Na_{N-1} \neq 0$. Thus every solution is their linear combination. Direct substitution of the residues (12.2) gives the constant and linear coefficients of Π_n; matching those two coefficients in this two-dimensional space yields

$$V_n(z) = b_n^2 z^N P_n(1/z), \tag{12.7}$$

where

$$P_n(t) = -\sum_{k=0}^N (-1)^k \binom{N}{k}\binom{n+k}{k}\left((12n+17)\binom{n+k}{k} + (21n+29)\binom{n+k}{k-1}\right)t^k. \tag{12.8}$$

The convention is $\binom{n+k}{k-1} = 0$ at $k = 0$. In particular, the second binomial is $\binom{n+k}{k-1}$, not $\binom{n+k-1}{k-1}$.

The first two coefficients follow from the endpoint residues. Put $\eta_m = -2H_m - H_{2m+1}$, $\rho_m = 2m^2(m+1)$, and $\kappa_m = 3 + 2/m - 1/(2m+2)$. Their ratios are

$$\frac{v_{m,m}}{u_{m,m}} = \eta_m, \qquad \frac{u_{m,m-1}}{u_{m,m}} = \rho_m, \qquad \frac{v_{m,m-1}}{u_{m,m-1}} = \eta_m + \kappa_m.$$

For $n \ge 1$, expanding $V_n = B_{n+1}C_n - B_nC_{n+1}$ at its top two degrees yields, with $\pi_j = [t^j]\Pi_n$,

$$\begin{aligned}\pi_0 &= u_{n,n}u_{n+1,n+1}(\eta_{n+1} - \eta_n),\\ \pi_1 &= u_{n,n}u_{n+1,n+1}\big((\rho_n + \rho_{n+1})(\eta_{n+1} - \eta_n) - \rho_n\kappa_n + \rho_{n+1}\kappa_{n+1}\big).\end{aligned}$$

Here $u_{n,n} = -(n+1)b_n$, $u_{n+1,n+1} = -2(2n+3)b_n$, and $\eta_{n+1} - \eta_n = -5/(2(n+1)) - 1/(2n+3)$. Substitution yields $\pi_0 = -b_n^2(12n+17)$ and

$$\pi_1 = b_n^2(2n+1)(n+1)\big((n+1)(12n+17) + (21n+29)\big).$$

For $n = 0$, the residues give the same formulas directly: $\pi_0 = -17$ and $\pi_1 = 46$. For the Casoratian calculation below, put $h_m = 3H_m - H_{3m+1}$ and

$$\delta_m = h_{m+1} - h_m = \frac{2(27m^2 + 54m + 23)}{3(m+1)(3m+2)(3m+4)},$$

The values of π_0, π_1 are the first two coefficients of $-(12n+17)J_n - (21n+29)tJ_n'/(n+1)$. The identity $\frac{k}{n+1}\binom{n+k}{k} = \binom{n+k}{k-1}$ gives (12.8). The dimension bound proves (12.7) for every n, with no finite-range restriction.

For reference, the boundary residues used in this matching can be checked directly from (12.2). For $m \geq 1$,

$$u_{m,m-1} = -\frac{(2m+2)!}{((m-1)!)^2}, \qquad u_{m,m} = -\frac{(2m+1)!}{(m!)^2}, \tag{12.9}$$

$$\frac{u_{m,m-1}}{u_{m,m}} = (2m+2)m^2 = 2m^2(m+1), \tag{12.10}$$

$$\frac{v_{m,m-1}}{u_{m,m-1}} - \frac{v_{m,m}}{u_{m,m}} = 3 + \frac{2}{m} - \frac{1}{2m+2}, \tag{12.11}$$

$$\frac{u_{n,n}u_{n+1,n+1}}{b_n^2} = (2n+2)(2n+3). \tag{12.12}$$

These identities also cover the boundary-residue normalization in the three-form extraction below; the case $n = 0$ is read directly from (12.2).

Let $d_N = \operatorname{lcm}(1, \ldots, N)$ and $Q_n(z) = V_n(z)/b_n^2 \in \mathbb{Z}[z]$. Since $E_n = W_n + V_n \operatorname{Li}_2$ vanishes through degree N, its polynomial part satisfies

$$\frac{W_n(z)}{b_n^2} = -[Q_n(z)\operatorname{Li}_2(z)]_{\leq N}.$$

Each coefficient on the right has denominators among r^2 with $1 \leq r \leq N$, and $r \mid d_N$. Therefore

$$\frac{d_N^2}{b_n^2}W_n, \quad \frac{d_N^2}{b_n^2}V_n \in \mathbb{Z}[z]. \tag{12.13}$$

For every fixed integer a with $|a| \geq 2$, put

$$L_n(a) = a^N \frac{d_N^2}{b_n^2}E_n(1/a) = X_n(a) + Y_n(a)\operatorname{Li}_2(1/a), \qquad X_n(a), Y_n(a) \in \mathbb{Z}. \tag{12.14}$$

The factor a^N clears evaluation denominators because both polynomials in (12.13) have degree at most N.

12.2. **Remainder bound for both signs.** For $-1 < z < 1$, beta integration of (12.3) gives the exact identity

$$R_n(z) = \frac{(2n+1)!}{(n!)^2}z^{3n+2}\int_0^1\int_0^1 \frac{tu}{(1-ztu)^2}\left(\frac{t^2u^2(1-t)(1-u)}{(1-ztu)^2}\right)^n dt\,du. \tag{12.15}$$

To maximize the bracket, fix $p = tu$. Since $t + u \geq 2\sqrt{p}$, $(1-t)(1-u) = 1 - (t+u) + p \leq (1-\sqrt{p})^2$, with equality only at $t = u = \sqrt{p}$. Write $s = \sqrt{p}$. The bracket is at most

$$g_z(s) = \frac{s^4(1-s)^2}{(1-zs^2)^2}, \qquad 0 \leq s \leq 1.$$

For $|z| < 1$, its unique interior maximum satisfies $zs^3 - 3s + 2 = 0$: differentiating $\log g_z$ gives $2 - 3s + zs^3 = 0$, and the left side is strictly decreasing on $[0,1]$. At that root, $1 - zs^2 = 2(1-s)/s$, so $g_z(s) = s^6/4$. The prefactor $tu/(1-ztu)^2$ in (12.15) is bounded for fixed z with $|z| < 1$. Stirling's formula [13] and the unit area of the square therefore give

$$\limsup_{n\to\infty} |R_n(z)|^{1/n} \leq |z|^3 s(z)^6. \tag{12.16}$$

No cancellation estimate is used.

12.3. **Positive coefficient bound.** For $z = 1/q$, $q > 1$, all $u_{n,j}$ have the same (negative) sign. Put $x = j/n$ and

$$f_q(x) = (3-x)\log(3-x) - 2x\log x - 3(1-x)\log(1-x) - x\log q, \quad 0 \le x \le 1,$$

where $0\log 0 = 0$. Uniform Stirling estimates [13] in (12.2) give

$$\log\bigl(|u_{n,j}|q^{-j}\bigr) = nf_q(j/n) + O(\log(n+1))$$

uniformly for $0 \le j \le n$. Indeed, the coefficients of $n\log n$ and n cancel, and all factorial contributions are $O(\log(n+1))$, including at endpoints. Now

$$f_q'(x) = \log\frac{(1-x)^3}{qx^2(3-x)}, \qquad f_q''(x) = \frac{1}{3-x} - \frac{2}{x} - \frac{3}{1-x} < 0.$$

There is a unique maximizer $t(q) \in (0,1)$, determined by $(1-t)^3 = qt^2(3-t)$. Substitution of this equation gives

$$\exp(f_q(t)) = \left(\frac{3-t}{1-t}\right)^3.$$

Since the sum has only $n+1$ terms,

$$\limsup_{n\to\infty} |C_n(1/q)|^{1/n} \le \left(\frac{3-t(q)}{1-t(q)}\right)^3.$$

Also $|3H_{n-j} - 2H_j - H_{3n+1-j}| \le 6(1+\log(3n+1))$, so the same bound holds for $B_n(1/q)$. This upper-bound form of the discrete Laplace principle is all that is needed; no unproved ratio limit is used.

12.4. **Negative coefficient bound: contour and endpoint estimate.** Fix $q > 1$, put $z = -1/q$, and define the two logarithmic branches

$$F_n^{\pm}(z) = B_n(z) + C_n(z)(-\log q \pm i\pi).$$

Residues of $\Phi_n(w)e^{(-\log q \pm i\pi)w}$ give

$$F_n^{\pm}(z) = \frac{1}{2\pi i}\oint_{\Gamma_n^{\pm}} \Phi_n(w)e^{(-\log q\pm i\pi)w}\,dw, \tag{12.17}$$

where the rectangle encloses the poles $0, 1, \dots, n$. Set $\beta_q > 0$ equal to the absolute imaginary part of the lower nonreal root of

$$p_q(w) = (q+1)w^3 - 3(q+1)w^2 + 3w - 1 = 0. \tag{12.18}$$

For all sufficiently large n, let Γ_n^+ be the positively oriented rectangle with vertical sides $\Re w = -1/2, n+1/2$ and horizontal sides $\Im w = \pm n\beta_q$. It encloses precisely the poles $0, 1, \dots, n$. Its distance from each enclosed pole is at least $\min\{1/2, n\beta_q\}$, hence at least $1/2$ once $n\beta_q \ge 1/2$. All numerator zeros lie outside the rectangle at distance at least $1/2$. The rate proof uses only these sufficiently large indices. For F_n^- use the conjugate exponential branch on the same positively oriented rectangle.

We give the uniform sum-to-integral estimate needed on these contours. If $\xi = w/n$ lies on the scaled rectangle, then

$$\log|\Phi_n(n\xi)| = n\left(\int_1^3 \log|\xi - r|\,dr - 2\int_0^1 \log|\xi - r|\,dr\right) + O_q(\log(n+1)), \tag{12.19}$$

uniformly along all four sides. On a horizontal side $|\Im\xi| = \beta_q$, the functions $r \mapsto \log|\xi - r|$ have uniformly bounded derivative, so the ordinary left/right Riemann-sum error is $O_q(1)$. On the left vertical side $\Re\xi = -1/(2n)$, the real part lies outside both integration intervals by at least $1/(2n)$. On the right side $\Re\xi = 1 + 1/(2n)$ it lies outside $[0,1]$ by the same amount. For $[1,3]$ there, the only possible near-singularity lies in the first mesh cell: its real part is exactly halfway between the endpoint and the first mesh point. If $|\Im\xi| \le 1/n$, rescaling this one cell by $r = 1 + u/n$ shows that both its mesh contribution and n times its integral are $O(\log(n+1))$; the integral of $\log\sqrt{u^2+v^2}$ over a fixed interval is bounded uniformly for $0 \le v \le 1$. If $|\Im\xi| > 1/n$, or on the remaining cells, the total variation is bounded by $C_q + \int_{1/n}^{O_q(1)} dr/r = O_q(\log(n+1))$. Endpoint shifts of one mesh point in the numerator product contribute at most another $O_q(\log(n+1))$. This proves (12.19); in particular the logarithmic singularity is handled explicitly rather than hidden in a smooth-function Euler–Maclaurin assertion.

For completeness, the estimate in (12.19) is a bounded-variation Riemann-sum estimate, including the endpoint cell. Put $f_\xi(r) = \log|\xi - r|$. The numerator sum is the right Riemann sum on $[1,3]$, with the extra point $f_\xi(3+1/n)$, and the denominator sum is the right Riemann sum on $[0,1]$, with $f_\xi(0)$. On a horizontal edge the total variation is $O_q(1)$. On a vertical edge the only possible singular cell is the first half-mesh cell; after $r = 1 + u/n$ (or $r = u/n$) its mesh term and its n-scaled integral are $O(\log(n+1))$. The remaining variation is $O_q(1) + \int_{1/n}^{O_q(1)} dr/r = O_q(\log(n+1))$. Thus the estimate is uniform on all four edges and does not invoke a smooth Euler–Maclaurin formula across a logarithmic endpoint.

Combining (12.19) with the exact modulus of the exponential factor gives

$$\log|\Phi_n(n\xi)e^{(-\log q + i\pi)n\xi}| = nH_q(\xi) + O_q(\log(n+1)),$$

where

$$H_q(\xi) = \int_1^3 \log|\xi - r|\,dr - 2\int_0^1 \log|\xi - r|\,dr - \log q\,\Re\xi - \pi\Im\xi. \tag{12.20}$$

The discriminant of (12.18) is $-108q^2(q+1) < 0$, and $p_q(2) = 1 - 4q < 0 < p_q(3) = 8$. Thus the real root is in $(2,3)$; the conjugate roots have real part $u = (3-R)/2 \in (0,1/2)$. Write the lower root as $\xi_0 = u - i\beta_q$. Differentiating the analytic logarithmic phase underlying (12.20) shows $\partial_x H_q(\xi_0) = 0$ by the saddle equation $z(\xi-1)^3 = \xi^2(\xi-3)$. Along the lower side,

$$\frac{d^2}{dx^2}H_q(x - i\beta_q) = -6\Re\frac{1}{(x - i\beta_q)(x - 1 - i\beta_q)(x - 3 - i\beta_q)} < 0 \quad (0 \le x \le 1),$$

because the real part of the denominator is $x(x-1)(x-3) + \beta_q^2(4-3x) > 0$. Hence the unique lower-side maximum is ξ_0. On the vertical sides, parameterized downward,

$$\frac{d}{dy}H_q(-iy) = 3\arctan y - \arctan(y/3) > 0,$$

$$\frac{d}{dy}H_q(1 - iy) = \frac{\pi}{2} + 2\arctan y - \arctan(y/2) > 0.$$

On the lower halves of the limiting vertical sides, parameterized as $\xi = -iy$ and $\xi = 1 - iy$ with $0 < y \le \beta_q$, the displayed derivatives are strictly positive. On the upper halves use the exact reflection identity $H_q(x + iy) = H_q(x - iy) - 2\pi y$ for $y > 0$. Hence neither upper vertical half can exceed its reflected lower half, and the maxima on both lower vertical halves occur at their lower endpoints. The strictly concave lower horizontal phase has its unique maximum at the interior saddle ξ_0, strictly above both endpoints.

The logarithmic potentials defining H_q are continuous when ξ approaches the real integration intervals because their logarithmic singularities are integrable. They are therefore uniformly continuous on a compact neighborhood of the limiting rectangle. The maxima over the boundaries of the scaled finite rectangles converge to the maximum over the limiting boundary. The original rectangles have perimeter $O_q(n)$, while (12.19) contributes only a polynomial factor in n to the modulus bound. Taking n-th roots now gives the claimed rate bound.

$$\limsup_{n\to\infty} |F_n^{\pm}(-1/q)|^{1/n} \le e^{H_q(\xi_0)}. \tag{12.21}$$

To evaluate this number, choose logarithms continuously from the lower contour. An antiderivative gives the analytic phase

$$\Psi_q(\xi) = 3(\xi-1)\operatorname{Log}(\xi-1) - (\xi-3)\operatorname{Log}(\xi-3) - 2\xi\operatorname{Log}\xi + (-\log q + i\pi)\xi.$$

At the lower saddle choose the principal logarithms of $\xi_0, \xi_0 - 1, \xi_0 - 3$, whose arguments lie in $(-\pi, 0)$ (with the negative-real limiting value approached from below). The algebraic saddle equation implies that $\Psi_q'(\xi_0)$ is an integer multiple of $2\pi i$. As $q \to \infty$, the lower root satisfies $\xi_0 = -i/\sqrt{3q} + O(q^{-1})$; the three arguments tend respectively to $-\pi/2, -\pi, -\pi$. The corresponding imaginary-part combination is zero, so continuity in $q > 1$ forces this integer multiple to vanish identically. Substituting the saddle equation into Ψ_q gives

$$e^{H_q(\xi_0)} = \left|\frac{\xi_0 - 3}{\xi_0 - 1}\right|^3.$$

For $y = (\xi - 3)/(\xi - 1)$ the saddle equation becomes $y(y-3)^2 = 4z$. Its real root is $y_0 = -s_-^2/q$, where $s_-(q) \in (0, 2/3)$ is the unique solution of

$$-\frac{s_-^3}{q} - 3s_- + 2 = 0. \tag{12.22}$$

The product of the three roots of $y(y-3)^2 = 4z$ is $4z = -4/q$. Consequently the squared modulus of either nonreal root is $4/s_-^2$, and

$$e^{H_q(\xi_0)} = \frac{8}{s_-(q)^3}. \tag{12.23}$$

Finally

$$C_n(-1/q) = \frac{F_n^+ - F_n^-}{2\pi i}, \qquad B_n(-1/q) = \frac{F_n^+ + F_n^-}{2} + \log q\, C_n(-1/q).$$

Equations (12.21)–(12.23) thus give the same upper exponential rate $8/s_-^3$ for $B_n(-1/q)$.

12.5. **Integer-form rates and exact cutoffs.** The prime number theorem gives $\log d_N = N + o(N)$, while Stirling [13] gives $n^{-1} \log b_n \to 2 \log 2$. With $N = 2n + 1$,

$$\lim_{n\to\infty} \left(\frac{d_N^2}{b_n^2} \right)^{1/n} = \frac{e^4}{16}. \tag{12.24}$$

No finite global inequality for d_N is asserted. Combining (12.16), the coefficient bounds, and (12.4) yields

$$\limsup |L_n(q)|^{1/n} \le r_+(q) := \frac{e^4}{16q} \left(\frac{3 - t(q)}{1 - t(q)} \right)^3 s_+(q)^6, \qquad a = q > 1, \tag{12.25}$$

$$\limsup |L_n(-q)|^{1/n} \le r_-(q) := \frac{e^4 s_-(q)^3}{2q}, \qquad a = -q < -1, \tag{12.26}$$

where s_+ is the unique root in $(0,1)$ of $s_+^3/q - 3s_+ + 2 = 0$. The factors are as follows: q^2 from $|a|^N$, $e^4/16$ from (12.24), $q^{-3}s^6$ from the remainder, and the relevant coefficient growth rate. Thus no estimate for an uneliminated logarithmic form is being used.

For the positive rate set $z = 1/q$. Implicit differentiation of the two root equations gives

$$\frac{d \log r_+}{d \log z} = 1 + t + \frac{2z s_+^2}{1 - z s_+^2} > 0,$$

so $r_+(q)$ decreases with q. On the negative side, $q = s_-^3/(2 - 3s_-)$ is increasing in s_- and $r_-(q) = e^4(2 - 3s_-)/2$ is decreasing. It remains to check four rational endpoints. The Taylor sum for e^4 through degree 16 and a geometric tail bound give

$$\frac{6972316451}{127702575} < e^4 < \frac{377139364267}{6907548375}.$$

Opposite signs of the defining cubics at the following rational endpoints isolate the roots. The derivatives are $-3(1-t)^2 - q(6t - 3t^2) < 0$ for $t(q)$, $3s^2/q - 3 < 0$ for $s_+(q)$, and $-3s^2/q - 3 < 0$ for $s_-(q)$:

$t(11)$	$(14179/100000, 14180/100000)$
$s_+(11)$	$(67602/100000, 67604/100000)$
$t(12)$	$(13681/100000, 13683/100000)$
$s_+(12)$	$(67521/100000, 67523/100000)$
$s_-(7)$	$(65338/100000, 65339/100000)$
$s_-(8)$	$(65496/100000, 65497/100000)$

Exact rational interval substitution gives

$r_+(11)$	$(10937/10000, 10941/10000)$
$r_+(12)$	$(9834/10000, 9837/10000)$
$r_-(7)$	$(10877/10000, 10879/10000)$
$r_-(8)$	$(9587/10000, 9589/10000)$

and hence $r_+(12) < 1$ and $r_-(8) < 1$. Monotonicity proves $r_+(q) < 1$ for every integer $q \geq 12$ and $r_-(q) < 1$ for every integer $q \geq 8$.

12.6. **All-degree nonvanishing and conclusion.** It remains to show that the constructed integer forms do not vanish eventually. Let $U_m = (3m+1)!/(m!)^3$, $K_n = ((2n+1)!)^3/((3n+2)!)^2$, and $h_m = 3H_m - H_{3m+1}$. Direct extraction from (12.3) gives

$$\Phi_n(3n+2) = K_n, \qquad \frac{\Phi_n(3n+3)}{K_n} = \frac{8(n+1)}{9}.$$

For $m \geq 0$, put $M_m = B_m C_{m+1} - C_m B_{m+1}$; in particular $M_0 = B_0 C_1 - C_0 B_1$ is defined by the displayed residue polynomials. For $m \geq 1$, put $r_m = m^3/(3m+1)$ and $c_m = 2 + 3/m - 1/(3m+1)$. The lower endpoint residues give

$$u_{m,0} = -U_m, \quad u_{m,1} = r_m u_{m,0}, \quad v_{m,0} = h_m u_{m,0}, \quad v_{m,1} = r_m(h_m - c_m)u_{m,0}.$$

Expanding $B_m C_{m+1} - C_m B_{m+1}$ at degrees zero and one gives

$$M_m(0) = -U_m U_{m+1} \delta_m,$$

$$[z]M_m = U_m U_{m+1}\{-\delta_m(r_m + r_{m+1}) + r_{m+1}c_{m+1} - r_m c_m\},$$

The three-form determinant

$$\Delta_n(z) = \det \begin{bmatrix} A_n & B_n & C_n \\ A_{n+1} & B_{n+1} & C_{n+1} \\ A_{n+2} & B_{n+2} & C_{n+2} \end{bmatrix}$$

has degree at most $3n+3$. Replacing its first column by the corresponding R-column and using (12.3) shows its order is at least $3n+2$. Expanding along that column gives

$$\Delta_n = R_n M_{n+1} - R_{n+1}(B_n C_{n+2} - C_n B_{n+2}) + R_{n+2} M_n.$$

The last two terms start at degree $3n+5$. Also

$$R_n(z) = K_n z^{3n+2}\left(1 + \frac{8(n+1)}{9}z + O(z^2)\right).$$

It follows that

$$[z^{3n+2}]\Delta_n = K_n M_{n+1}(0), \qquad \frac{[z^{3n+3}]\Delta_n}{[z^{3n+2}]\Delta_n} = \frac{[z]M_{n+1}}{M_{n+1}(0)} + \frac{8(n+1)}{9}.$$

Substitution of the displayed coefficients gives the ratio $-\theta_n$, with

$$\theta_n = \frac{-\delta_{n+1}(r_{n+1} + r_{n+2}) + r_{n+2}c_{n+2} - r_{n+1}c_{n+1}}{\delta_{n+1}} - \frac{8(n+1)}{9}.$$

Substituting r_m, c_m, δ_m and clearing denominators gives

$$\theta_n = \frac{1728n^4 + 11520n^3 + 28140n^2 + 29965n + 11770}{18(3n+4)^2(27n^2 + 108n + 104)}. \tag{12.27}$$

Therefore

$$\Delta_n(z) = -K_n U_{n+1} U_{n+2} \delta_{n+1} z^{3n+2}(1 - \theta_n z), \tag{12.28}$$

The numerator is positive for $n \geq 0$; denominator minus numerator is

$$2646n^4 + 17640n^3 + 43140n^2 + 46067n + 18182 > 0.$$

Also $\delta_{n+1} > 0$. Therefore $\Delta_n(z) \neq 0$ for every real $0 < |z| < 1$.

For such z, the beta integral (12.15) gives $R_m(z) \neq 0$. If both $E_n(z)$ and $E_{n+1}(z)$ vanished, the two adjacent minors of the columns $A + \mathrm{Li}_2(z)C$ and B would vanish. Since $R_n(z)$ and $R_{n+1}(z)$ are nonzero, this makes the three columns in $\Delta_n(z)$ linearly dependent (the conclusion also holds when one of the corresponding B entries is zero, by using the other adjacent equation). Hence $\Delta_n(z) = 0$, contradicting (12.28). Thus no two consecutive $E_n(z)$ vanish.

Suppose now that $\mathrm{Li}_2(1/a) = r/s$ were rational, in lowest terms. By (12.14), $sL_n(a)$ is an integer for every n. The strict rate bounds imply $L_n(a) \to 0$, so $sL_n(a) = 0$ for all sufficiently large n. This contradicts the fact that no two consecutive $E_n(1/a)$, and hence no two consecutive $L_n(a)$, vanish. This proves Theorem 12.1.

13. Assembly of the proof

The positive construction supplies the criterion for $2 \le n \le 11$, the signed construction supplies it for $-7 \le n \le -2$, and the Hermite–Padé construction supplies it for $n \ge 12$ and $n \le -8$. Together with the two classical evaluations at $n = \pm 1$, these ranges prove Theorem 2.1.

Use of AI tools

The author devised the method. Codex (OpenAI, GPT-5.6 Sol) assisted with mathematical exploration and implementation. Claude Code (Anthropic, Claude Opus 5.5) assisted with review of the implementation, parameters, and printed data; that review is not a proof of the mathematical claims. The directed numerical bounds are computer-assisted: they were obtained with outward-rounded interval arithmetic as specified in Appendix C. The author is responsible for the mathematical statements and their interpretation.

Declarations

Competing interests. The author has no competing interests.

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Data availability. The ancillary files contain the printed-data replay and its logs. The source-range script checks text and table structure only; it does not certify the mathematical estimates.

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Appendix A. Positive endpoint data

The tables use ordinary decimal integers. Divide each mass list by the displayed denominator; the contour list gives the first half of the conjugate-symmetric Fejér weights, with zero filling at the remaining nodes.

Table A.1: $q = 2$ endpoint (27 independent masses).

item	exact value
denominator	5000000000000
pinned edges	$8, 12, 16, 20, 24, 28, 32, 36, 40$ divided by 32
pinned masses	16526887295, 294754833065, 416674605400, 489006355115, 568607559425, 625000000000, 625000000000, 625000000000
frequency edges	$1/2, 9/10, 13/10, 17/10, 21/10, 5/2, 29/10, 33/10, 37/10, 41/10, 9/2$

frequency masses	1839928019547, 925598240328, 604396185629, 446793550571, 346483328360, 273707087420, 215955707916, 166118856466, 118641146544, 62377877219
contour pair masses	103692576010, 101009799410, 96050653090, 89324463765, 81159146555, 71580231315, 60226824760, 46279839230, 20391345715

Table A.2: $q = 11$ endpoint (20 independent masses).

item	exact value
denominator	3000000000000
pinned edges	8, 12, 16, 20, 24, 28, 32, 36, 40 divided by 32
pinned masses	28, 0, 107702656325, 206939134529, 260854191850, 321658806124, 374999999996, 374999999996
frequency edges	1/2, 3/2, 5/2, 7/2, 9/2
frequency masses	2334546252656, 652078675810, 13375071534, 0
contour pair masses	105521636098, 103136532801, 98660112985, 92490547592, 84344850292, 75717735060, 60802883424, 55748307324

Appendix B. Exact signed reference data

Every entry below is an ordinary signed base-ten integer. For real chains, divide each length integer by 10^8 and each mass increment by 10^8. A single listed correction (j, r) adds $r/(5 \cdot 10^{13})$ to length j; $j = 0$ means no correction. Start each chain at its printed endpoint (units $1/(5 \cdot 10^{13})$), then accumulate lengths from left to right. Start each mass at zero and accumulate Δm_i. For Poisson rows divide all increments by 10^8, accumulate coordinates and masses from zero, then append the reversed conjugates of all but the final (real) prefix row. Entries are read left to right and then top to bottom. These conventions, together with the displayed masses and supports, determine every reference measure uniquely.

$q = 2$; $n = 41$; x; $start = 3349365000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	904764	1905814	711734	2075677	286937	4624699	2447859
2529220	-2005912	2659559	74649	2778512	56594	2885565	43329
2980263	33330	3062199	25792	3131021	19868	19467262	13206641
3228206	-13157141	3186436	336	3131021	-423	3062199	-1053
2980263	-1592	2885565	-1972	2778512	-2210	2659559	-2400
2529220	-2556	2388048	-2629	2236651	-2632	2075677	-2616
1905814	-2625	1727789	-2659	1542367	-2693	1350340	-2849
1152529	-3266	949785	-4562	742973	-5487	532979	-24509
320703	20464	4075	-2499124	11603	73573	17365	82752
20484	76534	20484	66626	17365	54610	11604	21958
4074	54225						

$q = 2$; $n = 48$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1385843	949785	680059	1152529	209735	1350340	128408
1542367	75603	1727789	50189	1905814	35139	2075677	26034
2236651	20335	2388048	16778	2529220	14604	2659559	13425
2778512	12980	2885565	13054	2980263	13531	3062199	14320

$q = 2$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
3131021	15294	16211108	11164507	3256154	-11054943	3228206	16310
3186436	19197	3131021	24778	3062199	19543	2980263	18296
2885565	16770	2778512	15360	2659559	13983	2529220	12669
2388048	11386	2236651	10131	2075677	8937	1905814	7774
1727789	6583	1542367	5377	1350340	4050	1152529	2463
949785	-131	742973	-2258	532979	-25599	320703	26042
4075	-2927729	11603	86195	17365	96969	20484	89696
20484	78091	17365	64014	11604	25747	4074	63559

$q = 2$; $n = 62$; $z \in [-q, -3/5]$; $start = -100000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
5704	244590	16245	-24961	24311	-10122	28678	-25136
28677	-30658	24312	-35202	16244	-38047	5704	-33807
448985	1148248	746170	-11922	1040162	7481	1329699	-2526
1613542	-4230	1890475	-6119	2159314	-7630	2418905	-9040
2668139	-10408	2905947	-11807	3131312	-13194	3343268	-14608
3540907	-16105	3723384	-17629	3889916	-19173	4039792	-20794
4172368	-22487	4287077	-24221	4383430	-26007	4461010	-27859
4519490	-29766	4558614	-31723	4578219	-33734	4578219	-35765
4558614	-37788	4519490	-39788	4461010	-41748	4383430	-43637
4287077	-45399	4172368	-46965	4039792	-48254	3889916	-49200
3723384	-49730	3540907	-49748	3343268	-49123	3131312	-47768
2905947	-45635	2668139	-42681	2418905	-38860	2159314	-34222
1890475	-28926	1613542	-23184	1329699	-17235	1040162	-11410
746170	-5884	448985	-2342	5704	-5971	16244	172
24312	144	28677	104	28678	57	24311	23
16245	-24	5704	28				

Poisson $q = 2$; 149 rows, 75 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
44433733	38967338	63870	-1298766	1432967	213587	-1344956	1389706	-12589
-1389706	1344956	48627	-1432967	1298766	23616	-1474695	1251185	24942
-1514843	1202264	20256	-1553368	1152057	18160	-1590231	1100615	15953
-1625391	1047995	14237	-1658810	994252	12744	-1690453	939446	11446
-1720285	883632	10308	-1748277	826873	9288	-1774394	769228	8392
-1798613	710760	7582	-1820906	651531	6854	-1841248	591603	6179
-1859618	531042	5554	-1875999	469913	4983	-1890369	408281	4450
-1902716	346211	3961	-1913024	283770	3494	-1921285	221026	3056
-1927488	158045	2636	-1931627	94895	2239	-1933698	31643	1862
-1933698	-31643	1497	-1931627	-94895	1146	-1927488	-158045	799
-1921285	-221026	469	-1913024	-283770	138	-1902716	-346211	-177
-1890369	-408281	-496	-1875999	-469913	-804	-1859618	-531042	-1117
-1841248	-591603	-1425	-1820906	-651531	-1735	-1798613	-710760	-2046
-1774394	-769228	-2354	-1748277	-826873	-2663	-1720285	-883632	-2973
-1690453	-939446	-3290	-1658810	-994252	-3613	-1625391	-1047995	-3941

Poisson $q = 2$ (continued)

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
-1590231	-1100615	-4270	-1553368	-1152057	-4601	-1514843	-1202264	-4933
-1474695	-1251185	-5276	-1432967	-1298766	-5623	-1389706	-1344956	-5977
-1344956	-1389706	-6332	-1298766	-1432967	-6685	-1251185	-1474695	-7038
-1202264	-1514843	-7399	-1152057	-1553368	-7767	-1100615	-1590231	-8140
-1047995	-1625391	-8506	-994252	-1658810	-8866	-939446	-1690453	-9218
-883632	-1720285	-9571	-826873	-1748277	-9924	-769228	-1774394	-10273
-710760	-1798613	-10606	-651531	-1820906	-10923	-591603	-1841248	-11241
-531042	-1859618	-11546	-469913	-1875999	-11823	-408281	-1890369	-12225
-346211	-1902716	-12281	-283770	-1913024	-13037	-221026	-1921285	-12682
-158045	-1927488	-14794	-94895	-1931627	-10897	-31643	-1933698	-46425

$q = 3$; $n = 40$; x; $start = 3349364905389$; $corr = (1, 94611)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	702301	1905814	840952	2075677	292770	7153919	4717567
2659559	-4169045	2778512	61508	2885565	47893	2980263	37685
3062199	29875	3131021	23786	19467262	13212594	3228206	-13139713
3186436	3120	3131021	2175	3062199	1404	2980263	759
2885565	234	2778512	-176	2659559	-503	2529220	-774
2388048	-980	2236651	-1137	2075677	-1274	1905814	-1398
1727789	-1531	1542367	-1720	1350340	-2032	1152529	-2570
949785	-4019	742973	-5056	532979	-24489	320703	21004
4075	-2536305	11603	74667	17365	83984	20484	77668
20484	67628	17365	55434	11604	22288	4074	55035

$q = 3$; $n = 48$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1317749	949785	719438	1152529	211244	1350340	132611
1542367	78595	1727789	53138	1905814	38062	2075677	29011
2236651	23361	2388048	19821	2529220	17664	2659559	16447
2778512	15893	2885565	15820	2980263	16082	3062199	16563
3131021	17168	16211108	11230618	3256154	-11118909	3228206	16001
3186436	18042	3131021	22282	3062199	17911	2980263	16743
2885565	15368	2778512	14113	2659559	12888	2529220	11695
2388048	10543	2236651	9427	2075677	8335	1905814	7258
1727789	6174	1542367	5040	1350340	3771	1152529	2252
949785	-308	742973	-2384	532979	-25690	320703	25964
4075	-2926999	11603	86175	17365	96942	20484	89662
20484	78075	17365	64003	11604	25740	4074	63540

$q = 3$; $n = 62$; $z \in [-q, -3/5]$; $start = -150000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
9779	292281	27847	-29820	41677	-12082	49162	-30034
49161	-36633	41677	-42073	27847	-45478	9779	-40415
769688	1372610	1279149	-13478	1783135	10100	2279483	-1453
2766072	-3165	3240815	-4980	3701681	-6421	4146695	-7731

$q = 3$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
4573952	-8963	4981624	-10270	5367963	-11546	5731316	-12864
6070127	-14287	6382943	-15728	6668428	-17236	6925357	-18871
7152631	-20584	7349276	-22367	7514451	-24276	7647446	-26320
7747696	-28470	7814767	-30740	7848376	-33156	7848376	-35715
7814767	-38394	7747696	-41196	7647446	-44117	7514451	-47142
7349276	-50225	7152631	-53312	6925357	-56337	6668428	-59222
6382943	-61852	6070127	-64064	5731316	-65661	5367963	-66430
4981624	-66146	4573952	-64554	4146695	-61406	3701681	-56550
3240815	-49983	2766072	-41820	2279483	-32497	1783135	-22656
1279149	-12806	769688	-3864	9779	-9271	27847	259
41677	267	49161	216	49162	143	41677	66
27847	-38	9779	70				

Poisson $q = 3$; 143 rows, 72 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
40400305	43134967	250203	-1432967	1298766	-21785	-1474695	1251185	48302
-1514843	1202264	21180	-1553368	1152057	23333	-1590231	1100615	18518
-1625391	1047995	16603	-1658810	994252	14511	-1690453	939446	12850
-1720285	883632	11458	-1748277	826873	10202	-1774394	769228	9118
-1798613	710760	8165	-1820906	651531	7310	-1841248	591603	6538
-1859618	531042	5829	-1875999	469913	5175	-1890369	408281	4582
-1902716	346211	4037	-1913024	283770	3532	-1921285	221026	3054
-1927488	158045	2599	-1931627	94895	2172	-1933698	31643	1772
-1933698	-31643	1396	-1931627	-94895	1034	-1927488	-158045	681
-1921285	-221026	341	-1913024	-283770	14	-1902716	-346211	-296
-1890369	-408281	-598	-1875999	-469913	-896	-1859618	-531042	-1188
-1841248	-591603	-1477	-1820906	-651531	-1756	-1798613	-710760	-2027
-1774394	-769228	-2293	-1748277	-826873	-2557	-1720285	-883632	-2820
-1690453	-939446	-3082	-1658810	-994252	-3340	-1625391	-1047995	-3593
-1590231	-1100615	-3842	-1553368	-1152057	-4087	-1514843	-1202264	-4334
-1474695	-1251185	-4579	-1432967	-1298766	-4823	-1389706	-1344956	-5064
-1344956	-1389706	-5299	-1298766	-1432967	-5529	-1251185	-1474695	-5755
-1202264	-1514843	-5979	-1152057	-1553368	-6203	-1100615	-1590231	-6422
-1047995	-1625391	-6637	-994252	-1658810	-6844	-939446	-1690453	-7047
-883632	-1720285	-7244	-826873	-1748277	-7435	-769228	-1774394	-7621
-710760	-1798613	-7803	-651531	-1820906	-7974	-591603	-1841248	-8148
-531042	-1859618	-8323	-469913	-1875999	-8461	-408281	-1890369	-8693
-346211	-1902716	-8754	-283770	-1913024	-9194	-221026	-1921285	-8885
-158045	-1927488	-11037	-94895	-1931627	-4863	-31643	-1933698	-49270

$q = 4$; $n = 40$; x; $start = 3349364905389$; $corr = (1, 94611)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	595630	1905814	908738	2075677	295753	7153919	4676244
2659559	-4111317	2778512	64043	2885565	50252	2980263	39935
3062199	32026	6317457	2623978	16280826	8015569	3228206	-10528099

$q = 4$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
3186436	4690	3131021	3670	3062199	2817	2980263	2109
2885565	1520	2778512	1028	2659559	618	2529220	284
2388048	1	2236651	-241	2075677	-447	1905814	-646
1727789	-864	1542367	-1122	1350340	-1514	1152529	-2140
949785	-3666	742973	-4797	532979	-24466	320703	21337
4075	-2557168	11603	75281	17365	84677	20484	78311
20484	68186	17365	55892	11604	22471	4074	55491

$q = 4$; $n = 48$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1297307	949785	731656	1152529	212004	1350340	134240
1542367	79909	1727789	54472	1905814	39425	2075677	30411
2236651	24786	2388048	21259	2529220	19082	2659559	17816
2778512	17179	2885565	16977	2980263	17068	3062199	17341
3131021	17699	16211108	11268944	3256154	-11159498	3228206	15339
3186436	16907	3131021	20352	3062199	16539	2980263	15453
2885565	14206	2778512	13060	2659559	11943	2529220	10864
2388048	9813	2236651	8788	2075677	7786	1905814	6791
1727789	5772	1542367	4708	1350340	3496	1152529	2023
949785	-476	742973	-2516	532979	-25732	320703	25849
4075	-2920506	11603	85984	17365	96728	20484	89461
20484	77901	17365	63859	11604	25679	4074	63399

$q = 4$; $n = 62$; $z \in [-q, -3/5]$; $start = -200000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
13853	318383	39451	-32482	59042	-13160	69646	-32714
69645	-39907	59042	-45825	39451	-49544	13853	-44029
1090391	1495445	1812128	-14257	2526108	11672	3229268	-732
3918601	-2319	4591155	-4113	5244047	-5442	5874485	-6637
6479766	-7753	7057301	-8919	7604614	-10088	8119364	-11284
8599346	-12563	9042503	-13903	9446940	-15303	9810923	-16812
10132893	-18430	10411475	-20146	10645471	-21989	10833883	-23979
10975902	-26118	11070921	-28412	11118532	-30883	11118532	-33553
11070921	-36424	10975902	-39500	10833883	-42788	10645471	-46299
10411475	-50015	10132893	-53909	9810923	-57943	9446940	-62053
9042503	-66147	8599346	-70081	8119364	-73647	7604614	-76580
7057301	-78542	6479766	-79118	5874485	-77858	5244047	-74310
4591155	-68112	3918601	-59129	3229268	-47655	2526108	-34426
1812128	-20681	1090391	-6612	13853	-11878	39451	348
59042	388	69645	335	69646	249	59042	139
39451	-29	13853	118				

Poisson $q = 4$; 141 rows, 71 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
38967338	44433733	23084	-1474695	1251185	211005	-1514843	1202264	-20189

Poisson $q = 4$ (continued)

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
-1553368	1152057	43666	-1590231	1100615	18891	-1625391	1047995	20970
-1658810	994252	16621	-1690453	939446	14911	-1720285	883632	12991
-1748277	826873	11496	-1774394	769228	10220	-1798613	710760	9078
-1820906	651531	8088	-1841248	591603	7202	-1859618	531042	6411
-1875999	469913	5700	-1890369	408281	5051	-1902716	346211	4451
-1913024	283770	3898	-1921285	221026	3388	-1927488	158045	2917
-1931627	94895	2476	-1933698	31643	2057	-1933698	-31643	1662
-1931627	-94895	1289	-1927488	-158045	940	-1921285	-221026	608
-1913024	-283770	286	-1902716	-346211	-26	-1890369	-408281	-325
-1875999	-469913	-609	-1859618	-531042	-881	-1841248	-591603	-1144
-1820906	-651531	-1405	-1798613	-710760	-1659	-1774394	-769228	-1905
-1748277	-826873	-2142	-1720285	-883632	-2370	-1690453	-939446	-2594
-1658810	-994252	-2818	-1625391	-1047995	-3037	-1590231	-1100615	-3252
-1553368	-1152057	-3460	-1514843	-1202264	-3660	-1474695	-1251185	-3857
-1432967	-1298766	-4052	-1389706	-1344956	-4243	-1344956	-1389706	-4430
-1298766	-1432967	-4611	-1251185	-1474695	-4790	-1202264	-1514843	-4964
-1152057	-1553368	-5134	-1100615	-1590231	-5295	-1047995	-1625391	-5452
-994252	-1658810	-5606	-939446	-1690453	-5760	-883632	-1720285	-5912
-826873	-1748277	-6053	-769228	-1774394	-6186	-710760	-1798613	-6315
-651531	-1820906	-6439	-591603	-1841248	-6573	-531042	-1859618	-6706
-469913	-1875999	-6807	-408281	-1890369	-6997	-346211	-1902716	-7014
-283770	-1913024	-7453	-221026	-1921285	-6970	-158045	-1927488	-9502
-94895	-1931627	-1362	-31643	-1933698	-54028			

$q = 5$; $n = 39$; x; $start = 3349364905389$; $corr = (1, 94611)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	529775	1905814	950484	2075677	297565	7153919	4650255
2659559	-4075339	2778512	65583	2885565	51687	2980263	41306
3062199	33350	22598283	15808801	3228206	-15688094	3186436	5690
3131021	4629	3062199	3737	2980263	2986	2885565	2352
2778512	1820	2659559	1366	2529220	978	2388048	649
2236651	365	2075677	99	1905814	-146	1727789	-407
1542367	-729	1350340	-1165	1152529	-1838	949785	-3439
742973	-4608	532979	-24443	320703	21547	4075	-2570516
11603	75674	17365	85119	20484	78722	20484	68542
17365	56189	11604	22591	4074	55779		

$q = 5$; $n = 48$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1293462	949785	734356	1152529	212459	1350340	134919
1542367	80571	1727789	55179	1905814	40164	2075677	31175
2236651	25561	2388048	22033	2529220	19832	2659559	18510
2778512	17795	2885565	17490	2980263	17449	3062199	17561
3131021	17739	16211108	11293776	3256154	-11187610	3228206	14672
3186436	15935	3131021	18854	3062199	15441	2980263	14424

$q = 5$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
2885565	13264	2778512	12207	2659559	11182	2529220	10177
2388048	9206	2236651	8262	2075677	7318	1905814	6389
1727789	5438	1542367	4408	1350340	3256	1152529	1832
949785	-645	742973	-2615	532979	-25760	320703	25723
4075	-2913154	11603	85771	17365	96482	20484	89231
20484	77704	17365	63701	11604	25616	4074	63235

$q = 5$; $n = 62$; $z \in [-q, -3/5]$; $start = -250000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
17928	334975	51054	-34169	76407	-13848	90129	-34416
90129	-41985	76408	-48219	51054	-52124	17928	-46322
1411093	1573532	2345108	-14716	3269080	12728	4179054	-223
5071131	-1687	5941494	-3489	6786414	-4692	7602274	-5783
8385580	-6855	9132977	-7878	9841266	-8937	10507413	-10055
11128565	-11205	11702063	-12425	12225451	-13736	12696488	-15135
13113157	-16626	13473673	-18243	13776492	-19996	14020319	-21891
14204109	-23934	14327074	-26166	14388688	-28606	14388688	-31250
14327074	-34124	14204109	-37263	14020319	-40672	13776492	-44372
13473673	-48364	13113157	-52655	12696488	-57239	12225451	-62081
11702063	-67096	11128565	-72195	10507413	-77209	9841266	-81865
9132977	-85820	8385580	-88586	7602274	-89520	6786414	-87936
5941494	-83107	5071131	-74470	4179054	-61970	3269080	-46202
2345108	-28901	1411093	-10183	17928	-14170	51054	432
76408	502	90129	464	90129	375	76407	239
51054	-3	17928	179				

Poisson $q = 5$; 137 rows, 69 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
35977800	46887182	112739	-1553368	1152057	95323	-1590231	1100615	9508
-1625391	1047995	30038	-1658810	994252	18683	-1690453	939446	17726
-1720285	883632	14825	-1748277	826873	13099	-1774394	769228	11485
-1798613	710760	10160	-1820906	651531	9018	-1841248	591603	8002
-1859618	531042	7122	-1875999	469913	6321	-1890369	408281	5610
-1902716	346211	4961	-1913024	283770	4371	-1921285	221026	3826
-1927488	158045	3330	-1931627	94895	2859	-1933698	31643	2432
-1933698	-31643	2018	-1931627	-94895	1644	-1927488	-158045	1280
-1921285	-221026	945	-1913024	-283770	622	-1902716	-346211	316
-1890369	-408281	22	-1875999	-469913	-257	-1859618	-531042	-525
-1841248	-591603	-782	-1820906	-651531	-1027	-1798613	-710760	-1268
-1774394	-769228	-1496	-1748277	-826873	-1723	-1720285	-883632	-1932
-1690453	-939446	-2147	-1658810	-994252	-2344	-1625391	-1047995	-2548
-1590231	-1100615	-2735	-1553368	-1152057	-2925	-1514843	-1202264	-3102
-1474695	-1251185	-3278	-1432967	-1298766	-3448	-1389706	-1344956	-3613
-1344956	-1389706	-3778	-1298766	-1432967	-3932	-1251185	-1474695	-4087
-1202264	-1514843	-4232	-1152057	-1553368	-4376	-1100615	-1590231	-4514

Poisson $q = 5$ (continued)

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
-1047995	-1625391	-4649	-994252	-1658810	-4780	-939446	-1690453	-4904
-883632	-1720285	-5027	-826873	-1748277	-5144	-769228	-1774394	-5260
-710760	-1798613	-5371	-651531	-1820906	-5478	-591603	-1841248	-5587
-531042	-1859618	-5696	-469913	-1875999	-5788	-408281	-1890369	-5954
-346211	-1902716	-5957	-283770	-1913024	-6390	-221026	-1921285	-5782
-158045	-1927488	-8662	-94895	-1931627	1153	-31643	-1933698	-58970

$q = 6$; $n = 41$; x; $start = 3349364905389$; $corr = (1, 94611)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	485088	1905814	978778	2075677	298775	7153919	4632391
2659559	-4050763	2778512	66616	2885565	52657	2980263	42225
3062199	34237	6317457	2619041	13024672	5366428	3256154	-7866090
3228206	7669	3186436	6386	3131021	5296	3062199	4377
2980263	3607	2885565	2942	2778512	2370	2659559	1892
2529220	1477	2388048	1106	2236651	791	2075677	500
1905814	208	1727789	-86	1542367	-433	1350340	-918
1152529	-1628	949785	-3267	742973	-4482	532979	-24417
320703	21698	4075	-2579794	11603	75952	17365	85421
20484	79011	20484	68789	17365	56392	11604	22674
4074	55984						

$q = 6$; $n = 47$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1296301	949785	733184	1152529	212765	1350340	135179
1542367	80931	1727789	55581	1905814	40597	2075677	31626
2236651	26018	2388048	22480	2529220	20254	2659559	18884
2778512	18096	2885565	17699	2980263	17554	3062199	17549
3131021	17593	19467262	14176287	3228206	-14059423	3186436	15114
3131021	17661	3062199	14540	2980263	13587	2885565	12500
2778512	11504	2659559	10551	2529220	9617	2388048	8700
2236651	7814	2075677	6932	1905814	6044	1727789	5138
1542367	4164	1350340	3049	1152529	1654	949785	-776
742973	-2713	532979	-25781	320703	25619	4075	-2906127
11603	85567	17365	96244	20484	89020	20484	77512
17365	63545	11604	25555	4074	63083		

$q = 6$; $n = 61$; $z \in [-q, -3/5]$; $start = -300000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
22002	346514	62657	-35341	93773	-14323	110613	-35605
110613	-43431	93773	-49880	62657	-53916	22003	-47931
1731796	1627865	2878087	-15016	4012053	13462	5128838	188
6223661	-1245	7291834	-2974	8328781	-4127	9330064	-5154
10291393	-6126	11208654	-7081	12077917	-8049	12895461	-9068
13657784	-10143	14361623	-11261	15003963	-12471	15582054	-13783
16093419	-15174	16535871	-16684	16907514	-18342	17206755	-20137

$q = 6$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
17432315	-22084	17583227	-24231	17658845	-26586	17658845	-29163
17583227	-31986	17432315	-35103	17206755	-38522	16907514	-42269
16535871	-46374	16093419	-50853	15582054	-55732	15003963	-60981
14361623	-66571	13657784	-72445	12895461	-78479	12077917	-84431
11208654	-89984	10291393	-94640	9330064	-97688	8328781	-98254
7291834	-95287	6223661	-87780	5128838	-75149	4012053	-57661
2878087	-37220	1731796	-14240	22003	-16337	156430	2342
110613	-636	110613	514	93773	360	62657	43
22002	248						

Poisson $q = 6$; 135 rows, 68 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
34424432	48039239	28503	-1590231	1100615	177007	-1625391	1047995	-15336
-1658810	994252	37497	-1690453	939446	16709	-1720285	883632	18252
-1748277	826873	14532	-1774394	769228	12997	-1798613	710760	11310
-1820906	651531	10007	-1841248	591603	8861	-1859618	531042	7858
-1875999	469913	6979	-1890369	408281	6194	-1902716	346211	5498
-1913024	283770	4861	-1921285	221026	4284	-1927488	158045	3746
-1931627	94895	3258	-1933698	31643	2807	-1933698	-31643	2392
-1931627	-94895	2001	-1927488	-158045	1631	-1921285	-221026	1282
-1913024	-283770	955	-1902716	-346211	650	-1890369	-408281	359
-1875999	-469913	82	-1859618	-531042	-186	-1841248	-591603	-439
-1820906	-651531	-681	-1798613	-710760	-911	-1774394	-769228	-1133
-1748277	-826873	-1349	-1720285	-883632	-1553	-1690453	-939446	-1756
-1658810	-994252	-1945	-1625391	-1047995	-2130	-1590231	-1100615	-2310
-1553368	-1152057	-2483	-1514843	-1202264	-2652	-1474695	-1251185	-2815
-1432967	-1298766	-2971	-1389706	-1344956	-3123	-1344956	-1389706	-3266
-1298766	-1432967	-3412	-1251185	-1474695	-3549	-1202264	-1514843	-3683
-1152057	-1553368	-3812	-1100615	-1590231	-3934	-1047995	-1625391	-4059
-994252	-1658810	-4170	-939446	-1690453	-4289	-883632	-1720285	-4390
-826873	-1748277	-4502	-769228	-1774394	-4598	-710760	-1798613	-4702
-651531	-1820906	-4798	-591603	-1841248	-4891	-531042	-1859618	-5000
-469913	-1875999	-5065	-408281	-1890369	-5245	-346211	-1902716	-5207
-283770	-1913024	-5685	-221026	-1921285	-4933	-158045	-1927488	-8166
-94895	-1931627	3172	-31643	-1933698	-63714			

$q = 7$; $n = 39$; x; $start = 3349364905389$; $corr = (1, 94611)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
1727789	452787	1905814	999210	2075677	299640	7153919	4619366
2659559	-4032921	2778512	67361	2885565	53357	2980263	42885
3062199	34872	22598283	15799919	3228206	-15668266	3186436	6897
3131021	5792	3062199	4853	2980263	4060	2885565	3377
2778512	2781	2659559	2282	2529220	1849	2388048	1454
2236651	1107	2075677	797	1905814	480	1727789	151
1542367	-218	1350340	-728	1152529	-1469	949785	-3133

$q = 7$ (continued)

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	-4382	532979	-24397	320703	21806	4075	-2586615
11603	76150	17365	85655	20484	79215	20484	68976
17365	56540	11604	22733	4074	56134		

$q = 7$; $n = 47$; y; $start = 480367989919$; $corr = (1, 10081)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
742973	1302156	949785	730240	1152529	212980	1350340	135224
1542367	81118	1727789	55813	1905814	40862	2075677	31904
2236651	26297	2388048	22743	2529220	20485	2659559	19072
2778512	18224	2885565	17752	2980263	17520	3062199	17419
3131021	17363	19467262	14190705	3228206	-14077572	3186436	14420
3131021	16690	3062199	13797	2980263	12896	2885565	11868
2778512	10919	2659559	10023	2529220	9147	2388048	8273
2236651	7435	2075677	6607	1905814	5755	1727789	4889
1542367	3946	1350340	2863	1152529	1514	949785	-901
742973	-2800	532979	-25782	320703	25499	4075	-2899676
11603	85372	17365	96033	20484	88821	20484	77342
17365	63401	11604	25497	4074	62944		

$q = 7$; $n = 62$; $z \in [-q, -3/5]$; $start = -350000000000000$; $corr = (0, 0)$

ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i	ℓ_i	Δm_i
26077	355036	74260	-36213	111139	-14672	131096	-36481
131097	-44500	111138	-51099	74261	-55251	26077	-49110
2052499	1667978	3411066	-15218	4755026	14007	6078623	506
7376191	-899	8642173	-2580	9871148	-3682	11057853	-4644
12197207	-5553	13284331	-6438	14314568	-7344	15283509	-8291
16187004	-9277	17021182	-10328	17782475	-11458	18467619	-12673
19073683	-13983	19598069	-15410	20038535	-16971	20393191	-18668
20660521	-20533	20839381	-22592	20929001	-24854	20929001	-27337
20839381	-30104	20660521	-33160	20393191	-36528	20038535	-40266
19598069	-44397	19073683	-48966	18467619	-53977	17782475	-59461
17021182	-65413	16187004	-71803	15283509	-78539	14314568	-85451
13284331	-92242	12197207	-98450	11057853	-103377	9871148	-106019
8642173	-105092	7376191	-99170	6078623	-87102	4755026	-68605
3411066	-45520	2052499	-18587	26077	-18457	74261	581
111138	702	131097	714	131096	659	111139	497
74260	106	26077	327				

Poisson $q = 7$; 131 rows, 66 prefix triples

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
31208810	50187849	185535	-1658810	994252	-8205	-1690453	939446	34749
-1720285	883632	17141	-1748277	826873	17765	-1774394	769228	14285
-1798613	710760	12707	-1820906	651531	11051	-1841248	591603	9767
-1859618	531042	8635	-1875999	469913	7659	-1890369	408281	6797
-1902716	346211	6035	-1913024	283770	5358	-1921285	221026	4734

Poisson $q = 7$ (continued)

Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i	Δx_i	Δy_i	Δm_i
-1927488	158045	4175	-1931627	94895	3651	-1933698	31643	3180
-1933698	-31643	2747	-1931627	-94895	2342	-1927488	-158045	1969
-1921285	-221026	1607	-1913024	-283770	1277	-1902716	-346211	960
-1890369	-408281	670	-1875999	-469913	392	-1859618	-531042	127
-1841248	-591603	-125	-1820906	-651531	-369	-1798613	-710760	-597
-1774394	-769228	-814	-1748277	-826873	-1023	-1720285	-883632	-1223
-1690453	-939446	-1415	-1658810	-994252	-1604	-1625391	-1047995	-1781
-1590231	-1100615	-1955	-1553368	-1152057	-2117	-1514843	-1202264	-2277
-1474695	-1251185	-2433	-1432967	-1298766	-2581	-1389706	-1344956	-2728
-1344956	-1389706	-2861	-1298766	-1432967	-2997	-1251185	-1474695	-3120
-1202264	-1514843	-3250	-1152057	-1553368	-3367	-1100615	-1590231	-3487
-1047995	-1625391	-3595	-994252	-1658810	-3705	-939446	-1690453	-3810
-883632	-1720285	-3909	-826873	-1748277	-4007	-769228	-1774394	-4099
-710760	-1798613	-4195	-651531	-1820906	-4282	-591603	-1841248	-4376
-531042	-1859618	-4469	-469913	-1875999	-4536	-408281	-1890369	-4698
-346211	-1902716	-4664	-283770	-1913024	-5147	-221026	-1921285	-4311
-158045	-1927488	-7829	-94895	-1931627	4850	-31643	-1933698	-68165

Appendix C. Exact certificate algorithm

All numerical bounds in this appendix are computer-assisted. The code and logs supplied as ancillary files use outward-rounded directed interval arithmetic; the rules below specify the calculation independently of those files. All inputs are the printed tables. Terminating decimals and fractions are parsed as exact rationals; a missing or duplicate field, invalid mass, failed strict inequality, or unresolved cell is fatal. No unprinted numerical data are part of the argument.

Shared quadratic duality. Let $Q(\boldsymbol{\mu}) = B(\boldsymbol{\mu}, \boldsymbol{\mu})$ be a quadratic form with symmetric polarization B, and let $\mathcal{E}(\boldsymbol{\mu}) = C + Q(\boldsymbol{\mu}) + L(\boldsymbol{\mu})$, where L is linear. For a fixed reference $\boldsymbol{\nu}$, let F_i^{ν} be the component potentials representing $L(\boldsymbol{\mu}) + 2B(\boldsymbol{\nu}, \boldsymbol{\mu})$, and assume μ_i and ν_i have the same prescribed mass in each component. With $\boldsymbol{\sigma} = \boldsymbol{\mu} - \boldsymbol{\nu}$, polarization gives the exact identity

$$\mathcal{E}(\boldsymbol{\mu}) = C - Q(\boldsymbol{\nu}) + \sum_i \int F_i^{\nu}\, d\mu_i + Q(\boldsymbol{\sigma}).$$

Thus a nonnegative (respectively nonpositive) zero-mass remainder gives the lower (respectively upper) dual bound. The P0 and V proofs below differ only in the sign proof for this remainder and in their explicit transfer constants.

P0 endpoint measures. Divide each printed integer mass by its table denominator. On every pinned or frequency interval $[b_{i-1}, b_i]$, use the uniform measure of that mass. The contour has $N = 64$ nodes

$$\theta_j = \pi(2j+1)/64, \quad c_{63-j} = c_j, \qquad F_{48}(\theta) = \frac{1}{49}\left|\sum_{k=0}^{48} e^{ik\theta}\right|^2.$$

The nine (respectively eight) printed pair masses at $q = 2$ (respectively $q = 11$) occupy the first nodes; zero-fill the rest of the first half, then reflect. The contour density is $(2\pi)^{-1}\sum_j c_j F_{48}(\theta - \theta_j)$. The algorithm verifies exact total masses, nonnegative weights, interval density bounds, and the printed ellipse (ξ, A, B). For an interval row use the primitive $H(u) = u(\log u - 1)$ with $H(0) = 0$; for the contour use the finite Fourier/Joukowski formulas below. Put

$$c = \frac{A+B}{2}, \qquad d = \frac{A-B}{2}, \qquad \rho = \frac{d}{c}, \qquad z(u) = \xi + ce^{iu} + de^{-iu},$$

and

$$m_n = \frac{49-n}{49}\sum_{j=0}^{63} c_j \cos(n\theta_j), \qquad \gamma_n = \frac{(1+\rho^n)m_n}{n} \quad (1 \le n \le 48).$$

These are exact algebraic numbers in the real subfield generated by $e^{\pi i/64}$ and are enclosed by directed algebraic (or Arb) intervals; they are not assumed to be rational. On the contour,

$$U_\nu(z(u)) = M\log c - \sum_{n=1}^{48} \gamma_n \cos(nu).$$

For a real point inside the ellipse set $w = (t-\xi)/c$ and $P_0(w) = 2,\ P_1(w) = w,\ P_n(w) = wP_{n-1}(w) - \rho P_{n-2}(w)$. Then

$$U_\nu(t) = M\log c - \sum_{n=1}^{48} \frac{m_n}{n} P_n(w).$$

For a real point $-t$ on the negative axis to the left of the ellipse, let

$$u = t + \xi, \qquad \Delta_t = \sqrt{u^2 - 4cd}, \qquad s_t = -\frac{2c}{u+\Delta_t},$$

where the positive real square root is chosen and $|s_t| < 1$. The exterior formula is

$$U_\nu(-t) = M\log\frac{u+\Delta_t}{2} - \sum_{n=1}^{48} \gamma_n s_t^n.$$

The same formulas are used for interval boxes, with square roots and logarithms evaluated by outward rounding. For a complex contour point and a real interval, use

$$\mathcal{I}(z;a,b) = \Re\{\mathcal{L}(z-a) - \mathcal{L}(z-b)\}, \qquad \mathcal{L}(y) = y(\log y - 1),$$

with the logarithm continued from the upper half-plane; the lower half is obtained by conjugation. On a real crossing, split at the crossing and use $H(u) = u(\log|u| - 1)$, $H(0) = 0$. Thus all mixed contour–interval potentials, their primitives, and the contour self-energy

$$I(\nu,\nu) = M^2\log c - \sum_{n=1}^{48} \frac{(1+\rho^n)m_n^2}{n}$$

are finite expressions in the printed masses and geometry. No unprinted recurrence or rationality assertion is used.

V reference tables. The V data in Section B are printed as signed decimal integers in ordinary tables. In each real chain, the header gives the initial endpoint in units of $1/(5\cdot 10^{13})$; each displayed length is in units of 10^{-8}, except for the single residual recorded by (j,r) in the header, which adds $r/(5\cdot 10^{13})$ to length j (the pair $(0,0)$ means no correction). Mass increments are in units of 10^{-8}. Starting from zero mass, accumulate each pair left to right and top to bottom. In each Poisson prefix, all three columns are successive increments of the real coordinate, imaginary coordinate, and mass, in units of 10^{-8}. The table header gives the prefix and full row counts; these must agree with Table 10.1. Complete the Poisson list by adjoining the reversed conjugates of all prefix rows except its final real row. The verification checks every count, increment, support, total mass, and conjugate pair. Real rows are uniform. A Poisson row is harmonic measure on $|z| = r = 3/5$, with density $w(r^2 - |c|^2)/(2\pi|re^{i\theta} - c|^2)$ and potential $w\log|r^2 - z\bar{c}| - w\log r$ inside, $w\log|z-c|$ outside, and their common boundary value. It requires the exact strict test $|c|^2 < r^2$.

Directed interval evaluation. For a real interval row the potential is evaluated by H; pair integrals use $G(u) = \frac{1}{2}u^2\log|u| - \frac{3}{4}u^2$, $G(0) = 0$. The V denominator field uses

$$\int_a^b \log(q+xt)\,dt = \frac{(q+xb)(\log(q+xb)-1) - (q+xa)(\log(q+xa)-1)}{x},$$

with value $(b-a)\log q$ at $x = 0$. Its energy series for $\log(q+xv)$ ends at degree 420; the absolute tail is at most $2M_xM_v/(421q^{421})$, since $0 \le xv/q \le 1/2$.

check	initial cover and evaluator	deterministic acceptance rule
P0 contour	$u/\pi \in [j/256,(j+1)/256]$, $0 \le j < 256$; Arb, 45 digits; upper half only, lower by exact conjugation	lower field endpoint $\ge h_\eta$; otherwise rational midpoint bisection, left first, depth at most 10
P0 integrals	8192 equal cells on $[1/4,5/4]$ and 16384 on $[1/2,81/10]$	each lower field bound gives $\lvert I\rvert(h-W_-)_+$; the printed tail test uses $W'_\mu > 1/50$
V endpoint boxes	$0 \le x,v \le \delta$, $\delta = 1/250$; use the printed $E_{x,q}, E_{v,q}$	accept the two boxes when $E_{x,q} < K_{x,q}$ and $E_{v,q} < K_{v,q}$
V real supports	$x,v \in [\delta,1]$, $z \in [-q,-3/5]$ and $[3/5,1]$; Arb, 90 digits	accept iff the directed upper endpoint is strictly below its printed threshold; otherwise bisect the rational midpoint, left first, to depth 30
V circle bounds	each half-circle parameter $z = (3/5)e^{i\pi t}$, $t \in [0,1]$; Arb, 90 digits	same strict threshold test and bisection; circle endpoints are checked directly

The V supports cover the entire admissible set. The lower circle follows by conjugation. The two real endpoint boxes are discharged by the displayed endpoint inequalities rather than by evaluating $\log 0$. Nonfinite balls, unvisited endpoints, or unresolved cells are rejected. Finally recompute Q_q and require the directed lower endpoint of

$$Q_{q,-} - \frac{3}{2}\log q - K_{x,q} - \frac{5}{4}K_{v,q} - K_{z,q}$$

to exceed the printed rational α_q for every $q = 2,\dots,7$.

All quantities needed for these evaluations are printed in the preceding tables or defined by the displayed formulas. The leaf intervals generated by the bisection rule are auxiliary computation, not additional data.

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