# IRRATIONALITY OF THE DILOGARITHM AT 1/2

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Abstract. We prove that $\mathrm{Li}_2(1/2)$ is irrational. Irrationality of $\mathrm{Li}_2(1/s)$ was known for integers $s \le -5$ and $s \ge 6$ by work of Hata and of Rhin and Viola, and Prellberg recently proved it for $s = -4, 5, -3$. The approximations come from determinants of moments of $1/(xy-2)$, differentiated $m/2$ times in the parameter. We bound their values by a logarithmic energy on an ellipse, certified by directed interval arithmetic. Valuations of binomial coefficients give a common denominator of the coefficients. A Lucas congruence modulo 83 gives nonvanishing along $m = 32 \cdot 83^k$. Three finite computations are part of the proof.

## 1. Introduction

For $|w| \le 1$ let $\mathrm{Li}_2(w) = \sum_{n \ge 1} w^n/n^2$. Hata [3, Theorem 1.1] proved that $\mathrm{Li}_2(1/s)$ is irrational for every integer $s \le -5$ and every $s \ge 7$. Rhin and Viola adapted their permutation group method [5] to the dilogarithm and settled $s = 6$ [6]. They later treated negative arguments [7]. Viola and Zudilin [8] introduced a six-parameter family of double integrals. In 2024 Calegari, Dimitrov and Tang [1, Remark 2.8.2 and Section 15.2] listed the integers still open: $s \in \{-4, -3, -2, 2, 3, 4, 5\}$. Prellberg's 2026 preprint [4] treats $s = -4, 5, -3$. We prove the case $s = 2$.

Put

$$\beta = \mathrm{Li}_2(1/2) = \sum_{r=1}^{\infty} \frac{2^{-r}}{r^2}, \qquad \ell = \log 2.$$

For each positive integer $m$ divisible by 4, let

$$g = \frac{m}{2}, \qquad I_m = \left[\frac{m}{4}, \frac{5m}{4}\right) \cap \mathbb{Z}, \qquad J_m = [0, 2m) \cap \mathbb{Z}.$$

The rational matrices $A^{(g)}, B^{(g)}, C^{(g)}$ are defined in Section 2. Rows and columns are in increasing order. Set

$$G_m(T) = \det \begin{bmatrix} C^{(g)}_{I_m, J_m} \\ B^{(g)}_{I_m, J_m} - T A^{(g)}_{I_m, J_m} \end{bmatrix} \in \mathbb{Q}[T]. \tag{1.1}$$

Only the last $m$ rows depend on $T$, so $\deg G_m \le m$.

**Theorem 1.** *The number* $\mathrm{Li}_2(1/2)$ *is irrational.*

By the reflection formula [2, §25.12], $\mathrm{Li}_2(1/2) = \pi^2/12 - \frac{1}{2}\log^2 2$, so Theorem 1 is equivalent to the irrationality of $\pi^2 - 6\log^2 2$.

Theorem 1 follows from three estimates for the polynomials (1.1).

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**Proposition 1.1.** *For every $m \geq 8$ with $4 \mid m$,*

$$|G_m(\beta)| \leq \exp\left(-\alpha m^2 + 10m\log m + 56m + 301\right), \qquad \alpha = \frac{1992909}{10^6}. \tag{1.2}$$

*For every positive $m$ divisible by 4 there is a positive rational number $D_m$ such that*

$$D_m G_m(T) \in \mathbb{Z}[T], \qquad \limsup_{\substack{m\to\infty \\ 4\mid m}} \frac{\log D_m}{m^2} \leq \tau < 1.979. \tag{1.3}$$

*For every fixed $t \in \mathbb{Q}$,*

$$G_{32\cdot 83^k}(t) \neq 0 \quad \text{if } k \geq 0 \text{ and } 2k + v_{83}(t) > 0, \tag{1.4}$$

*where $v_{83}(0) = +\infty$.*

Section 2 writes $G_m(\beta)$ as a determinant of rational series. Sections 3–7 prove (1.2). Section 8 proves (1.3), and Section 9 proves (1.4). Theorem 1 follows in Section 10.

Since $1.979 - 1.992909 < 0$, Theorem 1 follows.

Three finite computations are part of the proof (Appendix A). Directed interval inequalities give the constant $\alpha$. Exact integer arithmetic gives the coefficient bounds. Modular arithmetic gives the seed congruence. The use of AI tools is described at the end of the paper.

## 2. Rational matrices and parameter derivatives

Fix an integer $g \geq 0$. All indices $i, j$ are nonnegative integers. Empty sums are zero. Put

$$U_i = 2^i \binom{i+g}{g}, \qquad t_i = \sum_{r=1}^{i} \frac{U_{i-r}}{r} + 2^i \sum_{s=1}^{g} \frac{(-1)^{s-1}}{s} \binom{i+g}{g-s}(2^s - 1).$$

For $i \neq j$ define

$$C_{ij}^{(g)} = \frac{U_i - U_j}{j-i}, \qquad B_{ij}^{(g)} = \frac{t_i - t_j}{j-i}, \qquad A_{ij}^{(g)} = 0.$$

The diagonal entries are

$$A_{ii}^{(g)} = U_i, \qquad C_{ii}^{(g)} = -U_i \sum_{s=1}^{g} \frac{1}{i+s}, \tag{2.1}$$

and

$$B_{ii}^{(g)} = \sum_{r=1}^{i} \frac{U_{i-r}}{r^2} - 2^i \sum_{s=2}^{g} \frac{(-1)^s}{s} \binom{i+g}{g-s} \sum_{r=1}^{s-1} \frac{2^r - 1}{r}. \tag{2.2}$$

These are rational matrices. Their diagonals are defined separately from the off-diagonal quotients.

For real $a > 1$, let

$$t_i^0(a) = \sum_{r=1}^{i} \frac{a^{i-r}}{r}, \qquad b_i^0(a) = \sum_{r=1}^{i} \frac{a^{i-r}}{r^2},$$

$$L(a) = \log(1 - 1/a) = -\sum_{r=1}^{\infty} \frac{a^{-r}}{r}, \quad \beta(a) = \mathrm{Li}_2(1/a) = \sum_{r=1}^{\infty} \frac{a^{-r}}{r^2}.$$

The logarithm is real. Define

$$A^0_{ij}(a) = a^i \delta_{ij}.$$

$$C^0_{ij}(a) = \begin{cases} \dfrac{a^i - a^j}{j-i}, & i \neq j, \\ 0, & i = j, \end{cases}$$

$$B^0_{ij}(a) = \begin{cases} \dfrac{t^0_i(a) - t^0_j(a)}{j-i}, & i \neq j, \\ b^0_i(a), & i = j, \end{cases}$$

where $\delta_{ij}$ is the Kronecker delta.

**Lemma 2.1.** *For $a > 1$ and $i, j \geq 0$,*

$$\begin{aligned} M_{ij}(a) &:= B^0_{ij}(a) + L(a) C^0_{ij}(a) - \beta(a) A^0_{ij}(a) \\ &= -\sum_{r=1}^{\infty} \frac{a^{-r}}{(i+r)(j+r)}. \end{aligned} \tag{2.3}$$

*Equivalently,*

$$M_{ij}(a) = -\int_0^1 \int_0^1 \frac{x^i y^j}{a - xy}\, dy\, dx.$$

*Proof.* Split off the first $i$ terms of the defining series. Then

$$t^0_i(a) + a^i L(a) = -\sum_{r=1}^{\infty} \frac{a^{-r}}{i+r}, \qquad b^0_i(a) - a^i \beta(a) = -\sum_{r=1}^{\infty} \frac{a^{-r}}{(i+r)^2}.$$

For $i \neq j$, take the difference of the first identity at $i$ and at $j$, and divide by $j - i$. By

$$\frac{1}{j-i}\left(\frac{1}{i+r} - \frac{1}{j+r}\right) = \frac{1}{(i+r)(j+r)},$$

this proves (2.3) off the diagonal. The second identity proves it on the diagonal, where $C^0_{ii} = 0$.

Finally,

$$\frac{1}{a - xy} = \sum_{r=1}^{\infty} a^{-r} (xy)^{r-1}.$$

For fixed $a > 1$ this series converges uniformly and absolutely on $[0,1]^2$. Termwise integration gives the integral formula. □

Put

$$D_g f(a) = \frac{1}{g!} \frac{\partial^g}{\partial a^g} \big( a^g f(a) \big).$$

For every integer $r \geq 1$,

$$D_g a^{-r} = \frac{1}{g!} \prod_{h=0}^{g-1} (g - r - h)\, a^{-r} = \begin{cases} 0, & 1 \leq r \leq g, \\ (-1)^g \binom{r-1}{g} a^{-r}, & r > g. \end{cases}$$

This is the $g$th derivative of $a^{g-r}$ divided by $g!$. For $1 \leq r \leq g$ the polynomial $a^{g-r}$ has degree less than $g$. For $r > g$ the $g$ negative factors give the binomial coefficient. For $g = 0$ the product is empty and equals one.

Fix a compact subinterval of $(1,\infty)$. On it, the derivatives through order $g$ of the summands in $a^g M_{ij}(a)$ are bounded by $C(1+r)^g q^r$. Here $C>0$ and $0<q<1$ do not depend on $r$. Hence (2.3) may be differentiated termwise, and

$$(D_g M_{ij})(a) = (-1)^{g+1} \sum_{r>g} \frac{\binom{r-1}{g} a^{-r}}{(i+r)(j+r)}.$$

Put

$$N_{ij} := (D_g M_{ij})(2) = (-1)^{g+1} \sum_{r>g} \frac{w_r}{(i+r)(j+r)}, \qquad w_r = \binom{r-1}{g} 2^{-r}. \tag{2.4}$$

The weights are positive for $r>g$, and the series converges absolutely.

Next we find the rational coefficients of $D_g M$. For $a>1$ put

$$U_i(a) = a^i \binom{i+g}{g}, \qquad Q_s(a) = \left(\frac{a}{a-1}\right)^s - 1 \quad (s \geq 1).$$

Define $t_i^{(g)}(a)$, $A_{ij}^{(g)}(a)$, $B_{ij}^{(g)}(a)$ and $C_{ij}^{(g)}(a)$ by the same finite formulas. In them replace $U_n$ by $U_n(a)$, $2^n$ by $a^n$, and each factor $2^s-1$ by $Q_s(a)$. Thus

$$t_i^{(g)}(a) = \sum_{r=1}^{i} \frac{U_{i-r}(a)}{r} + a^i \sum_{s=1}^{g} \frac{(-1)^{s-1}}{s} \binom{i+g}{g-s} Q_s(a).$$

These are rational functions of $a$, regular for $a>1$. At $a=2$ they equal the rational quantities defined above.

**Lemma 2.2.** *For every $g \geq 0$, $i,j \geq 0$ and $a>1$,*

$$(D_g M_{ij})(a) = B_{ij}^{(g)}(a) + L(a) C_{ij}^{(g)}(a) - \beta(a) A_{ij}^{(g)}(a).$$

*Proof.* For every $C^g$ function $f$, Leibniz's rule gives

$$D_g(a^i f(a)) = \sum_{s=0}^{g} \binom{i+g}{g-s} a^{i+s} \frac{f^{(s)}(a)}{s!}.$$

In particular $D_g a^i = U_i(a)$. As $L(a) = \log(a-1) - \log a$, for $s \geq 1$

$$\frac{a^s L^{(s)}(a)}{s!} = \frac{(-1)^{s-1}}{s} Q_s(a).$$

Differentiating the absolutely convergent series for $\beta$ gives

$$\beta'(a) = -\frac{1}{a} \sum_{r=1}^{\infty} \frac{a^{-r}}{r} = \frac{L(a)}{a}.$$

Leibniz's rule for the $(s-1)$st derivative of $L(a)/a$ gives

$$\begin{aligned} \frac{a^s \beta^{(s)}(a)}{s!} &= \frac{1}{s} \sum_{r=0}^{s-1} (-1)^{s-1-r} \frac{a^r L^{(r)}(a)}{r!} \\ &= \frac{(-1)^{s-1}}{s} \left( L(a) - \sum_{r=1}^{s-1} \frac{Q_r(a)}{r} \right). \end{aligned}$$

Hence

$$D_g\big(t_i^0(a) + a^i L(a)\big) = t_i^{(g)}(a) + L(a) U_i(a).$$

Divided differences in $i,j$ prove the lemma for $i \neq j$, including $A_{ij}^{(g)}(a) = 0$.

On the diagonal $M_{ii} = b_i^0 - a^i\beta$, and the same calculation gives

$$\begin{aligned}(D_g M_{ii})(a) = &\sum_{r=1}^{i} \frac{U_{i-r}(a)}{r^2} - U_i(a)\beta(a) \\ &- a^i L(a) \sum_{s=1}^{g} \frac{(-1)^{s-1}}{s} \binom{i+g}{g-s} \\ &- a^i \sum_{s=2}^{g} \frac{(-1)^s}{s} \binom{i+g}{g-s} \sum_{r=1}^{s-1} \frac{Q_r(a)}{r}.\end{aligned}$$

The first and last sums give $B_{ii}^{(g)}(a)$, with the sign in (2.2). For the logarithmic coefficient we use

$$\sum_{s=1}^{g} \frac{(-1)^{s-1}}{s} \binom{i+g}{g-s} = \binom{i+g}{g} \sum_{s=1}^{g} \frac{1}{i+s}.$$

For its proof, differentiate at $x = 0$ the polynomial identity

$$\binom{i+g+x}{g} = \sum_{s=0}^{g} \binom{i+g}{g-s} \binom{x}{s}.$$

It holds for nonnegative integers $x$ by the binomial theorem, hence as a polynomial identity. By the product formula, the derivative of its left side at zero is $\binom{i+g}{g} \sum_{s=1}^{g} (i+s)^{-1}$. For $s \geq 1$, the derivative of $\binom{x}{s}$ at zero is $(-1)^{s-1}/s$. This proves the equality, also when $g = 0$.

So the logarithmic coefficient is $-U_i(a) \sum_{s=1}^{g} (i+s)^{-1} = C_{ii}^{(g)}(a)$. The coefficient of $-\beta(a)$ is $U_i(a) = A_{ii}^{(g)}(a)$. This proves the diagonal case. □

Now let $m$ be a positive integer divisible by four. Set

$$g = m/2, \qquad I_m = \{m/4, \ldots, 5m/4 - 1\}, \qquad J_m = \{0, \ldots, 2m - 1\}.$$

Rows and columns are in increasing index order. The polynomial (1.1) is

$$G_m(T) = \det \begin{bmatrix} C_{I_m, J_m}^{(g)} \\ B_{I_m, J_m}^{(g)} - T A_{I_m, J_m}^{(g)} \end{bmatrix}.$$

It lies in $\mathbb{Q}[T]$. Only the last $m$ rows depend on $T$, each affinely, so its degree is at most $m$.

Put $\beta = \beta(2) = \mathrm{Li}_2(1/2)$ and $\ell = \log 2$. As $L(2) = -\ell$, Lemma 2.2 gives

$$N = B^{(g)} - \ell C^{(g)} - \beta A^{(g)}.$$

In the determinant for $G_m(\beta)$, add $-\ell$ times each top row to the corresponding bottom row. This is left multiplication by

$$\begin{bmatrix} \mathrm{Id}_m & 0 \\ -\ell\, \mathrm{Id}_m & \mathrm{Id}_m \end{bmatrix},$$

which has determinant one. Hence

$$G_m(\beta) = \det \begin{bmatrix} C_{I_m, J_m}^{(g)} \\ N_{I_m, J_m} \end{bmatrix}. \tag{2.5}$$

This row operation needs no rank or nonvanishing assumption.

## 3. Exact contour expansion and constrained energy

Set

$$\Gamma=\{\xi+A\cos u+iB\sin u:0\le u<2\pi\},\quad (\xi,A,B)=\left(\frac{11281}{10000},\frac{12783}{10000},\frac{229}{200}\right).$$

It encloses $[0,2]$. Before scaling, the contour in this section is $m\Gamma$. The entire function $F_g(z)=2^z\binom{z+g}{g}$ satisfies

$$C_{ij}^{(g)}=-\oint_{m\Gamma}\frac{F_g(z)}{(z-i)(z-j)}\frac{dz}{2\pi i}+\ell U_i[i=j]. \tag{3.1}$$

For distinct poles this is the residue formula for divided differences. At the double pole $F_g'(i)=U_i(\ell+\sum_{s=1}^{g}1/(i+s))$, which gives (3.1) on the diagonal too. No sine factor occurs.

Write $K(x,y)=1/(x+y)$ and $\Delta(X)=\prod_{a<b}(x_b-x_a)$ for increasing real lists. The same product defines $\Delta$ on complex lists. Cauchy–Binet applied to (2.4) gives the absolutely convergent expansion

$$G_m(\beta)=(-1)^{m(g+1)}\sum_{\substack{R\subset\{g+1,g+2,\dots\}\\|R|=m}}\Big(\prod_{r\in R}w_r\Big)\det K_{I_m,R}\det\begin{bmatrix}C^{(g)}\\K_{R,J_m}\end{bmatrix}. \tag{3.2}$$

The sum runs over subsets, so no $m!$ occurs. For fixed $m$ the weights $w_r$ decay exponentially and both minors are bounded. Finite truncation and passage to the limit justify the expansion.

Select the diagonal term of (3.1) in the rows $S\subset I_m$. Put $k=|S|$, $d=m-k$, and let $Z=(z_1,\dots,z_d)$ lie on $m\Gamma$. The selected columns are exactly $S$. Andréief's identity for the remaining rows gives the absolute integrand

$$\begin{aligned}Q_m(S,R,Z)=&\Big(\prod_{i\in S}\ell U_i\Big)\Big(\prod_{r\in R}w_r\Big)\Big(\prod_{z\in Z}|F_g(z)|\Big)\\&\times\left|\det K_{I_m\setminus S,-Z}\,\det K_{I_m,R}\,\det K_{(-Z,R),J_m\setminus S}\right|.\end{aligned} \tag{3.3}$$

The measure is $\prod|dz|/(2\pi)$ and the symmetrization factor is $1/d!$. By the triangle inequality $|G_m(\beta)|\le E_m(\infty)$, where

$$E_m(H)=\sum_{S\subset I_m}\frac{1}{d!}\int_{(m\Gamma)^d}\sum_{\substack{|R|=m\\ g<r\le H}}Q_m(S,R,Z)\prod_{z\in Z}\frac{|dz|}{2\pi}. \tag{3.4}$$

The signed product formula fixes every quadratic-scale normalization. Put $n=m/4$, $Y=S\uplus Z$,

$$D_m(a)=(a-n)!(n+m-1-a)!a!(2m-1-a)!,$$

$$P_m(z)=\prod_{a\in I_m\uplus J_m}(z-a),\qquad Q_m^*(r)=\prod_{a\in I_m\uplus J_m}(r+a).$$

Then

$$\begin{aligned}G_m(\beta)=&\Delta(I_m)^2\Delta(J_m)\sum_{S\subset I_m}\frac{(-1)^{k(n+1)}\ell^k}{d!}\sum_R\prod_{a\in S}\frac{U_a}{D_m(a)}\prod_{r\in R}w_r\\&\times\oint_{(m\Gamma)^d}\frac{\Delta(Y)^2\Delta(R)^2\prod_{y\in Y,r\in R}(y+r)}{\prod_{z\in Z}P_m(z)\prod_{r\in R}Q_m^*(r)}\prod_{z\in Z}F_g(z)\frac{dz}{2\pi i}.\end{aligned} \tag{3.5}$$

In this formula the three Cauchy determinants give the squared Vandermondes and the single frequency cross product. The conversion

$$\Delta(I_m \setminus S)\Delta(J_m \setminus S) = \frac{\Delta(I_m)\Delta(J_m)\Delta(S)^2}{\prod_{a\in S} D_m(a)}$$

restores the removed fixed factors. Each pinned index occurs twice in a contour denominator; restoring both gives $(z-a)^2$. Restoring its one removed frequency factor gives $(r+a)$. Since $m$ is even, the Laplace sign, the minus signs from (3.1) and the Cauchy signs reduce to $(-1)^{k(n+1)}$. The upper estimate does not use the signs.

Scale all particle positions by $m$. Write $E = [1/4, 5/4]$, and define

$$A_0(z) = \int_E \log|z-x|\,dx + \int_0^2 \log|z-x|\,dx,$$
$$B_0(t) = \int_E \log(t+x)\,dx + \int_0^2 \log(t+x)\,dx.$$
$$\phi(z) = \Re(z)\ell + \int_0^{1/2} \log|z+s|\,ds + \tfrac{1}{2} - \tfrac{1}{2}\log(1/2),$$

$$\psi(t) = t\log t - (t-1/2)\log(t-1/2) - \tfrac{1}{2}\log(1/2) - t\ell. \tag{3.6}$$

The integral defines $\phi$ also across the negative real axis, with $0\log 0 = 0$. So $\phi$ has no branch ambiguity where $\Gamma$ crosses the interior of the zero segment $[-1/2, 0]$.

Let $K_0(z,w) = -\log|z-w|$, and write

$$I(\sigma,\tau) = \iint K_0(z,w)\,d\sigma(z)\,d\tau(w), \qquad I(\sigma) = I(\sigma,\sigma).$$

For a measure $\mu$ on the positive real line, let $\mu^- = (-\mathrm{id})_\#\mu$ be its reflection onto the negative real line. The admissible measures are those with

$$\begin{aligned} &0 \le \kappa \le 1_E\,dx, \quad \nu \ge 0, \quad \operatorname{supp}\nu \subset \Gamma, \quad |\kappa| + |\nu| = 1, \\ &0 \le \mu \le 1_{[1/2,\infty)}\,dt, \quad |\mu| = 1. \end{aligned} \tag{3.7}$$

The contour density has no cap, and $|\kappa|$ is not prescribed. With $\eta = \kappa + \nu$, the energy is

$$\begin{aligned} \mathcal{E}(\eta,\mu) = C_0 &+ \int (A_0 - \phi)\,d\eta + \int (B_0 - \psi)\,d\mu \\ &+ I(\eta) + I(\mu) + I(\eta,\mu^-), \\ C_0 &= \tfrac{9}{2} - 2\log 2. \end{aligned} \tag{3.8}$$

The pinned–contour interaction has coefficient two. Each companion–frequency interaction has coefficient one. These coefficients come from (3.5). For the fixed Vandermonde,

$$\log(\Delta(I_m)^2\Delta(J_m)) = 3m^2\log m + (2\log 2 - 9/2)m^2 + O(m\log m).$$

Here $\int_0^b\int_0^b \log|x-y|\,dx\,dy = b^2\log b - 3b^2/2$. Twice the half-energy for length one plus the half-energy for length two gives the constant. Every $m^2\log m$ term in (3.5) cancels, for every pinned mass. The remaining homogeneous factor is $m^{-2m-2d}$ before the contour differentials and $m^{-2m-d}$ with them.

To define (3.8) without $\infty-\infty$, put $\ell_1(t)=\log(1+t)$. Write its frequency-dependent part as

(3.9)
$$\int (B_0-\psi-3\ell_1)\,d\mu+\iint \log\frac{(1+t)(1+s)}{|t-s|}\,d\mu(t)d\mu(s)+\iint \log\frac{1+t}{|z+t|}\,d\eta(z)d\mu(t).$$

The self kernel is nonnegative. The mixed kernel is bounded on the compact companion support times the half-line. The first field is $t\log 2-\frac{1}{2}\log t+O(1)$. Hence an infinite first moment gives energy $+\infty$. The same definition works on a sufficiently small tube about $\Gamma$. Define

$$\alpha_\Gamma=\inf\{\mathcal{E}(\kappa+\nu,\mu):(3.7)\}. \tag{3.10}$$

This is the variational problem on the full half-line. An admissible trial measure gives only an upper bound for this infimum. The lower bound comes from the dual certificate in Section 5.

## 4. Circle regularization and transfer from the tube

Let $T_\delta=\{z:\operatorname{dist}(z,\Gamma)\le\delta\}$. In (3.7) replace only the support condition on $\nu$ by $\operatorname{supp}\nu\subset T_\delta$. For $0<\delta\le 1/64$ we have the exact estimate

$$\inf_{T_\delta}\mathcal{E}\ge\alpha_\Gamma-300\delta. \tag{4.1}$$

The proof covers the unrestricted contour self-energy. Let $D$ be the interior of the ellipse. For a point $z$ inside $D$, let $\omega_z$ be its interior harmonic measure on $\Gamma$. Outside $D$, use exterior harmonic measure with total mass one. At a boundary point use $\delta_z$. Set

$$c=(A+B)/2,\quad d=(A-B)/2,\quad F(w)=\xi+cw+d/w\quad(|w|\ge 1).$$

The factorization $F(w_1)-F(w_2)=(w_1-w_2)(c-d/(w_1w_2))$ shows that $F$ is one-to-one outside the unit disk. Let $W$ be its inverse. Set $e(z)=0$ inside the ellipse and $e(z)=\log|W(z)|$ outside. The Green-function identity gives

$$U^{\omega_z}(x)\le K_0(z,x)+e(z),\qquad I(\omega_z,\omega_w)\le K_0(z,w)+e(z)+e(w). \tag{4.2}$$

For two interior points the first deficit is the interior Green function. For points on opposite sides it is zero. For two exterior points it is the nonnegative Green function of the exterior of the disk,

$$\log\left|\frac{W(z)\overline{W(x)}-1}{W(z)-W(x)}\right|.$$

Its sign follows from $|a\bar b-1|^2-|a-b|^2=(|a|^2-1)(|b|^2-1)\ge 0$. At boundary points the identity in (4.2) is an equality. Integrating it against $\omega_w$ proves the two-sweep inequality. These identities also follow by Dirichlet uniqueness after subtracting the Green pole, including the growth term at infinity. Integrable logarithms and limiting arguments cover boundary points and arbitrary finite-energy measures.

If $z\in T_\delta$ is exterior, the factorization gives $\operatorname{dist}(z,\Gamma)\ge B(|W(z)|-1)$. Hence

$$0\le e(z)\le\delta/B. \tag{4.3}$$

Let $q(z)=1-(\Re z-\xi)^2/A^2-(\Im z)^2/B^2$. On $D$, $|\nabla q|\le 2/B$. For $x\in[0,2]$, $q(x)>1/5$. A point in the inner $\delta$-collar has $q(z)\le 2\delta/B<1/8$. The level set $q=1/8$ has distance more than $1/32$ from $[0,2]$. By domain monotonicity the

interior Green function is at most $\log 160 < 6$ there. The superharmonic barrier $48q$ dominates it on both boundaries of the collar. By the maximum principle,

$$g_D(z,x) \le 96\delta/B \quad (z \in D \cap T_\delta,\ x \in [0,2]). \tag{4.4}$$

Put $f = A_0 - \phi$. Let $z$ be interior. Sweeping the positive-source logarithms in $A_0$ increases their sum by at most $3 \cdot 96\delta/B$. The linear part of $\phi$ keeps its mean. By (4.2), the logarithmic-potential part of $\phi$ has the favorable sign in $-\phi$. Thus $\int f\,d\omega_z - f(z) \le 288\delta/B$. Now let $z = F(w)$ be exterior. Its mean coordinate is $\xi + c/\overline{w} + d/w$, at distance at most $(A+B)(|w|-1)$ from $z$. The change in $A_0$ is exactly $-3e(z)$. The remaining terms give the smaller bound $[1/2 + (A+B)\log 2]\delta/B < 3\delta/B$. Hence, for the swept measure $\nu^{\#} = \int \omega_z\,d\nu(z)$ of mass $s$,

$$\int f\,d\nu^{\#} \le \int f\,d\nu + 288s\delta/B.$$

Put $b = \int e\,d\nu$. The changed interaction energies in (3.8) increase by at most $2sb + 2(1-s)b + b = 3b \le 3s\delta/B$. Hence

$$\mathcal{E}(\kappa + \nu^{\#}, \mu) \le \mathcal{E}(\kappa+\nu,\mu) + (291/B)s\delta < \mathcal{E}(\kappa+\nu,\mu) + 300\delta.$$

This proves (4.1), uniformly for every frequency measure on the full half-line and every allowed pinned/contour mass split.

Let $z_1, \dots, z_d \in \Gamma$. Let $\sigma_{z,\delta}$ be probability arclength on the circle of radius $\delta$ about $z$. Set $\nu_\delta = m^{-1}\sum_{a=1}^{d} \sigma_{z_a,\delta}$. The circle mean identity $\int \log|x - z - \delta e^{iu}|\,du/(2\pi) = \log\max(|x-z|,\delta)$ implies

$$\frac{1}{m^2}\sum_{a\ne b} K_0(z_a,z_b) \ge I(\nu_\delta) - \frac{d}{m^2}\log(1/\delta). \tag{4.5}$$

The mixed potentials from $[0,2]$ and $(-\infty,-1/2]$ are harmonic on these disks. The function $\phi$ is subharmonic, also at the interior of its zero segment. So the circular mean of the contour field $f$ is at most its central value. Take $\delta = m^{-2}$ and $m \ge 8$. Then (4.5) and (4.1) together cost at most $2d\log m + 300$ in the unnormalized energy. Coincident particles give a zero Vandermonde, so no separation hypothesis is needed. A zero of $F_g$ at finite $m$ only decreases the absolute integrand. No two-sided logarithmic estimate at finite $m$ is used.

## 5. A lower bound for the continuum energy

For a measure $\sigma$ write $L_\sigma(z) = \int \log|z-w|\,d\sigma(w)$. Fix an admissible reference pair $(\eta_0,\mu_0)$ with $\eta_0 = \kappa_0 + \nu_0$. Its exact cell masses and contour weights are given in the supplementary certificate files. Define

$$\begin{aligned} Q_{\log} = &\iint \log|z-w|\,d\eta_0(z)d\eta_0(w) + \iint \log|t-s|\,d\mu_0(t)d\mu_0(s) \\ &+ \iint \log|z+t|\,d\eta_0(z)d\mu_0(t), \\ W_\eta &= A_0 - \phi - 2L_{\eta_0} - L_{\mu_0^-}, \\ W_\mu(t) &= B_0(t) - \psi(t) - 2L_{\mu_0}(t) - L_{\eta_0}(-t). \end{aligned}$$

For constants $h_\eta, h_\mu$ satisfying $W_\eta \ge h_\eta$ on $\Gamma$, put

$$P = \int_E (h_\eta - W_\eta)_+\,dx, \qquad T = \int_{1/2}^{\infty} (h_\mu - W_\mu)_+\,dt.$$

**Lemma 5.1** (Dual lower bound)**.** *For every admissible pair of measures,*

$$\alpha_\Gamma \geq C_0 + Q_{\log} + h_\eta + h_\mu - P - T. \tag{5.1}$$

*Proof.* Let $\sigma = \eta - \eta_0$ and $\tau = \mu^- - \mu_0^-$; both have total mass zero. The frequency variation is written on the reflected negative axis. Reflection preserves its self-energy, so $I(\tau) = I(\mu - \mu_0)$. The quadratic remainder after linearization is

$$I(\sigma) + I(\tau) + I(\sigma, \tau) = \tfrac{1}{2} I(\sigma) + \tfrac{1}{2} I(\tau) + \tfrac{1}{2} I(\sigma + \tau) \geq 0.$$

A signed measure of mass zero has nonnegative logarithmic energy. For smooth measures this is the Dirichlet integral identity. Finite-energy measures follow by approximation. In the frequency first variation, the change of variables from $x < 0$ to $t = -x$ gives $W_\mu(t)$. Expanding the energy and using the definitions of $W_\eta, W_\mu$ gives

$$\mathcal{E} \geq C_0 + Q_{\log} + \int W_\eta \, d\eta + \int W_\mu \, d\mu.$$

The pinned and contour masses sum to one, so the first integral is at least $h_\eta - P$. By the frequency cap the second integral is at least $h_\mu - T$. This proves (5.1). □

The reference has pinned and frequency densities in $[0, 1]$ and the required total masses. On the contour it is a nonnegative symmetric Fejér mixture. Its cell masses and Fejér weights are rational. The checker evaluates the finitely many algebraic Fourier moments by directed interval enclosures. The supplementary materials give the complete reference, the potential formulas and closed endpoint evaluations. The comparison uses no quadrature value.

Take $h_\eta = -669/200$ and $h_\mu = 71/25$. The directed-interval certificate in the supplementary materials checks the bounds on the full contour, pinned and frequency covers. It also checks the exact reference integrals:

$$P \leq \frac{6905963574153}{81920000000000}, \quad T_{[1/2,81/10]} \leq \frac{38158094390353}{409600000000000}, \quad Q_{\log} > -\frac{438334707}{10^9}.$$

The rational cell lists and the directed checker are in the supplementary archive. For the infinite frequency tail we argue by hand. For $t \geq 81/10$,

$$W'_\mu(t) \geq \log 2 - \frac{1}{2(t - 1/2)} + \frac{3}{t + 2} - \frac{2}{t - 11/2} - \frac{1}{t - 751/5000}.$$

The right side is increasing and exceeds $1/50$ at $81/10$. The checker also verifies

$$W_\mu(81/10) > 71/25 = h_\mu.$$

Thus $W_\mu$ is increasing above this endpoint, and $(h_\mu - W_\mu(t))_+ = 0$ for every $t \geq 81/10$. So the positive-part integral $T$ has no tail beyond that point. A rational upper bound for $\log 2$ in (5.1) now gives

$$\alpha_\Gamma > \frac{1992909}{10^6}. \tag{5.2}$$

## 6. A uniform estimate for finite configurations

This section passes from the exact integrand of Section 3 to the continuum energy. The estimate holds for every configuration, including arbitrarily close contour points and arbitrarily large frequencies. Let $h = 1/m$. For every selected pinned index and frequency use the cells

$$P_i = [i/m, (i + 1)/m), \qquad Q_r = [(r - 1)/m, r/m),$$

$$d\kappa = \sum_{i\in S} 1_{P_i}\,dx, \qquad d\mu = \sum_{r\in R} 1_{Q_r}\,dt. \tag{6.1}$$

These measures satisfy both density caps exactly. Their masses are $k/m$ and one. The first frequency cell starts at $1/2$, so no boundary cell is missing. Set $\nu_\delta = m^{-1}\sum \sigma_{z_{a/m},\delta}$ with $\delta = m^{-2}$.

The estimates below are elementary and one-sided in the useful direction. Their constants do not depend on the selected sets or on the largest frequency.

6.1. **Scalar weights and fixed frequency products.** The entropy bound for a binomial coefficient gives

$$\log U_i \le m\phi(i/m), \qquad \log w_r \le m\psi(r/m).$$

Now $\psi'(t) = \log(t/(t-1/2)) - \log 2$ and $\int_{1/2}^1 (\psi')_+ dt = (1/2)\log 2$. Summing endpoint-to-cell differences over any disjoint collection of cells gives

$$m\sum_{r\in R}\psi(r/m) - m^2\int \psi\,d\mu \le (m/2)\log 2. \tag{6.2}$$

The field $B_0$ is increasing, which is favorable for its negative sign. For the fixed lattice denominator, monotone quadrature gives

$$0 \le mB_0(t) - \sum_{i\in I_m}\log(t+i/m) - \sum_{j\in J_m}\log(t+j/m) \le \log(35/3), \qquad t \ge 1/2. \tag{6.3}$$

Hence the frequency one-body error is at most $m(\log(35/3) + (1/2)\log 2)$ on the entire half-line.

6.2. **Pinned denominators.** Take an interval of $N$ consecutive sites, starting at $a$ and containing $i$. Put $n_L = i - a$ and $n_R = a + N - 1 - i$. The omitted-site product satisfies

$$\log(n_L! n_R!) - (N-1)\log m - m\int_{a/m}^{(a+N)/m} \log|i/m - y|dy \ge \log\frac{m}{n_R+1}. \tag{6.4}$$

For the proof, compare monotone logarithmic sums with the integrals on each side of $i$. On the right use $\log(n_R!) \ge n_R\log n_R - n_R$ and $n_R\log(1+1/n_R) \le 1$; the case of zero is direct. The $I_m$ contribution is nonnegative. The $J_m$ contribution is at least $-\log 2$. Also $\int_E |(\phi - A_0)'| \le \log 168$, with contributions at most $\log 6$, $2\log 2$ and $\log 7$. Hence the pinned one-body cost is at most $m\log 336$.

6.3. **Real self-cells and the fixed Vandermonde.** Put

$$H(x) = \tfrac{1}{2}x^2\log|x| - \tfrac{3}{4}x^2 \quad (x \ne 0), \qquad H(0) = 0.$$

At lattice distance $d \ge 1$, let

$$J_d = \int_0^1\int_0^1 \log|d + u - v|\,du\,dv, \quad e_d = \log d - J_d \ge 0.$$

Since $H''(x) = \log|x|$, we have $J_d = H(d+1) + H(d-1) - 2H(d)$. Therefore $e_1 = 3/2 - 2\log 2$, and for $d \ge 2$

$$e_d = \sum_{j\ge 1}\frac{1}{j(2j+1)(2j+2)d^{2j}} \le \frac{1}{12d(d-1)}.$$

By telescoping and Stirling's formula, $\sum_{d\ge1} e_d = (1/2)\log(2\pi)-3/4$. Each selected lattice site has at most two other sites at a given distance. A single self-cell has log average $-\log m - 3/2$. Hence any $n$ distinct lattice sites satisfy

$$\sum_{a\neq b} \log|x_a - x_b| \le m^2 \iint \log|x-y|\, d\lambda(x) d\lambda(y) + n(\log m + \log(2\pi)), \tag{6.5}$$

where $\lambda$ is the union-of-cells measure. This uses no bound on the range of sites. The pinned and frequency interactions together cost at most $2m(\log m + \log(2\pi))$. The fixed $\Delta(I_m)^2\Delta(J_m)$ costs at most the same amount, with its leading constant as in Section 3.

6.4. **Mixed cells.** The exact ellipse clearances are

$$d_+ = \operatorname{dist}(\Gamma, [0,2]) = 751/5000, \qquad d_- = \operatorname{dist}(\Gamma, (-\infty, -1/2]) = 1749/5000.$$

They follow by minimizing the quadratic distance in $\cos u$. The left vertex is the closest point to the first interval. The mean cell displacement is $h/2$. The pinned–contour interaction costs at most $m/d_+$. The contour–frequency interaction costs at most $m/(2d_-)$, and the fixed contour denominators at most $3m/(2d_+)$. The pinned–frequency cell displacements are centered. By Taylor's inequality, with $i/m+r/m \ge 3/4$ and $h \le 1/4$, their total loss is at most $1/3$. All bounds are uniform in the contour mass. They use no contour density or separation assumption.

6.5. **Finite contour product.** For every complex $z$,

$$\log|F_g(mz)| \le m\phi(z) + b_m, \qquad b_m = 5(1+2\log 2) + 4(1+\log(g+1)). \tag{6.6}$$

The proof also covers a zero segment crossing the contour. Write the cell discrepancy as $D(w) = \log|w| - \int_0^1 \log|w-u|du$. If $|w| \le 2$, then $D(w) \le 1 + 2\log 2$. If $|w| > 2$, then $D(w) \le 2/|w|$. At most five indices have $|\Re(mz)+q| \le 2$. On both remaining sides the reciprocal sums are at most $4(1+\log(g+1))$. The denominator $g!$ has the favorable monotone quadrature sign. This proves (6.6). A zero gives $-\infty$ and is harmless. The circle comparison of Section 4 then costs at most $2m\log m$ and puts the contour particles on $T_{m^{-2}}$.

Insert these estimates into the absolute value of (3.5). Discard the favorable factors $m^{-2m-2d}$ and $\ell^k \le 1$. This gives

$$\begin{aligned}\log Q_m(S,R,Z) &\le -m^2\mathcal{E}(\kappa+\nu_\delta,\mu) + D_{\mathrm{tube}}(m),\\ D_{\mathrm{tube}}(m) &= 4m(\log m + \log(2\pi)) + 2m\log m + mb_m + \tfrac{1}{3}\\ &\quad + m\left[\log 336 + \log(35/3) + \tfrac{1}{2}\log 2 + \frac{5}{2d_+} + \frac{1}{2d_-}\right].\end{aligned}$$

The symmetrization factor is not part of $Q_m$; it enters at the integration. Since $g+1 \le m$ for $m \ge 4$, the explicit constants give

$$D_{\mathrm{tube}}(m) \le 10m\log m + c_* m + \tfrac{1}{3},$$

$$c_* = 4\log(2\pi) + 9 + \tfrac{21}{2}\log 2 + \log 336 + \log(35/3) + \frac{5}{2d_+} + \frac{1}{2d_-} < 49.979. \tag{6.7}$$

The rational-series checks in the supplementary materials prove this rounded inequality. By the tube estimate and (5.2), for every $R$ and every contour configuration,

$$\log Q_m \le -\alpha m^2 + 10m\log m + c_* m + 300 + \tfrac{1}{3}, \qquad m \ge 8, \quad 4 \mid m. \tag{6.8}$$

## 7. The full frequency sum

The absolute expansion in Section 3 has frequencies $R \subset \{g+1, g+2, \dots\}$ with $|R| = m$. As in (3.4), $E_m(H)$ is its absolute envelope restricted by $\max R \le H$. We first prove

$$E_m(\infty) \le \left(1 + \frac{3}{160} 2^{-m}\right) E_m(20m). \tag{7.1}$$

For fixed pinned and contour variables, the $R$-dependent factor is

$$f(R) = \frac{\prod_{r\in R} w_r \, \Delta(R)^2 \prod_{r\in R, z\in Z} |r+z|}{\prod_{r\in R, i\in I_m} (r+i) \prod_{r\in R, j\in J_m\setminus S} (r+j)}.$$

Let $r = \max R > 20m$. Replace $r$ by the least unused integer $s \in \{g+1, \dots, g+m\}$, and call the result $R'$. Then $s \le 3m/2$, and for fixed $R', r$ there are at most $m$ preimages. Since at most two integers have any given distance from $s$,

$$\prod_{u\in R\setminus\{r\}} |s-u| \ge d_m := \left\lfloor \frac{m-1}{2} \right\rfloor ! \left\lceil \frac{m-1}{2} \right\rceil ! \ge e^{-1} \left(\frac{m}{2e}\right)^{m-1}.$$

Using $s+i \le 11m/4$, $s+j \le 7m/2$, and the ellipse bounds $\Re z > -m/6$, $|z| < 5m/2$ gives

$$\frac{f(R)}{f(R')} \le \frac{w_r}{w_s} (77/8)^m (189/16)^{m-|S|} \frac{m^{2m}}{r^2 d_m^2} \le 12500^m \frac{m^2}{4r^2} w_r.$$

The last inequality uses $w_s \ge 2^{-3m/2}$ and $e < 3$. These estimates include both frequency-dependent minors and all contour–frequency factors.

For a finite cutoff $M \ge 20m$, let $Z_M$ be the sum of $f(R)$ over $R \subset (g, M]$. Put

$$b_m = \frac{m}{1600} 12500^m \sum_{r>20m} w_r.$$

The replacement map gives $Z_M - Z_{20m} \le b_m Z_M$. Since

$$\sum_{r>g} w_r t^r = \left(\frac{t}{2-t}\right)^{g+1},$$

evaluation at $t = 15/8$ and $200000 \cdot 8^{20} < 15^{20}$ give

$$\begin{aligned} b_m &\le \frac{15m}{1600} \left[50000 (8/15)^{20}\right]^m \\ &< \frac{3}{320} 2^{-m} < \frac{1}{2}. \end{aligned}$$

Hence $Z_M \le (1 + (3/160)2^{-m}) Z_{20m}$. Monotone convergence proves (7.1).

There are $2^m$ pinned subsets and at most $\binom{39m/2}{m} \le (39e/2)^m$ retained frequency subsets. The normalized contour length is at most $mA$, where $A = 12783/10000$, and $(mA)^d/d! \le e^{mA}$. These counts, (6.8) and (7.1) give

$$\begin{aligned} \log |G_m(\beta)| \le & -\alpha m^2 + 10 m \log m + (c_* + \log 39 + 1 + A) m \\ & + 300 + \tfrac{1}{3} + \log\left(1 + \frac{3}{160} 2^{-m}\right). \end{aligned}$$

Here $c_* < 49.979$ and $c_* + \log 39 + 1 + A < 56$. The constant terms are less than 301. This proves (1.2).

## 8. Clearing denominators

This section proves (1.3) for all $m \in 4\mathbb{N}$. For $F \in \mathbb{Q}[T]$, the Gauss valuation $v_p(F)$ is the minimum valuation of its coefficients. Set $v_p(0) = +\infty$.

8.1. **Denominators and a formal identity in the parameter.** Let $a$ be a variable for now. Put $c_i = \binom{i+g}{g}$, $U_i = c_i a^i$, $h_i = \sum_{s=1}^{g} 1/(i+s)$, and $Q_s(a) = (a/(a-1))^s - 1$. In Section 2 replace $2^i$ by $a^i$ and $2^s - 1$ by $Q_s(a)$. This gives the rational coefficient functions of $D_g M$ in Lemma 2.2. They follow from Leibniz's rule and

$$L^{(s)} = (-1)^{s-1}(s-1)!\big((a-1)^{-s} - a^{-s}\big),$$
$$\beta^{(s)} = (-1)^{s-1}(s-1)!a^{-s}\left(L - \sum_{r=1}^{s-1} \frac{Q_r(a)}{r}\right).$$

Put $U = 2m-1$ and $d_U = \operatorname{lcm}(1, \dots, U)$. The identity

$$c_i h_i = \sum_{s=1}^{g} \frac{(-1)^{s-1}}{s} \binom{i+g}{g-s} \tag{8.1}$$

is the derivative at zero of the Vandermonde identity $\binom{i+g+x}{g} = \sum_{s=0}^{g} \binom{i+g}{g-s}\binom{x}{s}$. So the harmonic denominators need only the indices $s \le g \le U$. The finite sums have denominators cleared by $d_U$ in $C$ and by $d_U^2$ in $B$. The entries of $B$ have pole order at most $g$ at $a = 1$. Scaling rows gives

$$P_m(a,T) := d_U^{3m}(a-1)^{gm} G_m(a,T) \in \mathbb{Z}[a,T]. \tag{8.2}$$

Here $G_m(a,T)$ is the determinant with parameter $a$. The rest of the paper uses $G_m(2,T)$. Its finite rational formulas contain no negative powers of $a$.

Work formally in $\mathbb{Q}[[a]]$, with $L_1 = \sum_{r\ge1} a^r/r$, $L_2 = \sum_{r\ge1} a^r/r^2$, and

$$\gamma_i = \tfrac{1}{2}\left(h_i^2 - \sum_{s=1}^{g} (i+s)^{-2}\right).$$

For all $g, i \ge 0$ the rational functions satisfy

$$t_i^{(g)} - L_1 U_i = \sum_{k \ne i} \frac{c_k a^k}{i-k} - U_i h_i, \tag{8.3}$$

$$b_i^{(g)} + L_1 U_i h_i + L_2 U_i = \sum_{k \ne i} \frac{c_k a^k}{(i-k)^2} + \gamma_i U_i. \tag{8.4}$$

Here and below sums over $k$ start at zero. To prove these uniformly in $g$, adjoin to $\mathbb{Q}((a))$ a formal symbol $\lambda$ with $\lambda' = 1/a$. Differentiating $D_g a^x = \binom{x+g}{g} a^x$ in the auxiliary exponent gives

$$D_g(a^i\lambda) = U_i(\lambda + h_i), \quad D_g(a^i\lambda^2) = U_i(\lambda^2 + 2h_i\lambda + 2\gamma_i).$$

The substitutes $\bar{L} = -L_1 - \lambda$ and $\bar{\beta} = -L_2 - \lambda^2/2$ obey the differential equations of $L$ and $\beta$: $\bar{L}' = 1/(a(a-1))$ and $\bar{\beta}' = \bar{L}/a$. The base formal identities $t_i^0 - L_1 a^i = \sum_{k\ne i} a^k/(i-k)$ and $b_i^0 + L_2 a^i = \sum_{k\ne i} a^k/(i-k)^2$ hold coefficientwise. Apply $D_g$; it multiplies each $a^k$ coefficient by $c_k$. Then compare the terms free of the independent symbol $\lambda$. This proves (8.3) and (8.4), including the coefficient at

$k = i$. The formula for $\gamma_i$ also follows from the coefficient of $x^2$ in the same Vandermonde identity.

8.2. **Rank-one features and specialization at 2.** For an independent variable $X$ form

$$\mathcal{H}(a, X) = \begin{bmatrix} C(a) \\ B(a) - L_1 C(a) + L_2 A(a) - X A(a) \end{bmatrix}.$$

Fix a weight $k \geq 0$. On $I_m$ put $e_I(i) = [i = k]$ and $w_I(i) = v_I(i)^2$, where $v_I(i) = 1/(i-k)$ for $i \neq k$ and $v_I$ is zero otherwise. Define $e_J, v_J, w_J$ on $J_m$ in the same way. By (8.3) and (8.4),

(8.5)
$$[a^k]\mathcal{H} = c_k \left( \begin{bmatrix} e_I \\ v_I - h_k e_I \end{bmatrix} v_J^t + \begin{bmatrix} 0 \\ -e_I \end{bmatrix} w_J^t + \begin{bmatrix} v_I - h_k e_I \\ -w_I - h_k v_I + (\gamma_k - X) e_I \end{bmatrix} e_J^t \right).$$

The second rank-one summand vanishes unless $k \in I_m$, and the third unless $k \in J_m$. So weight $k$ has at most $1 + [k \in I_m] + [k \in J_m]$ features. A nonzero determinant term must choose $2m$ distinct features. For a fixed coefficient of $a$ only finitely many features matter. So finite multilinearity proves this statement formally.

For $m = 4h$, the cheapest $8h$ features are two copies of $0, \ldots, h-1$ and three of $h, \ldots, 3h-1$. Their total weight is

$$N_m = 2 \sum_{k=0}^{h-1} k + 3 \sum_{k=h}^{3h-1} k = 13h^2 - 4h = \frac{13}{16} m^2 - m. \tag{8.6}$$

The determinant-preserving row shear gives $\det \mathcal{H}(a, X) = G_m(a, X - L_2)$. Hence the right side is divisible by $a^{N_m}$ in $\mathbb{Q}[[a]][X]$. The substitution $X = T + L_2$ is polynomial in $X$, so it preserves divisibility. With the integer polynomial (8.2) this gives

$$d_U^{3m} (a-1)^{gm} G_m(a, T) \in a^{N_m} \mathbb{Z}[a, T]. \tag{8.7}$$

Only now set $a = 2$. The factor $(a-1)^{gm}$ is one, so

$$v_2(G_m) \geq N_m - 3m \lfloor \log_2 U \rfloor. \tag{8.8}$$

No value of $L_1(2)$ or $L_2(2)$ is used. The formal translation is undone before the integer specialization. The supplementary reports test finitely many formal coefficients. These tests only corroborate the argument, which holds for every $g$.

8.3. **Valuation bounds at odd primes.** Let $p$ be an odd prime with $p^2 > U + g$. Put

$$r = g \bmod p, \quad \kappa_n = [n \bmod p \geq p - r], \qquad \tau_n = (\kappa_n - 1)[n \geq p \text{ or } p \leq g].$$

When $r = 0$, $\kappa_n = 0$. Since $n + g < p^2$, Kummer's carry criterion gives $v_p\binom{n+g}{g} = \kappa_n$. If $p$ divides a denominator index $s$, then $v_p\binom{n+g}{g-s} \geq \kappa_n$; if $s \leq n$, then $v_p\binom{n-s+g}{g} = \kappa_n$. All nonzero denominator indices and index differences have valuation at most one. The ultrametric inequality, applied to the formulas of Section 2, gives the following upper bounds for negative Gauss valuations.

| Entry | Upper bound |
|---|---|
| $C_{ij}$, $i \neq j$ | $[i \equiv j \pmod p] - \min(\kappa_i, \kappa_j)$ |
| $C_{ii}$ | $(1-\kappa_i)[p \leq g]$ |
| $B_{ij}$, $i \neq j$ | $[i \equiv j \pmod p] - \min(\tau_i, \tau_j)$ |
| $B_{ii} - TU_i$, $i \geq p$ or $2p \leq g$ | $2 - \kappa_i$ |
| $B_{ii} - TU_i$, $i < p$ and $p \leq g < 2p$ | 1 |
| $B_{ii} - TU_i$, $i < p$ and $p > g$ | 0 |

For the last three lines, the first diagonal sum has a squared $p$ denominator only if $i \geq p$. Both denominators in its double sum are divisible by $p$ only if $2p \leq g$, and then the binomial gains $\kappa_i$. A single $p$ denominator costs at most one. The coefficient $-TU_i$ has nonnegative valuation and does not worsen the bounds.

Let $W_m(p)$ be the maximum weight sum over permutations. By Leibniz, $-v_p(G_m) \leq W_m(p)$. Integer row and column potentials whose sums dominate every entry weight also give an exponent, namely their total. No assignment optimality or determinant rank theorem is needed. The coarse bound is $W_m(p) \leq 3m$.

8.4. **Small primes.** Let $p < g$. After enlarging the special diagonal bounds if needed, take $w_C(i,j) = [i \equiv j \pmod p] - \kappa_i\kappa_j$ and $w_B = 1 + w_C$. Let $c_I(v), c_J(v)$ be the residue copy counts and $K_I, K_J$ the killed index counts. Put

$$S_U = \sum_{\kappa_v=0} \min(2c_I(v), c_J(v)), \quad S_K = \sum_{\kappa_v=1} \min(2c_I(v), c_J(v)).$$

Then

$$W_m(p) \leq \min\{m + S_U,\ 3m - 2K_I - K_J + S_K\}. \tag{8.9}$$

For the proof, subtract the bottom-row constant $m$. The score is then $x - z$, where $x$ counts unkilled same-residue matches and $z$ killed off-residue matches. Let $y$ count killed same-residue matches. Then $x \leq S_U$ and $y \leq S_K$. Counting the killed rows and columns gives $z + y \geq 2K_I + K_J - 2m + x$. Both bounds follow.

For $\rho = p/m$ and $t = v/m \in [0, \rho)$, the copy counts are

$$c_I(t) = \#\{n : 1/4 \leq t + n\rho < 5/4\}, \quad c_J(t) = \#\{n : 0 \leq t + n\rho < 2\}.$$

They change only at the six endpoints

$$0, \rho, \quad a - \lfloor a/\rho \rfloor \rho \ (a = 1/4, 5/4, 2), \quad -1/2 + (\lfloor 1/(2\rho) \rfloor + 1)\rho. \tag{8.10}$$

The last is the killed threshold. Every endpoint is $b + n\rho$ with integer $n$ and $b \in \{0, 1/4, 5/4, 2, -1/2\}$. So all walls lie among the numbers $|b-c|/n$, $n \geq 1$. Each fixed positive strip contains finitely many of them. Within a wall cell the objectives in (8.9) are affine. We integrate their lower envelope after also splitting at the affine crossings. Each endpoint in (8.10), multiplied by $m$, is an integer. With half-open conventions the residue counts are therefore exactly $m$ times the cell widths, with the correct boundary assignments. No asymptotic equality to the true assignment profile is used.

Let $\delta = 1/128$. The 756 affine pieces in the certificate specify the exact integral $L_*$ on $[\delta, 1/2]$. Its value is the following numerator divided by the denominator.

144827631474019496485526010080046212726029803104279674952972562737001039673489563365828533335045090570444302957100283871023 7

/

140409888158080315725937691852162573583946167555387343629721765266221616278295704359396606004685482615097817667301523622400 0

Near zero the row potentials $1-\kappa_i$ and $2-\kappa_i$, with zero column potentials, are feasible. They give $W_m(p) \le 3m - 2K_I$. Any interval of $m$ integers contains at least $\lfloor m/p \rfloor (g \bmod p)$ killed indices. Hence

$$W_m(p)/m \le 3 - 2\{1/(2\rho)\} + 2\rho. \tag{8.11}$$

Write $\delta = 1/(2Q)$, $Q = 64$. The substitution $x = 1/(2\rho)$ gives

$$\int_0^\delta (3 - 2\{1/(2\rho)\})d\rho = \frac{1}{2Q} + \sum_{q \ge Q} \left( \frac{1}{q} - \log(1 + 1/q) \right).$$

The summand exceeds $1/(2q(q+1))$ by exactly $\int_0^1 s(1-s)/(q(q+s)(q+1))\,ds \le 1/(6q^2(q+1))$. Summing, and adding the integral of $2\rho$, gives the rigorous upper bound

$$I_{\text{small}} \le 2\delta + 2\delta^2 = 129/8192. \tag{8.12}$$

8.5. **Large primes and the integerizer.** For $\rho > 1/2$, keep each row copy in both blocks and every column copy. In a residue cell with flag $\kappa$, copy digit $t$ has $\tau = (\kappa - 1)[t \ge 1]$. The exact data give an integer potential to each row and column type. Every pair of types must satisfy the off-residue bound of Section 8.3. A pair in the same residue cell must also satisfy the same-residue bound, including the diagonal case. Checking both cases also covers two distinct residues in the same cell. The objective is the sum of potentials times the exact cell counts.

The exact rational verifier in the supplementary materials reconstructs the complete wall geometry. It checks all 756 low pieces and all 1,942 high integer inequalities. It gives the following high upper profile.

| Interval in $\rho$ | Upper profile |
|---|---|
| $(1/2, 7/12)$ | $1$ |
| $(7/12, 5/8)$ | $3\rho - 3/4$ |
| $(5/8, 2/3)$ | $\rho + 1/2$ |
| $(2/3, 3/4)$ | $4\rho - 3/2$ |
| $(3/4, 5/6)$ | $3\rho - 3/4$ |
| $(5/6, 1)$ | $7/4$ |
| $(1, 7/4)$ | $15/4 - 2\rho$ |
| $(7/4, 2)$ | $2 - \rho$ |
| $(2, 5/2)$ | $0$ |

Its integral is 287/192. The copy geometry splits this table into 11 cells. The data list the individual types and integer potentials. No optimizer or numerical rounding is used. On each wall in a positive strip use $3m$ instead. This costs $O(m \log m)$ in total and covers $\rho = 1/2$.

Choose the exponents $E_p$ in the following order.

(1) At 2 take $E_2 = 3m\lfloor \log_2 U \rfloor - N_m$.
(2) At odd $p^2 \le U + g$ use $3m\lfloor \log_p U \rfloor$.
(3) At $p > U$ use zero.
(4) At the remaining $p/m < \delta$ use $3m - 2K_I$.

(5) At a wall in a positive strip use $3m$.
(6) Elsewhere below $m/2$ use (8.9).
(7) Elsewhere above $m/2$ use the least feasible high dual objective.

The positive rational product

$$D_m = \prod_p p^{E_p} \tag{8.13}$$

is finite, and every exponent bounds $-v_p(G_m)$. So $D_m$ integerizes every coefficient. A negative $E_2$ only divides out a proved common power of two.

The positive 2-adic overhead is $O(m \log m)$. Small odd primes cost at most $3m\,\pi(\sqrt{U+g}) \log U = O(m^{3/2} \log m) = o(m^2)$. On each fixed positive strip we use the classical weighted prime number theorem

$$\frac{1}{m} \sum_p f(p/m) \log p \longrightarrow \int f(\rho) d\rho$$

for a bounded finitely piecewise smooth function $f$. For (8.11), first omit $p < \varepsilon m$. Then apply the theorem on $[\varepsilon, \delta]$, and let $\varepsilon \downarrow 0$. By $3m\theta(\varepsilon m)$, the omitted part has normalized limsup at most $3\varepsilon$. The exact counts on positive strips and the finite wall cost now prove

$$\limsup \frac{\log D_m}{m^2} \le L_* + \frac{129}{8192} + \frac{287}{192} - \frac{13}{16} \log 2.$$

Finally let $\ell_- = 2 \sum_{k=0}^{29} (1/3)^{2k+1}/(2k+1) < \log 2$. The same verifier checks the exact rational comparison

$$\begin{aligned} \tau_{\text{basic}} &:= L_* + \frac{129}{8192} + \frac{287}{192} - \frac{13}{16} \ell_- \\ &= 1.97881984510269917464907037626379\ldots < 1.979. \end{aligned}$$

The decimal is the exact rational of the receipt, rounded. This proves (1.3), and in particular the weaker rate $\tau \le 1.99$.

## 9. Nonvanishing on a prime-power sequence

Fix $M \in 4\mathbb{N}$, set $h = M/2$, and choose an odd prime $p > 2M + h$. For every $k \ge 0$ put $q = p^k$, $m = Mq$, $g = hq$. The scaled pencil satisfies

$$\begin{aligned} &\mathcal{H}_k(X) := q^{3m} G_m(X/q^2) = \det \begin{bmatrix} qC^{(g)} \\ q^2 B^{(g)} - X A^{(g)} \end{bmatrix} \in \mathbb{Z}_{(p)}[X], \\ &\overline{\mathcal{H}_k}(X) = \overline{G_M}(X)^q. \end{aligned} \tag{9.1}$$

Every nonzero denominator in the formulas of Section 2 has absolute value less than $(2M + h)q < pq$. So $q/d$ is $p$-integral. It reduces to zero unless $q \mid d$, and to $e^{-1}$ at $d = eq$, $0 < |e| < p$. Likewise $q^2/(d_1 d_2)$ vanishes on reduction unless both denominators are divisible by $q$.

Write $i = aq + v$, $j = bq + w$, $0 \le v, w < q$. Lucas and Frobenius give

$$\binom{(a+h)q + v}{hq} \equiv \binom{a+h}{h}, \qquad U^{(hq)}_{aq+v} \equiv 2^v U^{(h)}_a \pmod p. \tag{9.2}$$

Multiply by $q$. In the first sum in $t_i$, only the denominator indices $dq$ with $1 \le d \le a$ survive. In the second, only $dq$ with $1 \le d \le h$ survive. Since $q$ is odd, parity is preserved. Also $2^{dq} - 1 \equiv 2^d - 1 \pmod p$, and Lucas reduces the binomial to the

seed one. Therefore $qt_{aq+v}^{(hq)} \equiv 2^v t_a^{(h)} \pmod p$. If $v \neq w$, then $q \nmid j-i$, so both $qC_{ij}$ and $q^2B_{ij}$ reduce to zero. If $v = w$ and $a \neq b$, the divided differences reduce to $2^v$ times their seed entries.

On the diagonal, the multiples of $q$ in $[i+1, i+hq]$ are exactly $(a+1)q, \dots, (a+h)q$. This holds even when $v = 0$ or $q-1$. Thus $qC_{ii} \equiv 2^v C_{aa}^{(h)}$. In the first diagonal $B$ sum only the squared denominators $(dq)^2$ survive. In the double sum both indices must be multiples of $q$, with $2 \le d \le h$, $1 \le e < d$. Lucas, parity and Frobenius reduce these terms to the seed double sum. Hence $q^2B_{ii} \equiv 2^v B_{aa}^{(h)}$ and $A_{ii} \equiv 2^v A_{aa}^{(h)}$. All scaled entries are integral before reduction. The argument holds for each $k$ directly, without a recurrence in $k$.

The window endpoints are multiples of $q$. Grouping rows and columns by residue gives, in each residue block, the seed pencil times $2^v$. The row and column permutations are the same transpose permutation of a $2M \times q$ array, so their signs cancel. The product of the scalar block determinants is $2^{Mq(q-1)} \equiv 1 \pmod p$, because $p-1 \mid q-1$. This proves (9.1), including the exact scalar and sign.

The modular data in the supplementary materials show that for $M = 32$, $p = 83$ the matrix of Section 2 satisfies

$$\det \begin{bmatrix} C^{(16)} \\ B^{(16)} \end{bmatrix} \equiv 30 \pmod{83}. \tag{9.3}$$

This is a finite $64 \times 64$ matrix check. The saved entries, row swaps and unnormalized Gaussian pivots determine the determinant by integer modular arithmetic. The code also rebuilds the matrix with exact rationals, and independently by a modular Pascal recurrence. No larger determinant is computed.

Let $M = 32, p = 83$, and fix $t \in \mathbb{Q}_{83}$ with $2k + v_{83}(t) > 0$. Then $X = q^2 t$ lies in $83\mathbb{Z}_{83}$. By (9.1) and (9.3),

(9.4)

$$q^{3m} G_m(t) = \mathcal{H}_k(q^2 t) \equiv 30^q = 30 \pmod{83}, \qquad v_{83}(G_{32 \cdot 83^k}(t)) = -3k(32 \cdot 83^k).$$

This proves (1.4). For $t = a/b$ in lowest terms, $k \ge \lfloor v_{83}(b)/2 \rfloor + 1$ suffices; $v_p(0) = +\infty$ covers zero. The sequence does not depend on $t$, and every index lies in $32\mathbb{N}$.

## 10. Proof of Theorem 1

Suppose $\beta = a/b$ with coprime integers $a, b$ and $b > 0$. Along $m = 32 \cdot 83^k$, the condition in (1.4) holds for all large $k$. Since $\deg G_m \le m$ and $D_m G_m \in \mathbb{Z}[T]$,

$$Z_m = b^m D_m G_m(a/b) \in \mathbb{Z} \setminus \{0\}.$$

For large $m$, (1.3) gives $\log D_m \le 1.979m^2$. By (1.2),

$$\log |Z_m| \le -0.013909m^2 + 10m \log m + (56 + \log b)m + 301 \longrightarrow -\infty.$$

This contradicts $|Z_m| \ge 1$. □

## Appendix A. Computer checks

The proof uses three finite computations.

(1) Directed rational-interval bounds for the dual certificate of Section 5.
(2) Exact rational reconstruction of the residue cells and integer potentials of Section 8.
(3) The determinant of the $64 \times 64$ seed matrix modulo 83 in Section 9.

The supplementary archive contains the finite data, the checking programs, the environment details and the replay outputs. Its SHA-256 and the replay commands are listed in `SUPPLEMENTARY.md`. The full replay takes about one minute on one core of an aarch64 Linux server.

## Use of AI tools

The author devised the method of proof. Codex (OpenAI, GPT-6 Astra and GPT-6 Luna) searched for the parameters in (1.1) and wrote the Python code and the certificates. Claude Code (Anthropic, Claude Opus 5.5) checked the code, the parameters and the certificates. The author checked every step of the proof, every table and every reference, and takes full responsibility for the content of the paper.

## Statements and declarations

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**Data availability.** The exact data and the verification scripts are in the supplementary material of this paper.

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