THE GOLDBACH EXCEPTIONAL SET: AN EXPONENT 11/20

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Abstract. We prove $E(X) \ll_\varepsilon X^{11/20+\varepsilon}$ for every $\varepsilon > 0$. Here $E(X)$ counts the positive even integers at most X that are not sums of two primes. A four-level Harman decomposition gives a signed prime minorant. Exact rational bounds for four discarded families give a positive lower bound for its logarithmically weighted mass. We prove a Bombieri–Vinogradov estimate at level $X^{9/20-\eta}$, with both maxima and a divisor weight. We also bound the exponential sum of the same weight on the minor arcs. The major arcs use classical zero-density estimates and the zero-free half-plane $\Re s > 7/8$ of OpenAI [6, Theorem 1.1].

1. Introduction

Let $\mathbb{P}$ denote the set of primes, and put

$$E(X) = \#\{n \in \mathbb{N} : n \le X,\ 2 \mid n,\ n \ne p + p' \text{ for all } p, p' \in \mathbb{P}\}. \tag{1.1}$$

The two primes need not be distinct.

Theorem 1. *For every $\varepsilon > 0$,*

$$E(X) \ll_\varepsilon X^{11/20+\varepsilon}. \tag{1.2}$$

Pintz and Althöfer obtain the exponent 7/12 from the zero-free theorem (F1) below, and Pintz's method gives 3/5 under $H_{7/8}$. Theorem 1 improves both exponents:

$$\frac{11}{20} < \frac{7}{12} < \frac{3}{5}.$$

The hypothesis and the precise citations are stated below. The 7/8 zero-free theorem is an external input and is not reproved here.

1.1. The zero-free hypothesis.

Hypothesis $H_{7/8}$. For every Dirichlet character χ,

$$L(s, \chi) \ne 0 \qquad (\Re s > 7/8). \tag{$H_{7/8}$}$$

This includes the character modulo 1, whose L-function is $\zeta(s)$. The functional equation then gives

$$\frac{1}{8} \le \beta \le \frac{7}{8} \tag{1.3}$$

for every nontrivial zero $\varrho = \beta + i\gamma$. An imprimitive character has the same nontrivial zeros as its inducing primitive character.

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Theorem 2. *Assume $H_{7/8}$. For every $\varepsilon > 0$,*

$$E(X) \ll_\varepsilon X^{11/20+\varepsilon}. \tag{1.4}$$

(F1) [6, Theorem 1.1] Every Dirichlet L-function, including $\zeta(s)$, is zero-free in $\Re s > 7/8$.

(F2) [8, Theorem 1 and Corollary 2] Theorem 1 assumes a common zero-free half-plane H_{θ_0}. Corollary 2 invokes [6, Theorem 1.1] and gives the exponent $7/12$ without a hypothesis in its statement. The exponent obtained from Pintz's method under $H_{7/8}$ is $3/5$; see [7, Sections 6–8].

Theorem 1 follows from Theorem 2 and (F1); the short proof is in Section 11.

1.2. **The method.** We construct a signed minorant on $I = (X/8, X/4]$. Every discarded term is nonnegative, and every negative term is retained or expanded. The loss of mass comes from four discarded families, and their rational upper bounds give

$$\sum_{m\in I} (\log m)\rho(m) \ge \Big(\frac{c_{\mathrm{cert}}}{8} - o(1)\Big) X, \qquad c_{\mathrm{cert}} = \frac{342155430899}{2250000000000} > \frac{3}{20}. \tag{1.5}$$

Only a lower bound is asserted.

Put $\lambda(m) = (\log m)\rho(m)$; this weight is used throughout. Its distribution estimate keeps both maxima inside the modulus sum and includes the factor $2^{\omega(d)}$. The minor-arc estimate is

$$\sup_{\alpha\in\mathfrak{m}} |S_\lambda(\alpha)| \ll X^{31/40+\eta/2} L^{9/2}.$$

Under $H_{7/8}$ the major arcs contribute at least $0.025X$, and the resulting dyadic exceptional-set bound is

$$|\mathcal{E}_X| \ll \frac{\mathcal{S}^2 L}{X} \ll X^{31/20+\eta-1} L^{10} = X^{11/20+\eta} L^{10} = X^{1-t} L^{10}.$$

The major-arc argument follows [7, Section 8]. We derive the required mean square from zero-density estimates and do not use [7, (8.25)]. Heath-Brown's $20/9$ estimate limits this argument to levels $t < 9/20$. The mass, distribution and minor-arc estimates are all unconditional.

2. Notation and parameters

Write $L = \log X$ and $e(x) = e^{2\pi i x}$. Given $\varepsilon > 0$, choose

$$\begin{gathered} \eta = \min(\varepsilon/2, 1/100), \qquad T = \frac{9}{20}, \qquad t = \frac{9}{20} - \eta, \\ \varepsilon_0 = \frac{\eta}{10}, \qquad X_1 = X^{1-\varepsilon_0}, \qquad \varepsilon_{\mathrm{HB}} = \frac{\eta}{90}. \end{gathered} \tag{2.1}$$

The decomposition level T is fixed, and the analysis level is t. Put

$$z = X^{1/10}, \qquad A = X^{9/20}, \qquad D = X^{11/20}, \qquad I = (X/8, X/4]. \tag{2.2}$$

Then

$$Az = D, \qquad AD = X, \qquad z^4 < A < z^5, \qquad z^{10} = X. \tag{2.3}$$

Let P be any integer satisfying

$$\frac{1}{2}X^t \le P \le X^t. \tag{2.4}$$

All estimates are uniform in this choice. We keep the three parameters

$$Q_P = X/P, \qquad Q_B = X^t, \qquad Q_0 = \lceil 2X/P \rceil \tag{2.5}$$

distinct; they govern arc widths, distribution and Dirichlet approximation. The choice of ε_0 gives

$$1 - \varepsilon_0 - \frac{20}{9}t = \frac{191}{90}\eta > 0. \tag{2.6}$$

For large X,

$$\begin{aligned} 2P^2 < X, \qquad D \le X/P \le 2X^{11/20+\eta}, \\ P < X^{1/2} < X_1 \le X/8, \qquad A^{3/2} = X^{27/40}. \end{aligned} \tag{2.7}$$

Implied constants may depend on fixed saving exponents and on η. They are uniform in all variables within the stated ranges.

Let $P^-(v)$ be the least prime factor of v, with $P^-(1) = +\infty$, and put

$$\psi(v,w) = \mathbf{1}_{P^-(v)\ge w}.$$

Thus $\psi(1,w) = 1$, and we call v with $\psi(v,w) = 1$ *w-rough*. The function $\Omega(v)$ counts prime factors with multiplicity. In $2^{\omega(d)}$ the function $\omega(d)$ counts distinct prime factors; elsewhere $\omega(u)$ denotes Buchstab's function:

$$\omega(u) = 1/u \quad (1 \le u \le 2), \qquad (u\omega(u))' = \omega(u-1) \quad (u > 2). \tag{2.8}$$

Buchstab's function is continuous on $[1,\infty)$.

For $f : \mathbb{N} \to \mathbb{C}$, put

$$\Delta_f(d) = \max_{(a,d)=1} \max_{y \le X} \left| \sum_{\substack{m \le y \\ m \equiv a \pmod d}} f(m) - \frac{1}{\varphi(d)} \sum_{\substack{m \le y \\ (m,d)=1}} f(m) \right|. \tag{2.9}$$

Both maxima are part of this definition, and subadditivity and cutoff monotonicity give

$$\Delta_{f\mathbf{1}_{[1,Y]}} \le \Delta_f, \qquad \Delta_{f\mathbf{1}_{(Y_1,Y_2]}} \le 2\Delta_f \tag{2.10}$$

for endpoints at most X.

Let $\|x\|$ denote the distance to the nearest integer, and define

$$\begin{aligned} \mathfrak{M}(q,a) &= \{\alpha \in \mathbb{R}/\mathbb{Z} : \|\alpha - a/q\| \le 1/(qQ_P)\}, \\ \mathfrak{M} &= \bigcup_{q \le P} \bigcup_{\substack{a \bmod q \\ (a,q)=1}} \mathfrak{M}(q,a), \qquad \mathfrak{m} = (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}. \end{aligned} \tag{2.11}$$

These arcs are disjoint, since distinct reduced fractions are at distance at least $1/(qq')$ and $q + q' \le 2P < Q_P$. Once λ is defined, put

$$S(\alpha) = \sum_{X_1 < p \le X} (\log p) e(p\alpha), \qquad S_\lambda(\alpha) = \sum_{m \in I} \lambda(m) e(m\alpha). \tag{2.12}$$

For $\mathfrak{B} = \mathfrak{M}$ or $\mathfrak{m}$, write

$$I_{\mathfrak{B}}(n) = \int_{\mathfrak{B}} S(\alpha) S_\lambda(\alpha) e(-n\alpha)\, d\alpha.$$

Finally,

$$c_q(v) = \sum_{\substack{a \bmod q \\ (a,q)=1}} e(av/q).$$

A star on a character sum restricts it to primitive characters.

3. The four-level Harman weight

3.1. **Two exact identities.**

Lemma 3.1. *For every integer $m > z^2$,*

$$\mathbf{1}_{\mathbb{P}}(m) = \psi(m,z) - \sum_{\substack{p \mid m \\ p \ge z,\ p^2 \le m}} \psi(m/p, p). \tag{3.1}$$

For $k \ge 1$ and $z \le y$,

$$\psi(k,y) = \psi(k,z) - \sum_{\substack{p \mid k \\ z \le p < y}} \psi(k/p, p). \tag{3.2}$$

Proof. If m is prime, the sum in (3.1) is empty. If m is composite and z-rough, only $p = P^-(m)$ contributes, and this prime satisfies $p^2 \le m$. If $P^-(m) < z$, every term on the right vanishes; this case includes prime powers.

For (3.2), the case $k = 1$ is immediate. Otherwise put $q = P^-(k)$. For $q \ge y$ the sum is empty, and for $z \le q < y$ only $p = q$ contributes. For $q < z$, every term vanishes. □

Both identities hold when their left sides vanish, so we expand zero-valued terms as well.

3.2. **The stopping rule.**

For $m \in I$, a node of depth j consists of primes

$$p_1 > p_2 > \cdots > p_j \ge z, \qquad P_j := p_1 \cdots p_j \mid m, \qquad p_1^2 \le m.$$

It carries $(-1)^j \psi(m/P_j, p_j)$, and expanding it by (3.2) gives

$$\begin{aligned}(-1)^j \psi(m/P_j, p_j) &= (-1)^j \psi(m/P_j, z) \\ &\quad + \sum_{\substack{p_{j+1} \mid m/P_j \\ z \le p_{j+1} < p_j}} (-1)^{j+1} \psi(m/P_{j+1}, p_{j+1}).\end{aligned} \tag{3.3}$$

The first term is a retained rough leaf, denoted R_j, and the root is $R_0(m) = \psi(m,z)$.

At depth 1, stop when $A \le p_1 \le D$, which gives the leaf B_1, and expand when $p_1 < A$. At depth 2, stop when $A \le P_2 \le D$, which gives B_2. Discard the nodes with $P_2 > D$, which form U_2, and expand every node with $P_2 < A$. At depth 3, stop when $A \le P_3 \le D$, giving B_3, and expand both ranges $P_3 < A$ and $P_3 > D$. At depth 4, stop when $A \le P_4 \le D$, giving B_4. Discard all other nodes; they form L_4, U_4 and V_4 as specified below.

Every depth-1 node has $p_1 \le \sqrt{X/4} < D$. Every depth-3 node has $P_2 < A$, and therefore

$$p_3 < p_2 < P_2^{1/2} < A^{1/2}, \qquad P_3 < A^{3/2} = X^{27/40}. \tag{3.4}$$

If $P_3 > D$, then every child has $P_4 > D$, so B_4 and L_4 can arise only from $P_3 < A$. There are no nodes of depth 5.

Table 3.1 defines every family. Each entry is summed over decreasing primes at least z with $P_j \mid m$. The condition $p_1^2 \le m$ is imposed in every nonempty row. For $p_1 < A$, this condition is automatic on I:

$$p_1^2 < X^{9/10} < X/8 < m \tag{3.5}$$

for $X > 8^{10}$.

TABLE 3.1. Leaves of the decomposition.

Family	Prefix conditions	Residual	Sign
R_0	Empty prefix	$\psi(m, z)$	$+$
R_1	$p_1 < A$	$\psi(m/p_1, z)$	$-$
R_2	$p_1 < A$, $P_2 < A$	$\psi(m/P_2, z)$	$+$
R_3	$p_1 < A$, $P_2 < A$, and $P_3 < A$ or $P_3 > D$	$\psi(m/P_3, z)$	$-$
B_1	$A \le p_1 \le D$	$\psi(m/p_1, p_1)$	$-$
B_2	$p_1 < A$, $A \le P_2 \le D$	$\psi(m/P_2, p_2)$	$+$
B_3	$p_1 < A$, $P_2 < A$, $A \le P_3 \le D$	$\psi(m/P_3, p_3)$	$-$
B_4	$p_1 < A$, $P_2 < A$, $P_3 < A$, $A \le P_4 \le D$	$\psi(m/P_4, p_4)$	$+$
U_2	$p_1 < A$, $P_2 > D$	$\psi(m/P_2, p_2)$	$+$
L_4	$p_1 < A$, $P_2 < A$, $P_3 < A$, $P_4 < A$	$\psi(m/P_4, p_4)$	$+$
U_4	$p_1 < A$, $P_2 < A$, $P_3 < A$, $P_4 > D$	$\psi(m/P_4, p_4)$	$+$
V_4	$p_1 < A$, $P_2 < A$, $P_3 > D$	$\psi(m/P_4, p_4)$	$+$

Proposition 3.2. *For $m \in I$,*

$$\begin{aligned}\mathbf{1}_{\mathbb{P}}(m) = {} & R_0 - R_1 + R_2 - R_3 - B_1 + B_2 - B_3 + B_4 \\ & + U_2 + L_4 + U_4 + V_4.\end{aligned} \tag{3.6}$$

Proof. Start with (3.1) and apply (3.3) according to the stopping rule. The process terminates at depth 4, and Table 3.1 lists every terminal term with its sign. Every step is an exact identity. □

3.3. **The weight and its support.** On I, define

$$\begin{aligned}\rho &= R_0 - R_1 + R_2 - R_3 - B_1 + B_2 - B_3 + B_4 \\ &= \mathbf{1}_{\mathbb{P}} - U_2 - L_4 - U_4 - V_4.\end{aligned} \tag{3.7}$$

Set $\rho = 0$ off I, and put

$$\lambda(m) = (\log m)\rho(m), \qquad F_\lambda = \sum_{m \in I} \lambda(m). \tag{3.8}$$

Every discarded term is nonnegative. Hence

$$\rho(m) \le \mathbf{1}_{\mathbb{P}}(m), \qquad \lambda(m) \le (\log m)\mathbf{1}_{\mathbb{P}}(m). \tag{3.9}$$

The weight may be negative. Every nonzero term is z-rough, and since $m \le X/4 < z^{10}$, its support satisfies

$$\Omega(m) \le 9. \tag{3.10}$$

A node is determined by at most four distinct prime factors of m, and it contributes at most one retained term. Thus

$$|\rho(m)| \le \sum_{j=0}^{4} \binom{9}{j} = 256, \qquad |\lambda(m)| \le 2^8 L. \tag{3.11}$$

For B_j, write $u = P_j$ and $v = m/u$; the product u determines its ordered prime prefix. For B_1 we have $v > 1$, since $p_1 \le D < X/8$, and then $\psi(v, p_1) = 1$ forces $v \ge p_1$. Thus $p_1^2 \le m$ is automatic here as well. Every balanced family has

$$A \le u \le D, \qquad A/8 < v \le D/4. \tag{3.12}$$

Its only roughness coupling is $\psi(v, P^-(u))$, and no retained family is a bare prime sum.

4. The mass estimate

(F3) [1, Chapters 7 and 18] We use the prime number theorem and Mertens' estimate in the forms

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \qquad \pi(x) = \mathrm{li}(x) + O\left(\frac{x}{\log^3 x}\right),$$

$$\theta(x) = \sum_{p \le x} \log p = x + O(xe^{-c\sqrt{\log x}}),$$

$$\sum_{y < p \le x} \frac{1}{p} = \log \frac{\log x}{\log y} + O\left(\frac{1}{\log y}\right) \qquad (x \ge y \ge 2).$$

Here $\mathrm{li}(x) = \int_2^x dt/\log t$.

4.1. A Buchstab envelope.

Put $C_\omega = 71/125$ and

$$\bar{\omega}(u) = \begin{cases} 1, & u < 1, \\ \max(1/u, C_\omega), & u \ge 1. \end{cases} \tag{4.1}$$

Lemma 4.1. *Let Ω_0 be the positive root of $xe^x = 1$. Then*

$$\sup_{u \ge 2} \omega(u) = \Omega_0 < C_\omega.$$

The function $\bar{\omega}$ is continuous and non-increasing. For $u \ge \max(u_0, 1)$,

$$\omega(u) \le \bar{\omega}(u_0).$$

Also $0 \le \omega \le 1$, and ω is 1-Lipschitz.

Proof. On $[2, 3]$, equation (2.8) gives

$$u\omega(u) = 1 + \log(u - 1) \qquad (2 \le u \le 3).$$

With $w = u - 1$, differentiate $(1 + \log w)/(1 + w)$; its maximum occurs at $\log w = 1/w$ and equals $1/w = \Omega_0$. For $k \ge 3$ put $M_k = \max_{[k,k+1]} \omega$. Then, for $k \le u \le k + 1$,

$$u\omega(u) = k\omega(k) + \int_k^u \omega(t - 1)\, dt \le kM_{k-1} + (u - k)M_{k-1} = uM_{k-1}.$$

Thus $M_k \le M_{k-1}$, and the rational inequality

$$C_\omega \sum_{j=0}^{4} \frac{C_\omega^j}{j!} = \frac{733930882351}{732421875000} > 1 \tag{4.2}$$

proves $\Omega_0 < C_\omega$. The envelope now follows from $\omega(u) = 1/u$ on $[1,2]$. On this interval $|\omega'| \le 1$, and for $u > 2$,

$$|\omega'(u)| = \frac{|\omega(u-1) - \omega(u)|}{u} \le \frac{1}{2}.$$

Continuity then completes the Lipschitz assertion. □

4.2. **Rough numbers through $u = 10$.** Let

$$\Phi(x,y) = \#\{1 \le n \le x : \ p \mid n \Rightarrow p > y\}.$$

This count includes 1.

Lemma 4.2. *Uniformly for $y \ge 2$ and $y \le x \le y^{10}$,*

$$\Phi(x,y) = \frac{x\omega(u) - y}{\log y} + O\left(\frac{x}{\log^2 y}\right), \qquad u = \frac{\log x}{\log y}. \tag{4.3}$$

The implied constant is absolute.

Proof. Let H(k) denote (4.3) for $1 \le u \le k$. For $1 \le u \le 2$ no composite is counted, so

$$\Phi(x,y) = 1 + \pi(x) - \pi(y).$$

Then (F3) and $\omega(u) = 1/u$ prove H(2).

Assume H(k), where $2 \le k \le 9$, and take $y^k < x \le y^{k+1}$. Decomposition by the least prime factor gives

$$\Phi(x,y) = \Phi(x,\sqrt{x}) + \sum_{y < p \le \sqrt{x}} \Phi(x/p, p) + E_1, \qquad 0 \le E_1 \le \frac{2x}{y}. \tag{4.4}$$

The error counts the residuals divisible by the selected prime, and it is $O(x/\log^2 y)$. The base case gives

$$\Phi(x,\sqrt{x}) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 y}\right).$$

For $y < p \le \sqrt{x}$ the new parameter is at most k. Hence

$$\Phi(x/p,p) = g(p) - \frac{p}{\log p} + O\left(\frac{C_k x}{p \log^2 p}\right), \quad g(s) = \frac{x}{s \log s}\omega\left(\frac{\log x}{\log s} - 1\right). \tag{4.5}$$

Here C_k is the induction constant, and by (F3),

$$\sum_{p \le \sqrt{x}} \frac{p}{\log p} \ll \frac{x}{\log^2 x}, \qquad \sum_{y < p \le \sqrt{x}} \frac{C_k x}{p \log^2 p} \le C_k \frac{x}{\log^2 y}\left(\log 10 + O\left(\frac{1}{\log y}\right)\right).$$

The function g is absolutely continuous, and Lemma 4.1 gives, almost everywhere,

$$\begin{aligned} |g'(s)| &\le \frac{x}{s^2 \log s}\left(1 + \frac{1}{\log s}\right) + \frac{x}{s \log s} \cdot \frac{\log x}{s \log^2 s} \\ &\le \frac{x}{s^2 \log s}\left(1 + \frac{1+u}{\log 2}\right) \le \frac{17x}{s^2 \log s}. \end{aligned}$$

Stieltjes integration against $\pi = \mathrm{li} + R$ yields

$$\sum_{y<p\le\sqrt{x}} g(p) = \int_y^{\sqrt{x}} \frac{g(s)}{\log s}\,ds + O\left(\frac{x}{\log^3 y}\right).$$

With $s = x^{1/v}$, the integral equals

$$\begin{aligned}\frac{x}{\log x}\int_2^u \omega(v-1)\,dv &= \frac{x}{\log x}\big(u\omega(u)-1\big)\\ &= \frac{x\omega(u)}{\log y} - \frac{x}{\log x}.\end{aligned}$$

Insert this in (4.4). Since $u > 2$, the term $y/\log y$ is absorbed by the error, and this proves $\mathrm{H}(k+1)$. The recurrence $C_{k+1} \ll 1 + C_k(1+\log 10)$ gives an absolute C_{10}. □

The subtraction of y in (4.3) is needed near $x = y$, so we retain it in the interval estimate.

Lemma 4.3. *Let $w \ge 3$ be an integer and $w' = w - 1/2$. Uniformly for $1 \le y_1 < y_2 \le (w')^{10}$,*

$$\begin{aligned}N(y_1,y_2;w) &:= \#\{y_1 < v \le y_2 : v \ge 2,\ P^-(v) \ge w\}\\ &= \frac{1}{\log w'}\int_{\max(y_1,w')}^{y_2} \omega\left(\frac{\log t}{\log w'}\right)dt + O\left(\frac{y_2}{\log^2 w}\right).\end{aligned} \tag{4.6}$$

The integral is defined to be zero when $y_2 \le w'$.

Proof. For integer v the conditions $P^-(v) \ge w$ and $P^-(v) > w'$ coincide, so $N = \Phi(y_2,w') - \Phi(y_1,w')$. If $y_2 \le w'$, this is zero. If $y_1 \ge w'$, the two terms $-w'$ cancel exactly. If $y_1 < w' < y_2$, then $\Phi(y_1,w') = 1$, and we use Lemma 4.2 and $\omega(1) = 1$. In both of the last two cases, with $y_a = \max(y_1,w')$,

$$N(y_1,y_2;w) = \frac{1}{\log w'}\left[y\,\omega\left(\frac{\log y}{\log w'}\right)\right]_{y_a}^{y_2} + O\left(\frac{y_2}{\log^2 w}\right).$$

Almost everywhere,

$$\frac{d}{dy}\left(y\,\omega\left(\frac{\log y}{\log w'}\right)\right) = \omega(u) + \frac{\omega'(u)}{\log w'}, \qquad u = \frac{\log y}{\log w'}.$$

Now Lemma 4.1 proves (4.6). □

4.3. **Absolute mass and square mass.**

Lemma 4.4. *We have*

$$\#\{m \le X : P^-(m) \ge z\} \le 1 + \frac{10C_\pi X}{L}\sum_{j=0}^{9} 3^j \ll X/L, \tag{4.7}$$

where $\pi(v) \le C_\pi v/\log v$ for $v \ge 2$. Consequently,

$$\sum_{m\in I}|\lambda(m)| \ll X, \qquad \sum_{m\in I}|\lambda(m)|^2 \ll XL. \tag{4.8}$$

Proof. For $1 < m \le X$ with $P^-(m) \ge z$, write $m = np$ with $p = P^+(m)$. Then $\Omega(n) \le 9$ and $X/n \ge z$, and the map $m \mapsto (n, p)$ is injective. The count is therefore at most

$$1 + \sum_{j=0}^{9} \sum_{\substack{n:\ \Omega(n)=j,\ P^-(n)\ge z \\ X/n \ge z}} \pi(X/n) \le 1 + \frac{10C_\pi X}{L} \sum_{j=0}^{9} \left(\sum_{z \le q \le X} \frac{1}{q} \right)^j.$$

By (F3), the inner sum is $\log 10 + O(1/L) \le 3$. This proves (4.7), and (3.11) then gives (4.8). □

4.4. **The four loss regions.** For $\mathcal{F} \in \{U_2, L_4, U_4, V_4\}$, let $k = 2$ for U_2 and $k = 4$ otherwise. Let $\mathcal{R}_\mathcal{F}$ consist of its prime prefixes in Table 3.1, without $p_1^2 \le m$. For a prefix put

$$\Pi = P_k, \quad w = p_k, \quad x_i = \frac{\log p_i}{L}, \quad s = \sum_{i=1}^{k} x_i = \frac{\log \Pi}{L}, \quad e = x_k = \frac{\log w}{L}.$$

Define

$$\mathcal{R}^*_\mathcal{F} = \{(p_1, \dots, p_k) \in \mathcal{R}_\mathcal{F} : 4\Pi w \le X\}. \tag{4.9}$$

Put $\delta = 1/10$ and $s_j = x_1 + \cdots + x_j$. The four regions are

$$\begin{aligned}
\mathcal{D}_{U_2} &= \{\delta \le b < a < \tfrac{9}{20},\ a + b > \tfrac{11}{20},\ a + 2b \le 1\}, \\
\mathcal{D}_{L_4} &= \{\delta \le e < c < b < a,\ s_4 < \tfrac{9}{20}\}, \\
\mathcal{D}_{U_4} &= \{\delta \le e < c < b < a,\ s_3 < \tfrac{9}{20},\ s_4 > \tfrac{11}{20}\}, \\
\mathcal{D}_{V_4} &= \{\delta \le e < c < b < a,\ a + b < \tfrac{9}{20}, \\
&\qquad\qquad s_3 > \tfrac{11}{20},\ s_4 + e \le 1\}.
\end{aligned} \tag{4.10}$$

Here a, b, c, e are the ordered coordinates, and every tuple in (4.9) lies in its corresponding region. For

$$u(x) := \frac{1 - s}{e},$$

the ranges are

$\mathcal{F}$	U_2	L_4	U_4	V_4
$u(x)$	$< 9/2$	$44/9 < u(x) < 6$	$8/3 < u(x) < 9/2$	$< 7/2.$

(4.11)

For example, $s_4 > 4e$ and $e < 9/80$ on $\mathcal{D}_{L_4}$. On $\mathcal{D}_{U_4}$ we have $s_4 < 9/20 + e$ and $e < 3/20$. These give the two lower bounds in (4.11), and the remaining upper bounds follow from (4.10).

Lemma 4.5. *For each discarded family,*

$$\sum_{m \in I} (\log m) \mathcal{F}(m) \le \frac{X}{8} \sum_{\mathcal{R}^*_\mathcal{F}} \frac{\bar{\omega}(u_*)}{e\,\Pi} (1 + o(1)) + O(X/L), \tag{4.12}$$

where

$$u_* := \max\left(1, \frac{\log(X/(8\Pi))}{\log(w - \frac{1}{2})}\right).$$

All errors are uniform over the indicated tuples.

Proof. For U_2 and V_4 we have $\Pi < A^2$, and for L_4 we have $\Pi < A$. For U_4, the inequalities $P_3 < A$ and $p_4 < P_3^{1/3}$ give $\Pi < A^{4/3}$. Thus $\Pi < X^{9/10}$ in every case, and for large X the residual $v > X/(8\Pi)$ is at least 2. By (3.5),

$$\sum_{m\in I} \mathcal{F}(m) = \sum_{\mathcal{R}_{\mathcal{F}}} N\left(\frac{X}{8\Pi}, \frac{X}{4\Pi}; w\right). \tag{4.13}$$

If $4\Pi w > X$, a counted residual would satisfy $v \geq w > X/(4\Pi)$. Its count is therefore exactly zero, and we delete these tuples before making any estimate.

For an actual tuple,

$$\frac{\log(X/(c\Pi))}{\log(w-1/2)} = u(x) - \frac{\log c}{eL} + O\left(\frac{1}{zL}\right) \qquad (c = 4, 8). \tag{4.14}$$

By (4.11), $X/(4\Pi) \leq (w-1/2)^{10}$ for large X. Apply Lemma 4.3 to the remaining tuples. The integration interval has length at most $X/(8\Pi)$, and Lemma 4.1 gives

$$N\left(\frac{X}{8\Pi}, \frac{X}{4\Pi}; w\right) \leq \frac{1}{\log(w-\frac{1}{2})} \cdot \frac{X}{8\Pi}\,\bar{\omega}(u_*) + O\left(\frac{X}{\Pi L^2}\right).$$

The summed error is $O(X/L^2)$ by (F3). Multiply by $\log m \leq L$ on the nonnegative losses, and use $L/\log(w-1/2) = (1+o(1))/e$. □

4.5. **Rectangle majorants.** Fix $n \geq 1$ and $k \in \{2, 4\}$. Put

$$\mathfrak{h} := \frac{9/20 - 1/10}{n} = \frac{7}{20n}, \quad K = \prod_{i=1}^{k} [x_{i,0}, x_{i,1}), \quad x_{i,0} = \frac{1}{10} + j_i\mathfrak{h}, \quad x_{i,1} = x_{i,0} + \mathfrak{h},$$

where $n > j_1 \geq \cdots \geq j_k \geq 0$. Every point of a loss region lies in exactly one such cell. Define

$$u_0(K) = \frac{1 - \sum_i x_{i,1}}{x_{k,1}}, \qquad W(K) = \frac{\bar{\omega}(u_0(K))\mathfrak{h}^k}{x_{1,0}\cdots x_{k-1,0}x_{k,0}^2}. \tag{4.15}$$

Lemma 4.6. *For $x \in K$ with $s + e \leq 1$,*

$$\frac{\omega(u(x))}{x_1\cdots x_{k-1}e^2} \leq \frac{\bar{\omega}(u_0(K))}{x_{1,0}\cdots x_{k-1,0}x_{k,0}^2}.$$

Proof. Write $s' = x_1 + \cdots + x_{k-1}$, so $u = (1-s')/e - 1$. First raise e to $x_{k,1}$. The value decreases, since $1 - s' \geq 2e > 0$ at the starting point. Then raise the other coordinates to their upper endpoints; each has slope $-1/x_{k,1}$. Thus $u(x) \geq u_0(K)$, and also $u(x) \geq 1$. Use the non-increasing envelope and the lower endpoints in the denominator. If $u_0(K) < 1$, the envelope equals 1, so the same bound holds. □

For U_2, take $n = 256$ and retain cells satisfying

$$a_0 < 9/20, \qquad a_1 + b_1 > 11/20, \qquad a_0 + 2b_0 \leq 1. \tag{4.16}$$

For V_4, take $n = 64$ and retain cells satisfying

$$a_0 + b_0 < 9/20, \qquad a_1 + b_1 + c_1 > 11/20, \qquad a_0 + b_0 + c_0 + 2e_0 \leq 1. \tag{4.17}$$

Write these cell sets as $\mathcal{K}_{U_2}$ and $\mathcal{K}_{V_4}$. Each exclusion removes only cells disjoint from the corresponding region. Define

$$\mathbf{U}_2 = \sum_{K\in\mathcal{K}_{U_2}} W(K), \qquad \mathbf{V}_4 = \sum_{K\in\mathcal{K}_{V_4}} W(K). \tag{4.18}$$

The exact rational arithmetic in Appendix A gives

$$\begin{aligned} |\mathcal{K}_{U_2}| &= 11093, \qquad \mathbf{U}_2 \le \frac{732661466777}{10^{12}}, \\ |\mathcal{K}_{V_4}| &= 18286, \qquad \mathbf{V}_4 \le \frac{98012508379}{10^{12}}. \end{aligned} \tag{4.19}$$

Proposition 4.7. *We have*

$$\sum_{m\in I}(\log m)U_2(m) \le (\mathbf{U}_2+o(1))\frac{X}{8}, \qquad \sum_{m\in I}(\log m)V_4(m) \le (\mathbf{V}_4+o(1))\frac{X}{8}. \tag{4.20}$$

Proof. Group the tuples in (4.12) by cell. By (4.14) and Lemma 4.6,

$$u_* \ge u_0(K) - \frac{10\log 8}{L} - O\left(\frac{1}{z}\right).$$

Uniform continuity of $\bar\omega$ gives $\bar\omega(u_*) \le \bar\omega(u_0(K))+o(1)$, and $1/e \le 1/x_{k,0}$. For each coordinate, (F3) gives

$$\sum_{X^{x_0}\le p<X^{x_0+\mathfrak{h}}}\frac{1}{p} = \log\frac{x_0+\mathfrak{h}}{x_0} + O\left(\frac{1}{L}\right) \le \frac{\mathfrak{h}}{x_0} + O\left(\frac{1}{L}\right).$$

Enlarge the sum to all prime tuples in the box product; every summand is nonnegative, including in diagonal cells. Each cell then contributes at most $W(K)+o(1)$. The number of cells is finite and independent of X, so summing proves the proposition. □

4.6. **The two volume bounds.**

Lemma 4.8. *We have*

$$\begin{aligned} \sum_{m\in I}(\log m)L_4(m) &\le \left(\frac{71}{115200}+o(1)\right)\frac{X}{8}, \\ \sum_{m\in I}(\log m)U_4(m) &\le \left(\frac{213}{12800}+o(1)\right)\frac{X}{8}. \end{aligned} \tag{4.21}$$

Proof. By (4.11) and (4.14), $u_* > 2$ for these two families. Hence $\bar\omega(u_*) = C_\omega$ and $1/e \le 10$, and Lemma 4.5 gives the bound

$$\sum_{m\in I}(\log m)\mathcal{F}(m) \le \frac{X}{8}\,C_\omega\,10 \sum_{(p_i)\in\mathcal{R}^*_{\mathcal{F}}}\prod_{i=1}^{4}\frac{1}{p_i}(1+o(1)) + O\left(\frac{X}{L}\right).$$

Cover the fixed region by cubes of side σ anchored at $1/10$. By (F3), each cube contributes at most $10^4\sigma^4 + O(1/L)$, and their total volume tends to $\mathrm{vol}(\mathcal{D}_{\mathcal{F}})$ as $\sigma\to 0$. Letting first $X\to\infty$ and then $\sigma\to 0$, we get

$$\sum_{(p_i)\in\mathcal{R}^*_{\mathcal{F}}}\prod_i\frac{1}{p_i} \le 10^4\,\mathrm{vol}(\mathcal{D}_{\mathcal{F}}) + o(1).$$

Use the gap coordinates

$$e = \delta + y_4, \quad c = e + y_3, \quad b = c + y_2, \quad a = b + y_1.$$

The Jacobian is 1, with $y_1, y_2, y_3 > 0$ and $y_4 \ge 0$. For L_4 the condition is

$$y_1 + 2y_2 + 3y_3 + 4y_4 < \frac{9}{20} - 4\delta = \frac{1}{20} =: y_L.$$

Thus

$$\mathrm{vol}(\mathcal{D}_{L_4}) = \frac{(1/20)^4}{576}. \tag{4.22}$$

For U_4, put $Y = y_1 + 2y_2 + 3y_3$ and $x_U = 3/20$; the two conditions become

$$Y + 3y_4 < \frac{3}{20} =: x_U, \qquad Y + 4y_4 > \frac{11}{20} - 4\delta = \frac{3}{20} = x_U.$$

For $0 < Y < x_U$ the y_4-interval has length $(x_U - Y)/12$, and the density in Y is $Y^2/12$. Hence

$$\mathrm{vol}(\mathcal{D}_{U_4}) = \int_0^{x_U} \frac{Y^2(x_U - Y)}{144}\, dY = \frac{(3/20)^4}{1728}. \tag{4.23}$$

Multiplying these volumes by $C_\omega 10^5$ gives $71/115200$ and $213/12800$, respectively. □

4.7. **The mass lower bound.**

Proposition 4.9. *Put*

$$\begin{aligned} c_{\mathrm{cert}} &= 1 - \frac{732661466777}{10^{12}} - \frac{71}{115200} - \frac{213}{12800} - \frac{98012508379}{10^{12}} \\ &= \frac{342155430899}{2250000000000} > \frac{3}{20}. \end{aligned} \tag{4.24}$$

Then

$$F_\lambda \geq \left(\frac{c_{\mathrm{cert}}}{8} - o(1)\right) X, \qquad |F_\lambda| \leq \sum_{m \in I} |\lambda(m)| \ll X. \tag{4.25}$$

Proof. By (F3), the prime contribution is

$$\theta(X/4) - \theta(X/8) = \frac{X}{8} + O\left(Xe^{-c\sqrt{L}}\right).$$

Subtract the four nonnegative losses in (3.7), using Proposition 4.7, (4.19) and Lemma 4.8. This retains the exact weight $\log m$ on primes, and on the losses the upper bound $\log m \leq L$ has the required direction. The absolute bound follows from Lemma 4.4. □

5. Duality

For even n, put

$$r_\lambda(n) := \sum_{X_1 < p \leq X} (\log p)\, \lambda(n-p) = \int_0^1 S(\alpha) S_\lambda(\alpha) e(-n\alpha)\, d\alpha = I_{\mathfrak{M}}(n) + I_{\mathfrak{m}}(n). \tag{5.1}$$

Both integrals are real, since the arcs are invariant under $\alpha \mapsto -\alpha$ and the coefficients are real. If n is not a sum of two primes, then

$$r_\lambda(n) \leq 0. \tag{5.2}$$

This follows term by term from (3.9). For $n \in (X/2, X]$ and $m \in I$ we also have

$$n - m \in [X/4, 7X/8] \subset (X_1, X]. \tag{5.3}$$

Proposition 5.1. *Let $\mathcal{E}_X$ be the exceptional even integers in $(X/2, X]$. Suppose that $I_{\mathfrak{M}}(n) \ge c_1 X$ for every even n in this interval, and that $\sup_{\alpha\in\mathfrak{m}} |S_\lambda(\alpha)| \le \mathcal{S}$. Then*

$$|\mathcal{E}_X| \le c_1^{-2}\mathcal{S}^2 L/X.$$

Proof. Put $Z(\alpha) = \sum_{n\in\mathcal{E}_X} e(n\alpha)$. By (5.2),

$$c_1 X \le I_{\mathfrak{M}}(n) \le -I_{\mathfrak{m}}(n) \qquad (n \in \mathcal{E}_X).$$

Parseval gives $\|Z\|_2^2 = |\mathcal{E}_X|$, and by (F3), $\|S\|_2^2 \le XL$ for large X. Squaring proves the assertion. □

6. Minor arcs

This section is unconditional. For intervals $J \subset I$ put

$$T_\rho(\alpha; J) = \sum_{m\in J} \rho(m) e(m\alpha).$$

We first bound these sums uniformly in J.

6.1. **Rational approximation and bilinear estimates.** For $\alpha \in \mathfrak{m}$, Dirichlet's approximation theorem gives coprime a, s with

$$P < s \le Q_0, \qquad |\alpha - a/s| \le (sQ_0)^{-1} \le s^{-2}. \tag{6.1}$$

A denominator $s \le P$ would place α on a major arc, since $(sQ_0)^{-1} \le P/(2sX)$. Thus

$$X/s < X/P \le 2X^{11/20+\eta}, \qquad s \le 3X/P \le 6X^{11/20+\eta}, \qquad D \le X/P. \tag{6.2}$$

(F4) [9, Lemma 2.2] If $(a, s) = 1$ and $|\alpha - a/s| \le s^{-2}$, then

$$\sum_{u\le U} \min\left(\frac{2X}{u}, \frac{1}{2\|\alpha u\|}\right) \ll (X/s + U + s)\log(2Xs),$$

$$\sum_{h\le H} \min\left(Y, \frac{1}{\|\alpha h\|}\right) \ll (1 + H/s)(Y + s\log s).$$

Lemma 6.1. *Let $|a_u|, |b_v| \le 1$, and let $J \subset (0, X]$ be an interval. Then*

$$\left|\sum_{u\le U} a_u \sum_{k:uk\in J} e(\alpha uk)\right| \ll (X/s + U + s)\log(2Xs). \tag{6.3}$$

If $MN \ll X$, then

$$\left|\sum_{M<u\le 2M}\sum_{v\le N} a_u b_v \mathbf{1}_J(uv) e(\alpha uv)\right| \ll X^{1/2}(X/s + M + N + s)^{1/2} L^{1/2}. \tag{6.4}$$

Proof. The inner sum in (6.3) is a geometric progression with modulus at most $\min(2X/u, (2\|\alpha u\|)^{-1})$, so (F4) applies. For (6.4), apply Cauchy–Schwarz in u.

For fixed v, v', the conditions $uv, uv' \in J$ restrict u to an interval. Group the off-diagonal terms by $h = |v - v'|$; the second estimate in (F4) gives

$$\begin{aligned} |T|^2 &\ll MN\left(M + \sum_{1\le h<N} \min\left(2M, \frac{1}{\|\alpha h\|}\right)\right) \\ &\ll MN\left(M + \left(1 + \frac{N}{s}\right)(M + s\log s)\right) \\ &\ll MN\left(\frac{MN}{s} + M + N + s\right)L, \end{aligned}$$

where T denotes the left side of (6.4). Since $MN \ll X$, this proves (6.4). □

By (6.2), the estimates needed below are

$$U \le X^{27/40}: \quad (6.3) \ll X^{27/40}L, \tag{6.5}$$

$$M, N \ll X/P: \quad (6.4) \ll X^{31/40+\eta/2}L^{1/2}. \tag{6.6}$$

The first of these uses $11/20 + \eta \le 0.56 < 27/40$.

6.2. **Exact separation of comparisons.** Let $K = 2\lceil X\rceil + 1$, and let H be the indicator of $\{0, \dots, \lceil X\rceil\}$ modulo K. Fourier inversion gives, for integers $x \in [-X, X]$,

$$\mathbf{1}[x \ge 0] = \sum_{g \bmod K} \widehat{H}(g)e(gx/K), \qquad \sum_g |\widehat{H}(g)| \le 2 + \log K \ll L, \tag{6.7}$$

where $\widehat{H}(g) = K^{-1}\sum_{h=0}^{\lceil X\rceil} e(-gh/K)$. The norm bound follows from the geometric-sum bound $|\widehat{H}(g)| \le \min(1, (2K\|g/K\|)^{-1})$. Consequently, for $2 \le u, v \le X$,

$$\psi(v, P^-(u)) = \sum_g \widehat{H}(g)e(gP^-(v)/K)e(-gP^-(u)/K). \tag{6.8}$$

If $F(u) \in [0, X]$ is integer-valued, then

$$\mathbf{1}[w \le F(u)] = \sum_g \widehat{H}(g)e(gF(u)/K)e(-gw/K) \qquad (1 \le w \le X). \tag{6.9}$$

These identities include equality at each endpoint, and each separated comparison costs $O(L)$.

6.3. **The stopped Legendre identity.**

Lemma 6.2. *Let $r < A$ be a positive integer, and let $k \ge 1$. Then*

$$\begin{aligned} \psi(k, z) = &\sum_{\substack{d\mid k,\ d\mid\Pi(z)\\ rd<A}} \mu(d) \\ &+ \sum_{\substack{d\mid k,\ d\mid\Pi(z),\ d>1\\ rd\ge A>rd/P^-(d)}} \mu(d)\psi(k/d, P^-(d)), \end{aligned} \tag{6.10}$$

where $\Pi(z) = \prod_{p<z} p$. Every term in the second sum has $A \le rd < Az = D$.

Proof. Iterate

$$\psi(k', y) = 1 - \sum_{\substack{p<y \\ p\mid k'}} \psi(k'/p, p).$$

The selected primes decrease strictly, so their product d is squarefree. Expand a node precisely when $rd < A$. At a stopped node, $q = P^-(d) < z$ and $rd/q < A \le rd < Aq < Az$. The terminal terms give (6.10). The rule depends only on r, d, and no restriction is placed on repeated prime factors of the residual. □

6.4. **The eight retained families.**

Proposition 6.3. *Uniformly for $\alpha \in \mathfrak{m}$ and intervals $J \subset I$,*

$$|T_\rho(\alpha; J)| \ll X^{31/40+\eta/2} L^{7/2}. \tag{6.11}$$

Proof. We use the eight terms in (3.7); the four discarded families require no exponential-sum estimate.

The families R_0, R_1, R_2. For $j = 0, 1, 2$ the sums have the form

$$\sum_{m\in J} R_j(m) e(\alpha m) = \sum_r c_r \sum_{k:\ rk\in J} \psi(k, z) e(\alpha r k),$$

where $r = P_j < A$, with $r = 1$ when $j = 0$. Here $c_r \in \{0, 1\}$ depends only on r, which is z-rough. Apply Lemma 6.2 and put $u = rd$. The rough and smooth parts of u determine r, d uniquely, so the resulting coefficients have modulus at most 1. The free terms have $u < A$ and contribute $O(X^{11/20+\eta} L)$ by (6.3). The stopped terms are

$$\sum_{A\le u<D} a'_u \sum_{k':\ uk'\in J} \psi(k', P^-(u)) e(\alpha u k'), \qquad |a'_u| \le 1, \quad P^-(u) = P^-(d).$$

Here $k' > X/(8D) > 2$. Split u into $O(L)$ intervals $(M, 2M]$ with $A/2 \le M < D$; then $k' \le X/(4M)$. By (6.8) and (6.6), the total is $O(X^{31/40+\eta/2} L^{5/2})$.

The family R_3. Write $r = P_2 = p_1 p_2 < A$ and $w = p_3$. The prime conditions are

$$p_1 > p_2 > w \ge z, \qquad rw < A \text{ or } rw > D.$$

Apply Lemma 6.2 with anchor $r = P_2$ to the residual $\psi(k, z)$ in $m = rwk$; the anchor is not P_3. For free terms the outer variable is $U = rwd$. Its rough part determines p_1, p_2, p_3, and its smooth part determines d, so the coefficient is bounded by 1. The inner variable is free on J/U. Since $w^2 < p_2^2 < r < A$,

$$U = (rd)w < A^{3/2} = X^{27/40}.$$

These terms contribute $O(X^{27/40} L)$, for both conditions on rw.

For stopped terms, put $u = rd \in [A, D)$, $q = P^-(d) < z$, and $V = wk'$. Since $w \ge z > q$, we have $\psi(k', q) = \psi(V, q)$, and the sum becomes

$$\sum_{A\le u<D} a_u \sum_{V: uV\in J} \psi(V, P^-(u)) \sum_{w\mid V,\ w\in\mathbb{P},\ w\ge z} \vartheta(u, w) e(\alpha u V). \tag{6.12}$$

Here $|a_u| \le 1$ includes the stopping condition, and the rough part of u is $r = p_1 p_2$. Also

$$\vartheta(u, w) = \mathbf{1}[w < p_2]\big(\mathbf{1}[rw < A] + \mathbf{1}[rw > D]\big).$$

The correspondence is bijective, since $k' = V/w$ is determined by u, V, w; this remains true when $w^2 \mid V$. Put

$$F_1(u) = \min\bigl(p_2 - 1, \lceil A/r \rceil - 1\bigr),$$
$$F_2(u) = p_2 - 1,$$
$$F_3(u) = \min(p_2 - 1, \lfloor D/r \rfloor).$$

In F_1, $\lceil \cdot \rceil$ is the ceiling function, and 1 is subtracted after rounding up. Set these to zero where $a_u = 0$. All three are integers in $[0, X]$, and

$$\vartheta(u, w) = \mathbf{1}[w \le F_1(u)] + \mathbf{1}[w \le F_2(u)] - \mathbf{1}[w \le F_3(u)]. \tag{6.13}$$

Separate the roughness condition by (6.8) and each comparison by (6.9). For fixed g, g', i, the two coefficients are

$$a_u e\Bigl(\frac{-gP^-(u) + g'F_i(u)}{K}\Bigr), \quad b_V(g, g') := e\Bigl(\frac{gP^-(V)}{K}\Bigr) \sum_{w \mid V,\ w \text{ prime},\ w \ge z} e\Bigl(\frac{-g'w}{K}\Bigr).$$

Their moduli are at most 1 and 5, since $V \le X/(4A) < z^6$. With two Fourier norms and $O(L)$ dyadic intervals, (6.6) now gives $O(X^{31/40+\eta/2}L^{7/2})$ for (6.12).

The families B_1, B_2, B_3, B_4. Each has the form

$$\sum_{m \in J} B_j(m) e(\alpha m) = \sum_{A \le u \le D} \mathbf{1}_{E_j}(u) \sum_{v:\ uv \in J} \psi(v, P^-(u)) e(\alpha uv).$$

The ordered prime factorisation of u determines the prefix conditions. For B_1, the condition $p_1^2 \le m$ follows from $v \ge p_1$. The residual satisfies $v > X/(8D) > 2$, and one Fourier separation with $O(L)$ dyadic intervals gives $O(X^{31/40+\eta/2}L^{5/2})$.

All Type I outer variables are below $X^{27/40}$. Both variables in every Type II piece are $\ll D \le X/P$, and the underlying products place them in $[A/8, D]$ up to endpoint constants. Adding the eight estimates proves the proposition. □

Corollary 6.4. *Uniformly for $\alpha \in \mathfrak{m}$,*

$$\sup_{\alpha \in \mathfrak{m}} |S_\lambda(\alpha)| \ll X^{31/40+\eta/2}L^{9/2} = X^{1-t/2}L^{9/2}.$$

Proof. Partial summation gives

$$S_\lambda(\alpha) = \log\frac{X}{4}\, T_\rho(\alpha; I) - \int_{X/8}^{X/4} T_\rho\bigl(\alpha; (X/8, y]\bigr)\frac{dy}{y}.$$

Its modulus is at most $2L$ times the largest prefix sum, and Proposition 6.3 applies. □

7. Bilinear distribution with both maxima

Lemma B. *Let X be large, $L = \log X$, $z = X^{1/10}$, and $1 \le Q < X^{1/2}$. Let $M, N \ge 1$. Suppose that α, β satisfy the following conditions. Their supports lie in $(M, 2M]$ and $(N, 2N]$, respectively, and consist of z-rough integers greater than 1. Their coefficients have modulus at most c_0. For some $\mathcal{A} \ge 1$, every $d \le L^{\mathcal{A}}$, $(b, d) = 1$ and real u satisfy*

$$\Biggl|\sum_{\substack{n \le u \\ n \equiv b \pmod d}} \beta_n - \frac{1}{\varphi(d)} \sum_{\substack{n \le u \\ (n,d)=1}} \beta_n\Biggr| \le C_{\mathrm{SW}} N L^{-\mathcal{A}}. \tag{7.1}$$

Put $f = \alpha * \beta$. For $1 \le \mathcal{D} \le \mathcal{A}$,

$$\sum_{q \le Q} \Delta_f(q) \ll XL^{2\mathcal{D}+1-\mathcal{A}} + X^{9/10}L^{\mathcal{D}+1} + XL^{2-\mathcal{D}} + X^{19/20}L^3 + QX^{1/2}L^2. \tag{7.2}$$

The implied constant depends only on c_0, C_{SW}. In particular, for $Q \le Q_B$ and $B' > 0$, take $\mathcal{D} = B' + 2$ and $\mathcal{A} = 3B' + 5$. Then the left side is $O(XL^{-B'})$.

(F5) [1, Chapter 17] For $x > 0$, $x \ne 1$, $c > 0$ and $T \ge 1$,

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s}\, ds - \mathbf{1}[x > 1] \right| \ll x^c \min\left(1, \frac{1}{T|\log x|}\right).$$

(F6) [4, Theorem 7.13] If c_n is supported on N' consecutive integers, then

$$\sum_{r \le R'} \frac{r}{\varphi(r)} \sideset{}{^*}\sum_{\chi \bmod r} \left| \sum_n c_n \chi(n) \right|^2 \le (N' + R'^2) \sum_n |c_n|^2.$$

The star restricts the sum to primitive characters.

Proof of Lemma B. We may assume that both sequences are nonzero and $MN < X$, since otherwise all prefix sums vanish. Roughness gives

$$z/2 \le M, N \le 2X/z = 2X^{9/10}, \qquad \|\alpha\|_2^2 \le 2c_0^2 M, \quad \|\beta\|_2^2 \le 2c_0^2 N. \tag{7.3}$$

For $\ell \le X$ in the support, $\Omega(\ell) \le 10$, hence $|f(\ell)| \le 2^{10}c_0^2$. Write

$$S_\chi(y; \alpha', \beta') = \sum_{mn \le y} \alpha'_m \beta'_n \chi(mn).$$

Orthogonality and induction. The principal character gives the exact coprime mean in Δ_f. Therefore

$$\sum_{q \le Q} \Delta_f(q) \le \Sigma := \sum_{q \le Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0}} \max_{y \le X} |S_\chi(y; \alpha, \beta)|. \tag{7.4}$$

Let χ be induced by a primitive character χ_1 modulo r, and write $q = rk$. For all integers ℓ,

$$\chi(\ell) = \chi_1(\ell)\mathbf{1}_{(\ell,k)=1}.$$

No coprimality condition between r, k is required. Put $\alpha_m^{(k)} = \alpha_m \mathbf{1}_{(m,k)=1}$ and $\beta_n^{(k)} = \beta_n \mathbf{1}_{(n,k)=1}$. The induction is a bijection onto the pairs with $r > 1$, $rk \le Q$. Since $\varphi(rk) \ge \varphi(r)\varphi(k)$,

$$\Sigma \le \sum_{k \le Q} \frac{1}{\varphi(k)} \sum_{1 < r \le Q/k} \frac{1}{\varphi(r)} \sideset{}{^*}\sum_{\chi_1 \bmod r} \max_{y \le X} |S_{\chi_1}(y; \alpha^{(k)}, \beta^{(k)})|. \tag{7.5}$$

We split this sum at $r = L^{\mathcal{D}}$.

Small conductors. Only here do we remove the restriction $(mn, k) = 1$. Since $k \le Q < z^5$, at most four primes at least z divide k, and for each such prime at most X/z products $mn \le X$ occur. Thus, uniformly in all cutoffs,

$$\left| S_{\chi_1}(y; \alpha^{(k)}, \beta^{(k)}) - S_{\chi_1}(y; \alpha, \beta) \right| \le 4 \cdot 2^{10} c_0^2 X/z.$$

For $1 < r \le L^{\mathcal{D}}$, expand β by reduced residue classes. The primitive character is nonprincipal, so (7.1) gives

$$\left|\sum_{n\le u}\beta_n\chi_1(n)\right| \le \varphi(r)C_{\mathrm{SW}}NL^{-\mathcal{A}}.$$

Summing over m gives $|S_{\chi_1}(y;\alpha,\beta)| \le 2c_0C_{\mathrm{SW}}rXL^{-\mathcal{A}}$, and there are at most $\varphi(r)$ primitive characters. Using $\sum_{k\le Q}\varphi(k)^{-1} \ll L$, we obtain

$$\Sigma_{\mathrm{small}} \ll XL^{2\mathcal{D}+1-\mathcal{A}} + X^{9/10}L^{\mathcal{D}+1}. \tag{7.6}$$

Perron's formula and the maximum in y. Put $c = 1/L$ and $T = X^3$. Every prefix can be represented with a half-integer $y = j + 1/2$. The convolution is supported in $(MN, 4MN] \subset [1, 4X]$, and its absolute mass is at most $4c_0^2X$. For the relevant ℓ, $|\log(y/\ell)| \ge 1/(10X)$ and $(y/\ell)^c \le 3$, so (F5) gives a uniform error $O(X^{-1})$. Define

$$A_k(s,\chi) = \sum_m \alpha_m^{(k)}\chi(m)m^{-s}, \qquad B_k(s,\chi) = \sum_n \beta_n^{(k)}\chi(n)n^{-s}.$$

Then

$$\max_{y\le X}|S_\chi(y;\alpha^{(k)},\beta^{(k)})| \le \frac{3}{2\pi}\int_{-T}^{T}\frac{|A_kB_k(c+it,\chi)|}{|c+it|}\,dt + O(X^{-1}).$$

Since $\int_{-T}^{T}|c+it|^{-1}dt \le 8L$ for large X, any finite character family $\mathcal{F}$ satisfies

$$\sum_{\chi\in\mathcal{F}}\max_y|S_\chi(y)| \le 4L\sup_{|t|\le T}\sum_{\chi\in\mathcal{F}}|A_kB_k(c+it,\chi)| + O(|\mathcal{F}|/X). \tag{7.7}$$

The same argument applies with positive character weights, and it keeps the maximum inside the sum over characters.

Large conductors. The intervals $(R_i, 2R_i]$, where $R_i = 2^iL^{\mathcal{D}}$, $i \ge 0$, cover every conductor greater than $L^{\mathcal{D}}$. Fix k, one such interval, and t. Cauchy–Schwarz with weights $r/\varphi(r)$, followed by (F6), gives

$$\begin{aligned}\sum_{R<r\le\min(2R,Q/k)}\frac{1}{\varphi(r)}\sideset{}{^*}\sum_{\chi_1 \bmod r}|A_kB_k| &\le \frac{1}{R}\big((2M+4R^2)2c_0^2M\big)^{1/2}\\ &\quad\times\big((2N+4R^2)2c_0^2N\big)^{1/2}\\ &\ll\Big(\frac{X^{1/2}}{R}+M^{1/2}+N^{1/2}+R\Big)X^{1/2}.\end{aligned}$$

The weighted Perron error is $O(R/X)$. Sum (7.7) over k and these intervals; the endpoint terms form geometric sums. Using (7.3), we get

$$\begin{aligned}\Sigma_{\mathrm{large}} &\ll L^2\Big(XL^{-\mathcal{D}} + LX^{1/2}(2X^{9/10})^{1/2} + QX^{1/2}\Big)\\ &\ll XL^{2-\mathcal{D}} + X^{19/20}L^3 + QX^{1/2}L^2.\end{aligned} \tag{7.8}$$

Equations (7.6) and (7.8) prove (7.2). With the stated choices of $\mathcal{D}, \mathcal{A}$, both logarithmic terms are $XL^{-B'}$, and the remaining three terms have a fixed power saving. □

7.1. **Siegel–Walfisz estimates for prime boxes. (F7)** [1, Chapter 22] For every $\mathcal{A} > 0$, there is $c_{\mathcal{A}} > 0$ such that

$$\left|\pi(w; d, b) - \frac{\pi(w)}{\varphi(d)}\right| \ll_{\mathcal{A}} w \exp\big(-c_{\mathcal{A}}\sqrt{\log w}\big),$$

uniformly for $(b, d) = 1$ and $d \le (\log w)^{\mathcal{A}+1}$.

Lemma C. *Let $\mathcal{A} > 0$. For $\beta = \mathbf{1}_{\mathbb{P}\cap[U,V)}$, where $z \le U < V \le 4U$, the left side of (7.1) is $O_{\mathcal{A}}(Ue^{-c_{\mathcal{A}}\sqrt{L}})$, uniformly for $d \le L^{\mathcal{A}}$, $(b, d) = 1$ and real u.*

Suppose β_1 satisfies (7.1) with scale N_1 and constant C_1. If γ is supported on $(N_2, 2^j N_2]$ and $|\gamma| \le c$, then $\beta_1 * \gamma$ satisfies it with scale N_1N_2 and constant $C_1 c 2^{j+1}$. Restricting any such sequence to $(N', 2N']$ multiplies its constant by at most 2.

Proof. For large X, $d \le L^{\mathcal{A}} < z$, so every prime in $[U, V)$ is coprime to d. Put $E(w) = \pi(w; d, b) - \pi(w)/\varphi(d)$. The discrepancy is zero for $u < U$. For $U \le u < V$ it equals $E(u) - E(U^-)$, and for $u \ge V$ it equals $E(V^-) - E(U^-)$; the left limits preserve both half-open endpoints. All arguments satisfy $\log w \ge L/10$, so for large X we have $d \le (\log w)^{\mathcal{A}+1}$, and (F7) applies.

For the convolution, fix n_2 in the support of γ. If $(n_2, d) > 1$, neither the reduced residue sum nor its coprime mean contributes. Otherwise the residue for n_1 is $bn_2^{-1} \pmod d$, and we apply (7.1) at the cutoff u/n_2. The errors sum to at most

$$C_1 N_1 L^{-\mathcal{A}} \sum_{n_2} |\gamma(n_2)| \le C_1 c\, 2^{j+1} N_1 N_2 L^{-\mathcal{A}}.$$

The main terms give the coprime mean of the complete convolution. Finally, a dyadic restriction is the difference of two prefix sums. □

8. Distribution of the signed weight

Proposition 8.1 (Distribution of the signed weight)**.** *For every $B > 0$,*

$$\sum_{d \le Q_B} 2^{\omega(d)} \Delta_\lambda(d) \ll_{B,\eta} XL^{-B}, \qquad Q_B = X^{9/20-\eta}. \tag{8.1}$$

Both maxima in (2.9) remain inside the modulus sum. This estimate is unconditional.

8.1. **Marked prefixes on the full support.** Extend each discarded family to all positive integers m. Use the prefix conditions in Table 3.1, including $p_1^2 \le m$, and the residual $\psi(m/P_j, p_j)$. Write their sum as $\widetilde{U}$, and put

$$\widetilde{U}^\flat(m) := \widetilde{U}(m)\mathbf{1}_{m \le X/4}.$$

On I this equals the sum of the four losses. It is supported on z-rough integers with $\Omega(m) \le 9$, and

$$0 \le \widetilde{U}^\flat(m) \le \binom{9}{2} + \binom{9}{4} = 162. \tag{8.2}$$

For each four-prime prefix, the three possibilities L_4, U_4, V_4 are disjoint.

Let $m = \ell_1 \cdots \ell_r$ be squarefree, with $\ell_1 > \cdots > \ell_r \ge z$. Put $t_i = \log \ell_i / L$ and $s = \sum_i t_i$. A marked prefix has residual roughness at its smallest selected

prime. This holds precisely when that prime is ℓ_r, also when the residual is 1. Thus $\widetilde{U}(m) = G_r(t)$, where

$$\begin{aligned} G_r(t) = &\sum_{i<r} \mathbf{1}[t_i < 9/20,\ t_i + t_r > 11/20,\ 2t_i \le s] \\ &+ \sum_{i_1<i_2<i_3<r} \mathbf{1}[t_{i_1} + t_{i_2} + t_{i_3} + t_r < 9/20,\ 2t_{i_1} \le s] \\ &+ \sum_{i_1<i_2<i_3<r} \mathbf{1}[t_{i_1} + t_{i_2} + t_{i_3} < 9/20, \\ &\qquad t_{i_1} + t_{i_2} + t_{i_3} + t_r > 11/20,\ 2t_{i_1} \le s] \\ &+ \sum_{i_1<i_2<i_3<r} \mathbf{1}[t_{i_1} + t_{i_2} < 9/20, t_{i_1} + t_{i_2} + t_{i_3} > 11/20,\ 2t_{i_1} \le s]. \end{aligned} \tag{8.3}$$

In particular, $G_1 = 0$ and $\widetilde{U}(1) = 0$. For $2 \le r \le 9$, let $\mathcal{H}_r$ consist of the hyperplanes

$$\sum_{i\in J} t_i = 9/20 \text{ or } 11/20 \quad (1 \le |J| \le 4), \quad 2t_i = \sum_k t_k, \quad t_i = t_j \quad (i \ne j). \tag{8.4}$$

These form a fixed finite rational arrangement. Off it, G_r is constant in each component of the strictly ordered cone, and the bound 162 holds there for $r \le 9$.

8.2. **Initial reductions and single primes.** Partial summation and (2.10) give

$$\Delta_\lambda(d) \le 2L\Delta_\rho(d), \qquad \Delta_\rho(d) \le \Delta_{\mathbf{1}_{\mathbb{P}}\mathbf{1}_I}(d) + \Delta_{\widetilde{U}^\flat \mathbf{1}_I}(d). \tag{8.5}$$

We split $\widetilde{U}^\flat$ into squarefree and nonsquarefree parts and add the divisor weight last.

(F8) [1, Chapter 28] For every $H > 0$, some $K(H) > 0$ satisfies

$$\sum_{d \le X^{1/2}L^{-K(H)}} \max_{(a,d)=1} \max_{y\le X} \left| \sum_{\substack{n\le y \\ n\equiv a \pmod d}} \Lambda(n) - \frac{y}{\varphi(d)} \right| \ll_H XL^{-H}. \tag{8.6}$$

Since $Q_B/(X^{1/2}L^{-K(H)}) = X^{-1/20-\eta}L^{K(H)} \to 0$, this covers all required moduli. Prime powers contribute $O(\sqrt{X}L)$ per modulus. Partial summation converts prime logarithmic sums to prime counts and retains the maximum in y. The modulus 1 case controls the corresponding total prime count. Removing primes that divide d costs $O(\omega(d)/\varphi(d))$, so the three conversions have total cost

$$Q_B X^{1/2} L + \sum_{d\le Q_B} \frac{\omega(d)}{\varphi(d)} \ll X^{19/20-\eta}L.$$

Consequently, for every $H > 0$,

$$\sum_{d\le Q_B} \Delta_{\mathbf{1}_{\mathbb{P}}}(d) \ll_H XL^{-H}. \tag{8.7}$$

Single primes are handled by this estimate; Lemma B is not used for them.

8.3. **Absolute-coordinate cubes.** Fix $C_0 > 1$. Put

$$\varpi := \frac{1}{10N_\varpi}, \qquad N_\varpi = \lceil L^{C_0} \rceil, \qquad \frac{1}{20}L^{-C_0} \le \varpi \le \frac{1}{10}L^{-C_0}, \qquad 1/10 \in \varpi\mathbb{Z}.$$

For $2 \le r \le 9$, use cubes

$$C = \prod_{i=1}^{r} [a_i, a_i + \varpi), \qquad a_i \in \tfrac{1}{10} + \varpi\mathbb{Z}_{\ge 0}, \quad a_i < 1, \quad a_1 \ge \cdots \ge a_r. \tag{8.8}$$

Let $\mathcal{B}_i(C) = \mathbb{P} \cap [X^{a_i}, X^{a_i+\varpi})$ and

$$\mathrm{conv}_C := \mathbf{1}_{\mathcal{B}_1(C)} * \cdots * \mathbf{1}_{\mathcal{B}_r(C)}.$$

Every prime is at least z, the box ratio is $X^\varpi \le \exp(L^{1-C_0}/10) \le 2$, and there are $O(L^{9C_0})$ cubes. An interior cube has closure disjoint from all hyperplanes in $\mathcal{H}_r$, and every other cube is a boundary cube. Only dimensions $2 \le r \le 9$ are used, since the discarded weight vanishes when $r \le 1$.

For an interior cube, the prime boxes are pairwise disjoint and ordered, so unique factorisation gives $\mathrm{conv}_C(m) \in \{0, 1\}$. It is 1 precisely for the squarefree integers with $\Omega(m) = r$ and $t(m) \in C$. Let $c_C \in [0, 162] \cap \mathbb{Z}$ be the constant value of G_r on C. Then, for every integer $m \ge 1$,

$$\widetilde{U}^\flat(m)\mathbf{1}_{\mathrm{sqf}}(m)\mathbf{1}_{t(m)\in C} = c_C \,\mathrm{conv}_C(m)\mathbf{1}_{m \le X/4}. \tag{8.9}$$

The cutoff on the right is retained, and c_C counts all admissible marked prefixes.

8.4. **Distribution of each convolution.** For any cube, including boundary cubes, split

$$\mathrm{conv}_C = \alpha * \beta, \qquad \alpha = \mathbf{1}_{\mathcal{B}_1(C)}, \qquad \beta = \mathbf{1}_{\mathcal{B}_2(C)} * \cdots * \mathbf{1}_{\mathcal{B}_r(C)}.$$

Both sequences are supported on rough integers greater than 1, and their coefficients are bounded by 1 and $(r-1)! \le 8!$. Lemma C supplies (7.1) at every fixed level. For $r = 2$, use its prime-box assertion. For $r > 2$, convolve one prime box with the other $r - 2$ boxes; the latter convolution has coefficients at most 7! and bounded support ratio. All constants are uniform over the cubes.

We specify the dyadic endpoints. Put $N_\star = \prod_{i=2}^{r} X^{a_i}$. The support of β lies in $[N_\star, 2^{r-1}N_\star)$, and we use the r intervals

$$(2^{j-1}N_\star, 2^j N_\star], \qquad 0 \le j \le r-1.$$

They begin at $N_\star/2$, so no integer at $N_\star$ is omitted. Use two intervals anchored at $X^{a_1}/2$ for α. Lemma C controls each restriction, and changing to the dyadic scale costs at most a factor 2^9. Lemma B therefore gives, uniformly in C,

$$\sum_{d \le Q_B} \Delta_{\mathrm{conv}_C \mathbf{1}_{[1,X/4]}}(d) \le \sum_{d \le Q_B} \Delta_{\mathrm{conv}_C}(d) \ll_{B'} XL^{-B'}. \tag{8.10}$$

The distribution estimate is applied to the full product m before truncation, and the modulus remains d throughout.

8.5. **Boundary cubes and positive transfer.** Define

$$\mathcal{E}(m) := \sum_{C \text{ boundary}} \widetilde{U}^\flat(m)\mathbf{1}_{\mathrm{sqf}}(m)\mathbf{1}_{t(m)\in C},$$

$$W := \sum_{C \text{ boundary}} \mathrm{conv}_C \geq 0, \qquad W^\flat := W\mathbf{1}_{[1,X/4]}.$$

Then $W \geq 0$ and $|\mathcal{E}| \leq 162W^\flat$. Every relevant sorted vector lies in exactly one cube. Hence

$$\widetilde{U}^\flat\mathbf{1}_{\mathrm{sqf}} = \sum_{C \text{ interior}} c_C \,\mathrm{conv}_C\, \mathbf{1}_{[1,X/4]} + \mathcal{E}. \tag{8.11}$$

Lemma 8.2. *If $|E| \leq \mathcal{C}W$ and $W \geq 0$, then*

$$\Delta_E(d) \leq \mathcal{C}\Delta_W(d) + \frac{2\mathcal{C}}{\varphi(d)} \sum_{m\leq X} W(m).$$

Proof. Bound each of the two terms defining the discrepancy of E by its absolute value, and replace $|E|$ by $\mathcal{C}W$. The residue sum of W is at most its coprime mean plus $\Delta_W(d)$, and the two means are each bounded by $\varphi(d)^{-1}\sum_{m\leq X} W(m)$. These bounds are uniform in the residue and the cutoff. □

By (8.10),

$$\sum_{d\leq Q_B} \Delta_{W^\flat}(d) \ll L^{9C_0}XL^{-B'}. \tag{8.12}$$

This includes cubes meeting the diagonals. For a term with $r \leq 9$ prime boxes, split off one box as α and put the remaining $r-1$ boxes in β. The β coefficients are at most $(r-1)! \leq 8!$, which is the bound used by Lemma B. The full convolution can have coefficients up to $r! \leq 9!$. There are no one-dimensional boundary convolutions.

Lemma 8.3. *We have*

$$\sum_{m\leq X} W(m) \ll \varpi LX + X/z \ll XL^{1-C_0} + X^{9/10}.$$

Proof. A tuple in a boundary cube lies within distance ϖ, in the maximum norm, of some hyperplane $c \cdot t = c_0$ in (8.4). Choose i_0 with $c_{i_0} \neq 0$, and fix the other primes, whose product is $M' \leq X/z$. The remaining prime lies in

$$[T, Te^{\kappa\varpi L}] \cap [z, X/M']$$

This interval contains $O(\varpi LX/M' + 1)$ integers, and by (F3),

$$\sum_{\text{tuples}} \frac{1}{M'} \leq \left(\sum_{z\leq p\leq X} \frac{1}{p}\right)^{r-1} \leq 3^8$$

for large X. There are at most $(r-1)!\lfloor X/z\rfloor \leq 8!X/z$ labelled tuples with product at most X/z. The number of hyperplanes and dimensions is fixed, so summing proves the assertion. □

Using $\sum_{d\leq Q_B}\varphi(d)^{-1} \ll L$, the last two lemmas give

$$\sum_{d\leq Q_B} \Delta_{\widetilde{U}^\flat\mathbf{1}_{\mathrm{sqf}}}(d) \ll L^{9C_0}XL^{-B'} + L(XL^{1-C_0} + X^{9/10}). \tag{8.13}$$

8.6. **Repeated prime factors.** The nonsquarefree part has modulus at most 162, and it is supported on $p^2 \mid m$ for some prime $p \ge z$. Fix $(a,d)=1$. If $p \mid d$, this divisibility is impossible in the reduced residue class. If $p \nmid d$, the Chinese remainder theorem gives

$$\#\{m \le X : m \equiv a \pmod d,\ p^2 \mid m\} \le X/(dp^2) + 1.$$

Summing over $z \le p \le \sqrt{X}$ gives $O(X/(dz) + \sqrt{X})$, and the coprime-average term is $O(X/(\varphi(d)z))$. Therefore

$$\sum_{d \le Q_B} \Delta_{\widetilde{U}^\flat \mathbf{1}_{\mathrm{nsqf}}}(d) \ll XL/z + Q_B\sqrt{X} = X^{9/10}L + X^{19/20-\eta}. \tag{8.14}$$

Repeated factors remain in the weight. This step counts multiples directly and needs no distribution estimate at modulus dp^2.

8.7. **The divisor weight and the parameter choices.** Since $|\lambda| \le 256L$,

$$\Delta_\lambda(d) \ll XL/\varphi(d) + L.$$

The elementary divisor estimates

$$\sum_{d \le Q} \frac{4^{\omega(d)}}{\varphi(d)} \ll L^4, \qquad \sum_{d \le Q} 4^{\omega(d)} \ll QL^3$$

give

$$\sum_{d \le Q_B} 4^{\omega(d)} \Delta_\lambda(d) \ll XL^5. \tag{8.15}$$

To prove them, expand $4^{\omega(d)}$ over squarefree divisors and use $\varphi(ab) \ge \varphi(a)\varphi(b)$; the corresponding Euler products have logarithmic exponents 3 and 4. Cauchy–Schwarz yields

$$\sum_{d \le Q_B} 2^{\omega(d)} \Delta_\lambda(d) \le \left(\sum_{d \le Q_B} \Delta_\lambda(d)\right)^{1/2} \left(\sum_{d \le Q_B} 4^{\omega(d)} \Delta_\lambda(d)\right)^{1/2}. \tag{8.16}$$

Proof of Proposition 8.1. Given $B > 0$, choose, in this order,

$$H := 2B + 8, \qquad C_0 := H + 3, \qquad B' := H + 9C_0 + 1,$$

then $\mathcal{D} = B' + 2$ and $\mathcal{A} = 3B' + 5$, and finally take X sufficiently large. Equations (8.7), (8.13), (8.14) and (8.5) give

$$\sum_{d \le Q_B} \Delta_\rho(d) \ll XL^{-H}, \qquad \sum_{d \le Q_B} \Delta_\lambda(d) \ll XL^{1-H}.$$

Combining (8.15) and (8.16) gives $XL^{3-H/2} = XL^{-B-1}$, which proves (8.1). □

9. Major arcs

Only this section uses $H_{7/8}$. We first prove the prime-side mean square. The argument follows [7, Section 8], with the density exponents computed below.

9.1. **Classical inputs.** Let $N^*(\sigma,T,Q')$ count, with multiplicity, the zeros $\varrho=\beta+i\gamma$ satisfying $\beta\ge\sigma$ and $|\gamma|\le T$, summed over primitive characters of conductor at most Q'.

(F9) [3, Theorem 1, p. 231] For $11/14\le\sigma\le1$,

$$N^*(\sigma,T,Q')\ll_\varepsilon (Q'^2T)^{2(1-\sigma)+\varepsilon}.$$

(F10) [3, Theorem 2(3), p. 232]; [7, Lemma 4.17]. Uniformly for $1/2\le\sigma\le1$,

$$N^*(\sigma,T,Q')\ll_\varepsilon (Q'^2T^{6/5})^{(20/9)(1-\sigma)+\varepsilon}.$$

(F11) [5, Theorem 12.2]; [7, Lemma 4.16]. For $1/2\le\sigma\le1$,

$$N^*(\sigma,T,Q')\ll (Q'^2T)^{3(1-\sigma)/(2-\sigma)}\log^9(Q'T).$$

(F12) [2, Lemma 1]; [7, Lemma 4.11]. For a finite sequence a_n and $\kappa>0$,

$$\int_{-\kappa}^{\kappa}\left|\sum_n a_n e(n\beta)\right|^2 d\beta\ll\kappa^2\int\left|\sum_{x<n\le x+1/(2\kappa)} a_n\right|^2 dx.$$

(F13) [4, Section 3.4]; [1, Chapter 9]. If $\chi \bmod q$ is induced by a primitive character of conductor r, then $|\tau(\chi)|^2\le r$. Also $\tau(\chi_{0,q})=\mu(q)$.

(F14) [1, Chapters 15–19]; [7, Lemma 4.15]. The explicit formula for primitive Dirichlet characters has the uniform truncation error

$$O\Big(\frac{x\log^2(rxT)}{T}+x^{1/4}\log x\Big).$$

The number of zeros in a unit height interval is $O(\log(r(|T|+2)))$.

9.2. **The prime-side mean square.** On $\mathfrak{M}(q,a)$, put

$$\mathcal{T}(\beta)=\sum_{X_1<k\le X} e(k\beta),\qquad S_1(a/q+\beta)=S(a/q+\beta)-\frac{\mu(q)}{\varphi(q)}\mathcal{T}(\beta). \tag{9.1}$$

The arcs are disjoint, so this defines S_1 on $\mathfrak{M}$.

Proposition 9.1. *Assume $H_{7/8}$. Let $0<\eta\le1/100$, $0<\varepsilon_0\le\eta/10$ and $X_1=X^{1-\varepsilon_0}$. Uniformly for $2\le P\le X^{9/20-\eta}$,*

$$\int_{\mathfrak{M}}|S_1(\alpha)|^2d\alpha\ll_\eta L^{18}X^{1-8\eta/9}+L^8P^2\ll_\eta L^{18}X^{1-8\eta/9}. \tag{9.2}$$

Proof. Put $\vartheta=\log P/L$, $Q=X/P$ and $\epsilon=\eta/90$, and write $\varrho=1-\delta+i\gamma$.
Character expansion. Every prime in S is coprime to $q\le P$, since $P<\sqrt{X}<X_1$. For the character χ^* inducing $\chi \bmod q$,

$$S(a/q+\beta)=\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(a)\tau(\bar\chi)\sum_{X_1<p\le X}\chi^*(p)\log p\,e(\beta p).$$

There is one exterior factor $1/\varphi(q)$. For primitive $\chi \bmod r$, put

$$\begin{aligned} S_1(\chi^*,\beta) &= \sum_n a_n(\chi^*)e(n\beta),\\ a_n(\chi^*) &= \mathbf{1}_{X_1<n\le X}\big(\mathbf{1}_{\mathbb{P}}(n)\chi^*(n)\log n - E(\chi^*)\big),\\ E(\chi^*) &= \mathbf{1}[r=1]. \end{aligned}$$

Orthogonality in a, followed by (F13), gives

$$\begin{aligned} \int_{\mathfrak{M}} |S_1|^2 &\ll L^2 \sum_{r\le P} \sideset{}{^*}\sum_{\chi \bmod r} W_1(\chi)^2,\\ W_1(\chi)^2 &= \int_{|\beta|\le 1/(rQ)} \left|\sum_n a_n(\chi)e(n\beta)\right|^2 d\beta. \end{aligned} \tag{9.3}$$

Here we used

$$\sum_{\substack{q\le P\\ r\mid q}} \frac{1}{\varphi(q)} \le \frac{1}{\varphi(r)} \sum_{l\le P/r} \frac{1}{\varphi(l)} \ll \frac{L}{\varphi(r)}, \qquad \frac{r}{\varphi(r)} \ll L.$$

This includes all moduli $q \le P$.

Short intervals. Apply (F12) with $Y = rQ/2 \le X/2$, and let $X_2 = \max(X_1, 6Y)$. Then

$$W_1(\chi)^2 \ll \int_{X_1-Y}^{X} \left| Y^{-1} \sum_{x<n\le x+Y} a_n(\chi)\right|^2 dx = I_1 + I_2 + I_3. \tag{9.4}$$

With empty intervals omitted, the three ranges are

$$I_1 = [X_1 - Y, \min(X_2, X-Y)], \qquad I_2 = [X_2, X-Y], \qquad I_3 = [X-Y, X].$$

The first has length at most $7Y$.

The explicit formula. Put $T_0 = 3\sqrt{X}$ and $\Lambda_{\mathbb{P}}(n) = \mathbf{1}_{\mathbb{P}}(n)\log n$. By (F14), uniformly for $X_1/2 \le u_1 < u_2 \le 9X$,

$$\sum_{u_1<n\le u_2} \big(\chi(n)\Lambda_{\mathbb{P}}(n) - E(\chi)\big) = -\sum_{\substack{\varrho:\ \beta\ge 1/2\\ |\gamma|\le T_0}} \frac{u_2^{\varrho} - u_1^{\varrho}}{\varrho} + O(\sqrt{X}L^2). \tag{9.5}$$

Here prime powers cost $O(\sqrt{X}L)$, the zeros with $\beta < 1/2$ cost $O(\sqrt{X}L^2)$, and the truncation costs $O(XL^2/T_0 + X^{1/4}L)$. The pole cancels the term $E(\chi)$ up to $O(1)$. Constant terms cancel in the difference, and their logarithmic terms cost $O(L)$. Hypothesis $H_{7/8}$ and the functional equation keep all nontrivial zeros in $1/8 \le \beta \le 7/8$, so every zero retained in (9.5) has

$$1/8 \le \delta \le 1/2. \tag{9.6}$$

The squared error in the two end ranges contributes at most

$$\sum_{r\le P} rY^{-1}XL^4 \ll L^4P^2.$$

The middle-range error is treated below.

Classes of zeros. Partition conductors into intervals $R < r \le 2R$ with dyadic R, including the initial interval $R = 1/2$ to cover $r = 1$. For $\mathcal{M} = 2^\mu$, $\mu \ge 0$, use the height intervals

$$(\mathcal{M} - 1)P/R \le |\gamma| \le (2\mathcal{M} - 1)P/R.$$

Shared endpoints are assigned once. Also use $h - 1/L < \delta \le h$, with $h \in L^{-1}\mathbb{Z}$. There are $O(L^3)$ classes, and $O(L^2)$ for each character. By (9.6), only $1/8 \le h \le 1/2 + 1/L$ occur. Within a class,

$$X^{-\delta} \le eX^{-h}, \qquad {X_1}^{-\delta} \le e\,{X_1}^{-h}, \qquad |\gamma| \le T_{\mathcal{M}} := 2\mathcal{M}P/R, \qquad T_{\mathcal{M}} \ge 2.$$

For one character the class contains $O((\mathcal{M}P/R)L)$ zeros, and over all characters it contains at most $N^*(\sigma_h, T_{\mathcal{M}}, 2R)$, where

$$\sigma_h = \max(1 - h, 1/2), \qquad \mathcal{K}(R, \mathcal{M}, h) = \mathcal{M}^{-1} N^*(\sigma_h, 2\mathcal{M}P/R, 2R)\, {X_1}^{-2h}. \tag{9.7}$$

For the terminal classes $h > 1/2$, the value $\sigma_h = 1/2$ is valid because (9.5) already restricts to $\beta \ge 1/2$. We also have

$$\min\left(1, \frac{2P}{r|\gamma|}\right) \le \frac{4}{\mathcal{M}}. \tag{9.8}$$

The upper end range. For $x \in [X - Y, X]$, the zero term is bounded by

$$\left|\frac{X^\varrho - x^\varrho}{Y\varrho}\right| \le \min\left(x^{-\delta}, \frac{2X^{1-\delta}}{Y|\gamma|}\right) \ll X^{-\delta} \min\left(1, \frac{2P}{r|\gamma|}\right) \ll \frac{X^{-h}}{\mathcal{M}}.$$

Apply Cauchy–Schwarz first to the classes for one character and then to the zeros in one class. For fixed $R, \mathcal{M}, h$, its contribution after summing characters is

$$\begin{aligned} L^2 \cdot RQ \cdot \frac{X^{-2h}}{\mathcal{M}^2} \cdot \frac{\mathcal{M}P}{R} L \cdot N^*(\sigma_h, T_{\mathcal{M}}, 2R) \\ = L^3 X \mathcal{M}^{-1} N^*(\sigma_h, T_{\mathcal{M}}, 2R) X^{-2h} \le L^4 X \mathcal{K}(R, \mathcal{M}, h). \end{aligned}$$

Summing all classes gives $O(L^7 X \max \mathcal{K})$.

The lower end range. Here the window is $(\max(x, X_1), x + Y]$, and its zero term is at most

$$\begin{aligned} \left|\frac{(x+Y)^\varrho - \max(x, X_1)^\varrho}{Y\varrho}\right| &\le \min\left({X_1}^{-\delta}, \frac{2X^{1-\delta}}{Y|\gamma|}\right) \\ &\le {X_1}^{-\delta} \min\left(1, \frac{2X}{Y|\gamma|}\right) \ll \frac{{X_1}^{-h}}{\mathcal{M}}. \end{aligned}$$

Here ${X_1}^{-\delta}$ is a power of $X_1 = X^{1-\varepsilon_0}$ with exponent $-\delta$. The numerator $2X^{1-\delta}$ is a power of X itself, with exponent $1 - \delta$.

The same calculation gives $O(L^7 X \max \mathcal{K})$. This range requires the factor ${X_1}^{-2h}$ in place of X^{-2h}.

The middle range. Partition it at $x_j = 2^j X_2$, using $x_j, \min(2x_j, X - Y)$. The final interval is truncated at $X - Y$, and every window remains inside X_1, X. Write

$$f(u) = \sum_{n \le u} \big(\chi(n)\Lambda_{\mathbb{P}}(n) - E(\chi)\big).$$

For $x \ge X_2 \ge 6Y$, the following estimate also bounds a truncated interval:

$$\int_x^{2x} \left|\frac{f(u+Y) - f(u)}{Y}\right|^2 du \le \frac{30}{Y^2} \max_{Y/(3x) \le \theta' \le 3Y/x} \int_x^{3x} |f(u + \theta' u) - f(u)|^2 du. \tag{9.9}$$

To prove it, for $2Y \le v \le 3Y$ write

$$g(u) = f(u+Y) - f(u) = \big[f(u+v) - f(u)\big] - \big[f((u+Y)+(v-Y)) - f(u+Y)\big].$$

Average over v and use Cauchy–Schwarz. In the first bracket substitute $\theta' = v/u$; its contribution is at most $(2/Y)(2x)(3Y/x) = 12$ times the maximum integral. In the second substitute $\theta' = (v-Y)/(u+Y)$; its contribution is at most $(2/Y)(3x)(3Y/x) = 18$ times that maximum. The common range is $Y/(3x), 3Y/x$, and division by Y^2 proves (9.9).

Here $\theta' \le 1$, and by (9.5) the zero coefficient is

$$f(u+\theta' u) - f(u) = -\sum_{\varrho} c_\varrho u^\varrho + O(\sqrt{X}L^2),$$

$$c_\varrho = \frac{(1+\theta')^\varrho - 1}{\varrho}, \qquad |c_\varrho| \le \min\left(\theta', \frac{3}{|\varrho|}\right).$$

Also

$$\left|\int_x^{3x} u^{\varrho+\bar{\varrho}'}\,du\right| \ll \frac{x^{3-\delta-\delta'}}{1+|\gamma-\gamma'|}.$$

The unit-interval zero count in (F14) gives $\sum_{\varrho'}(1+|\gamma-\gamma'|)^{-1} \ll L^2$ for one character. The arithmetic-geometric mean inequality in each pair yields

$$\begin{aligned}\frac{1}{Y^2}\int_x^{3x}\left|\sum_\varrho c_\varrho u^\varrho\right|^2 du &\ll \frac{L^2}{Y^2}\sum_\varrho x^{3-2\delta}\min(\theta', 3/|\varrho|)^2 \\ &\ll \frac{L^2}{Y}\sum_\varrho x^{2-2\delta}\min(\theta', 3/|\varrho|).\end{aligned}$$

Each zero contributes at most

$$L^2 x^{1-2\delta}\min\left(1, \frac{x}{Y|\gamma|}\right) \ll L^2\frac{X^{1-2h}}{\mathcal{M}}.$$

Here $X^{1-2h} = X\cdot X^{-2h}$ is a power of X itself. Since $X_1 < X$, the factor X^{-2h} is at most the separate class factor ${X_1}^{-2h}$ in (9.7). We used $1-2\delta \ge 0$ and $Y \ge RX/(2P)$. Summing the class counts and the $O(L)$ intervals gives the total zero contribution $O(L^6 X \max\mathcal{K})$. The error in (9.5) contributes, per interval, $O(X^2L^4/Y^2)$, and after the interval and character sums it is

$$\sum_{r\le P} r\cdot\frac{X^2L^5}{Y^2} \ll \sum_{r\le P} r\cdot\frac{X^2L^5}{(rX/(2P))^2} \ll L^6P^2.$$

Combining the three ranges gives

$$\sum_{r\le P}\sideset{}{^*}\sum_{\chi \bmod r} W_1(\chi)^2 \ll L^7 X\max\mathcal{K} + L^6P^2. \tag{9.10}$$

Density estimates for the classes. Recall $\vartheta \le 9/20 - \eta$, $\epsilon = \eta/90$ and $\varepsilon_0 \le \eta/10$. First suppose $1/8 \le h \le 3/14$, and use (F9) with $Q' = 2R$ and $T = 2\mathcal{M}P/R$. Then $Q'^2T \le 8P^2\mathcal{M}$, and the exponent $2h+\epsilon-1$ of $\mathcal{M}$ is negative. Thus $\mathcal{K} \ll X^{-s_3(h)}$, where

$$\begin{aligned} s_3(h) &= 2h(1-\varepsilon_0-2\vartheta) - 2\vartheta\epsilon \\ &\ge \frac{1}{4}\Big(\frac{1}{10} + 2\eta - \varepsilon_0\Big) - \frac{9}{10}\epsilon \ge \tfrac{1}{40} + \tfrac{93}{200}\eta \ge \tfrac{1}{40}. \end{aligned} \tag{9.11}$$

The exact positivity condition is $1-\varepsilon_0-2\vartheta>\vartheta\epsilon/h$, and at $h=1/8$ its right side is $8\vartheta\epsilon$.

Next suppose $3/14\le h\le 3/8-\epsilon$, and use (F10). Here

$$Q'^2T^{6/5}=4\cdot 2^{6/5}R^{4/5}(\mathcal{M}P)^{6/5}\le 10P^2\mathcal{M}^{6/5}.$$

The exponent of $\mathcal{M}$ is $(8/3)h+(6/5)\epsilon-1\le -22\epsilon/15<0$. Consequently $\mathcal{K}\ll_\epsilon X^{-s_2(h)}$, with

$$\begin{aligned} s_2(h)&=2h\Big(1-\varepsilon_0-\frac{20}{9}\vartheta\Big)-2\vartheta\epsilon\\ &\ge\Big(\frac{3}{7}\frac{191}{90}-\frac{1}{100}\Big)\eta\\ &\ge\frac{8}{9}\eta. \end{aligned}\tag{9.12}$$

The required margin is

$$1-\varepsilon_0-(20/9)\vartheta\ge 191\eta/90>0.\tag{9.13}$$

For $3/8-\epsilon\le h\le 1/2$, use (F11). The exponent $3h/(1+h)-1$ of $\mathcal{M}$ is nonpositive, and we obtain $\mathcal{K}\ll L^9X^{-s_1(h)}$, where

$$\begin{aligned} s_1(h)&=\frac{2h}{1+h}\Big(1+h-3\vartheta-\varepsilon_0(1+h)\Big)\\ &\ge\frac{2h}{1+h}\Big(\frac{1}{40}+3\eta-\epsilon-\frac{3}{2}\varepsilon_0\Big)\ge\frac{1}{75}. \end{aligned}\tag{9.14}$$

The threshold $\vartheta<11/24$ has margin $1/120+\eta$.

Finally, if $1/2<h\le 1/2+1/L$, then $\sigma_h=1/2$, and the elementary zero count gives

$$N^*(1/2,T,Q')\le N^*(0,T,Q')\ll Q'^2TL=8R\mathcal{M}PL.$$

Since $2h>1$,

$$\mathcal{K}\ll\mathcal{M}^{-1}\cdot 8R\mathcal{M}PL^9\cdot {X_1}^{-1}\le 8L^9P^2X^{-(1-\varepsilon_0)}\ll L^9X^{-1/10}.$$

All classes have now been covered. For $\eta\le 1/100$,

$$\max\mathcal{K}\ll_\eta L^9X^{-c(\eta)},\qquad c(\eta)=\min\big(\tfrac{1}{40},\tfrac{8}{9}\eta,\tfrac{1}{75},\tfrac{1}{10}\big)=\frac{8}{9}\eta.\tag{9.15}$$

Equations (9.3), (9.10) and (9.15) prove (9.2). The term L^8P^2 is absorbed because $P^2\le X^{9/10-2\eta}$. □

9.3. **A variant without the first density estimate.** One may omit (F9) and use (F10) throughout $1/8\le h\le 3/8-\epsilon$. The minimum in (9.12) is then at $h=1/8$, giving

$$s_2\ge\frac{1}{4}\cdot\frac{191}{90}\eta-\frac{\eta}{100}=0.5206\ldots\eta\ge\frac{\eta}{2}.$$

The mean-square saving is $\eta/2$ in place of $8\eta/9$, and the mixed error below becomes $X^{1-\eta/4}L^{19/2}$. All other steps are unchanged.

9.4. **The discrete pole and its kernel.** Parseval and Lemma 4.4, together with Proposition 9.1, give

$$\left|\int_{\mathfrak{M}} S_1(\alpha)S_\lambda(\alpha)e(-n\alpha)\,d\alpha\right| \ll X^{1-4\eta/9}L^{19/2}. \tag{9.16}$$

Put $V(\beta)=\int_{X_1}^{X} e(\beta u)\,du$.

(F15) [9, Lemma 2.8] Euler–Maclaurin summation gives $|\mathcal{T}(\beta)-V(\beta)|\ll 1+X|\beta|$.

On $\mathfrak{M}(q,a)$ this is $O(1+P/q)$. Therefore

$$\int_{\mathfrak{M}}\left|\frac{\mu(q)}{\varphi(q)}(\mathcal{T}-V)\right|^2 \ll \sum_{q\le P}\frac{1}{\varphi(q)}\cdot\frac{P}{qX}\left(1+\frac{P}{q}\right)^2 \ll \frac{P^3}{X}.$$

Its mixed contribution with S_λ is

$$O(P^{3/2}L^{1/2}). \tag{9.17}$$

Let $B_q=P/(qX)$ and $\mathrm{Si}(t)=\int_0^t(\sin u)/u\,du$. The remaining pole term is

$$R_{\mathrm{pole}}(n)=\sum_{q\le P}\frac{\mu(q)}{\varphi(q)}\sum_{m\in I}\lambda(m)c_q(n-m)K_q(n-m), \tag{9.18}$$

$$K_q(v)=\frac{1}{\pi}\Big[\mathrm{Si}\big(2\pi B_q(X-v)\big)+\mathrm{Si}\big(2\pi B_q(v-X_1)\big)\Big]. \tag{9.19}$$

This follows by integrating $V(\beta)e(-\beta v)$ over $[-B_q,B_q]$. By (5.3), both arguments have positive gaps at least $X/8$. Here, since $\varepsilon_0>0$ is fixed and $X_1/X=X^{-\varepsilon_0}\to 0$, we have $X_1\le X/8$ for all sufficiently large X. Differentiating (9.19) gives

$$K_q'(v)=\frac{1}{\pi}\left(\frac{\sin\big(2\pi B_q(v-X_1)\big)}{v-X_1}-\frac{\sin\big(2\pi B_q(X-v)\big)}{X-v}\right). \tag{9.20}$$

On $[X/4,7X/8]$, each of $X-v$ and $v-X_1$ ranges within $[X/8,7X/8]$. Hence

$$\mathrm{Var}_{[X/4,7X/8]}K_q=\int_{X/4}^{7X/8}|K_q'(v)|\,dv\le\frac{2\log 7}{\pi}<2.$$

The elementary estimates $|\mathrm{Si}(t)-\pi/2|\le 2/t$ and $\sup_{t\ge0}\mathrm{Si}(t)<1.86$ give

$$K_q(v)=1+O(q/P),\qquad \|K_q\|_\infty\le 1.19,\qquad \mathrm{Var}_{[X/4,7X/8]}K_q<2. \tag{9.21}$$

We keep K_q until the residue-class calculation is done.

9.5. **Reduced residue classes.** Only squarefree moduli contribute to (9.18). All non-squarefree arcs are already included in (9.16), since there $\mu(q)=0$ and $S_1=S$. In the rest of the pole calculation, q is squarefree.

For $(r,q)=1$, partial summation and (9.21) give

$$\sum_{\substack{m\in I\\ m\equiv r \pmod q}}\lambda(m)K_q(n-m)=\frac{1}{\varphi(q)}\sum_{\substack{m\in I\\(m,q)=1}}\lambda(m)K_q(n-m)+O(7\Delta_\lambda(q)). \tag{9.22}$$

The bounded variation keeps a factor P/q out of this estimate. For squarefree q, multiplicativity gives

$$\begin{aligned}\sum_{(r,q)=1} c_q(n-r) &= \mu(q)c_q(n),\\ \sum_{(r,q)=1} |c_q(n-r)| &\le \sum_{r \bmod q} |c_q(n-r)| = 2^{\omega(q)}\varphi(q).\end{aligned} \tag{9.23}$$

For a prime p, use $c_p(n) = p-1$ when $p \mid n$, and $c_p(n) = -1$ otherwise; the Chinese remainder theorem gives the product identities. Consequently, the coprime part of (9.18) is

$$\sum_{q\le P} \frac{\mu^2(q)c_q(n)}{\varphi(q)^2} \sum_{\substack{m\in I\\(m,q)=1}} \lambda(m)K_q(n-m) + O\left(\sum_{q\le Q_B} 2^{\omega(q)}\Delta_\lambda(q)\right). \tag{9.24}$$

Proposition 8.1 bounds the error by $O(XL^{-B})$, since it applies to this same weight, with both maxima, at level $Q_B \ge P$.

9.6. **Non-coprime classes.** The non-coprime part satisfies

$$|R_{\mathrm{nc}}(n)| \le \sum_{q\le P} \frac{\mu^2(q)}{\varphi(q)} \sum_{\substack{m\in I\\(m,q)>1}} |\lambda(m)c_q(n-m)K_q(n-m)|. \tag{9.25}$$

If a summand is nonzero, some prime $\ell \ge z$ divides both m, q. Since $q < z^5$, there are at most four such primes in q. Write $q = \ell s$. Squarefreeness gives $(\ell, s) = 1$ and

$$c_q(n-m) = c_\ell(n-m)c_s(n-m) = c_\ell(n)c_s(n-m).$$

Also $|c_s(v)| \le (s,v) \le \sum_{d\mid s,\ d\mid v} d$. For $d \mid s$ the conditions $\ell \mid m$ and $d \mid n-m$ are coprime, and their count up to $X/4$ is at most $X/(\ell d)+1$. Thus, termwise in the absolute sum,

$$\begin{aligned}\left|\sum_{\substack{m\in I\\ \ell\mid m}} \lambda(m)c_q(n-m)K_q(n-m)\right| &\ll L|c_\ell(n)| \sum_{d\mid s} d\left(\frac{X}{\ell d}+1\right)\\ &= L|c_\ell(n)|\left(\frac{\tau(s)X}{\ell} + \sigma(s)\right).\end{aligned}$$

$$\begin{aligned}\text{RHS of (9.25)} &\ll L \sum_{z\le\ell\le P} \frac{|c_\ell(n)|}{\varphi(\ell)} \sum_{s\le P/\ell} \frac{\tau(s)X/\ell + \sigma(s)}{\varphi(s)}\\ &\ll L^3 \sum_{z\le\ell\le P} \frac{|c_\ell(n)|}{\varphi(\ell)} \cdot \frac{X+P}{\ell}\\ &\ll \frac{XL^3}{z} = X^{9/10}L^3.\end{aligned}$$

Using $\sum_{s\le S} \tau(s)/\varphi(s) \ll L^2$ and $\sigma(s)/\varphi(s) \ll L$, we obtain

$$|R_{\mathrm{nc}}(n)| \ll L^3 \sum_{z\le\ell\le P} \frac{|c_\ell(n)|}{\varphi(\ell)} \frac{X+P}{\ell} \ll XL^3/z = X^{9/10}L^3. \tag{9.26}$$

If $\ell \nmid n$, sum $1/(\ell\varphi(\ell)) \ll 1/z$. If $\ell \mid n$, its contribution is at most $2X/z$, and at most ten primes at least z divide $n \le X = z^{10}$. This proves uniformity in n.

We next remove $(m,q)=1$ from (9.24). On the support of λ, at most $4(X/z+1)$ non-coprime integers occur. For squarefree q, write $d=(q,n)$ and $q=dr$. Then $|c_q(n)|=\varphi(d)$ and

$$\sum_{q} \frac{\mu^2(q)|c_q(n)|}{\varphi(q)^2} \le \sum_{d\mid n} \frac{1}{\varphi(d)} \sum_{r\ge 1} \frac{1}{\varphi(r)^2} \ll_\nu X^\nu. \tag{9.27}$$

Together with $|\lambda K_q| \ll L$, this bounds the removal error by

$$O_\nu(X^{9/10+\nu}) \qquad (\nu>0). \tag{9.28}$$

The divisor-bound exponent is arbitrary, so it absorbs the logarithm.

9.7. **Completion of the singular series.** The same squarefree factorisation gives, for every $\nu>0$,

$$\sum_{q\le P} \frac{\mu^2(q)q|c_q(n)|}{\varphi(q)^2} \ll_\nu X^\nu, \qquad \sum_{q>P} \frac{\mu^2(q)|c_q(n)|}{\varphi(q)^2} \ll_\nu X^\nu/P. \tag{9.29}$$

For the first bound, use $\sum_{r\le Y} r/\varphi(r)^2 \ll \log(2Y)$ and sum $d/\varphi(d)$ over $d \mid n$. For the second, use $\sum_{r>Y}\varphi(r)^{-2} \ll 1/Y$. These elementary estimates follow by expanding the factors over squarefree divisors. The resulting Euler products converge after the indicated logarithmic factor is removed.

Both steps below use the absolute mass bound of Lemma 4.4, restated in (4.25):

$$\sum_{m\in I} |\lambda(m)| \ll X, \qquad |F_\lambda| \le \sum_{m\in I}|\lambda(m)| \ll X.$$

By (9.21), replacing K_q by 1 costs $O_\nu(X^{1+\nu}/P)$, since $\sum_m |\lambda(m)| \ll X$. Completing the q-sum has the same cost, since $|F_\lambda| \ll X$. The completed series is

$$\begin{aligned} \mathfrak{S}(n) &= \sum_{q\ge 1} \frac{\mu^2(q)c_q(n)}{\varphi(q)^2} \\ &= 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p\mid n\\ p>2}}\frac{p-1}{p-2} \ge 2C_2 \ge 1.32 \qquad (2\mid n), \end{aligned} \tag{9.30}$$

where $C_2=\prod_{p>2}(1-(p-1)^{-2})$ and $2C_2 = 1.3203\ldots$.

9.8. **The major-arc asymptotic and positivity.**

Proposition 9.2. *Assume $H_{7/8}$ and the parameter choices of Section 2. Let $\lambda = (\log m)\rho(m)$ be the weight of Section 3. Thus it is supported on z-rough $m\in I$, with $|\lambda|\le 256L$, $\sum|\lambda|^2 \ll XL$ and $\sum|\lambda| \ll X$. Use Proposition 8.1 for this weight at $Q_B \ge P$. For $B>0$ and $0<\nu\le 1/20$, uniformly for even $n\in(X/2,X]$,*

$$\begin{aligned} I_{\mathfrak{M}}(n) = \mathfrak{S}(n)F_\lambda + O_{\eta,B,\nu}\big(&X^{1-4\eta/9}L^{19/2} + P^{3/2}L^{1/2} + XL^{-B} \\ &+ X^{9/10}L^3 + X^{9/10+\nu} + X^{1+\nu}/P\big). \end{aligned} \tag{9.31}$$

In particular, $I_{\mathfrak{M}}(n) = \mathfrak{S}(n)F_\lambda + o(X)$.

Proof. Use (9.16) and (9.17), and evaluate the pole by (9.24) and (9.26). Then (9.28) and (9.29) give exactly the six errors in (9.31). At the stipulated P,

$$P^{3/2}L^{1/2} \le X^{27/40-3\eta/2}L^{1/2}, \qquad X^{9/10+\nu} \le X^{19/20},$$
$$X^{1+\nu}/P \le 2X^{11/20+\eta+\nu} \le 2X^{61/100}.$$

Each error is $o(X)$, and this conclusion does not require a sign for F_λ. □

Corollary 9.3. *Assume $H_{7/8}$. For all large X and every even $n \in (X/2, X]$,*

$$I_{\mathfrak{M}}(n) \ge 0.025X.$$

Proof. Proposition 4.9 gives $F_\lambda \ge 0$ for large X. By (9.30) and Proposition 9.2,

$$\mathfrak{S}(n)F_\lambda \ge 1.32F_\lambda \ge 1.32\Big(\frac{c_{\mathrm{cert}}}{8} - o(1)\Big)X.$$

The exact rational (4.24) satisfies $1.32c_{\mathrm{cert}}/8 = 0.0250913982659\ldots > 0.025$, and the positive margin absorbs the error terms. □

10. Proof of Theorem 2

Proof of Theorem 2. Let $\varepsilon > 0$ and use (2.1). At each large scale X, choose P as in (2.4). Corollary 9.3 supplies $c_1 = 0.025$ in Proposition 5.1. Corollary 6.4 gives $\mathcal{S} \ll X^{31/40+\eta/2}L^{9/2}$ at the same P, and both use the same signed weight λ. Therefore

$$|\mathcal{E}_X| \ll \frac{\mathcal{S}^2 L}{X} \ll X^{31/20+\eta-1}L^{10} = X^{11/20+\eta}L^{10} = X^{1-t}L^{10}.$$

Choose $X_0 = X_0(\varepsilon)$ so that this holds at every scale at least X_0, and apply it to $Y = X/2^k$ with its own cutoff $P(Y)$. The constants are uniform, and the geometric sum gives

$$E(X) \le X_0 + \sum_{k\ge 0} C_\varepsilon (X/2^k)^{11/20+\eta}(\log X)^{10} \ll_\varepsilon X^{11/20+\eta}(\log X)^{10} \ll_\varepsilon X^{11/20+\varepsilon}.$$

Since $\eta < \varepsilon$, this is $O_\varepsilon(X^{11/20+\varepsilon})$. □

11. Proof of Theorem 1

Proof of Theorem 1. Theorem 2, together with the zero-free half-plane $\Re s > 7/8$ in [6, Theorem 1.1], gives the assertion. □

Appendix A. The exact rational mass certificate

This appendix gives the finite arithmetic for (4.19). The regions are exactly (4.10), and the cell tests are (4.16) and (4.17). No limiting numerical integration is used.

A.1. **Integer form of the cell weights.** For a cell of dimension k on the grid of size n, put

$$r_i = 2n + 7j_i, \qquad r_i^+ = r_i + 7, \qquad q_K = 20n - \sum_{i=1}^{k} r_i^+, \qquad s_K = r_k^+.$$

Thus $x_{i,0} = r_i/(20n)$ and $u_0(K) = q_K/s_K$. Write the rational envelope value as a_K/b_K, using

$$\frac{a_K}{b_K} = \begin{cases} 1, & q_K < s_K, \\ s_K/q_K, & q_K \geq s_K \text{ and } 125s_K \geq 71q_K, \\ 71/125, & q_K \geq s_K \text{ and } 125s_K < 71q_K. \end{cases} \tag{A.1}$$

Reduction to lowest terms is unnecessary, and (4.15) is exactly

$$W(K) = \frac{A_K}{B_K}, \qquad A_K = 7^k(20n)a_K, \qquad B_K = b_K r_1 \cdots r_{k-1} r_k^2. \tag{A.2}$$

For U_2, take $(n, k) = (256, 2)$ and $n > j_1 \geq j_2 \geq 0$; the integer tests are

$$r_1 < 9n, \qquad r_1 + r_2 + 14 > 11n, \qquad r_1 + 2r_2 \leq 20n. \tag{A.3}$$

For V_4, take $(n, k) = (64, 4)$ and $n > j_1 \geq j_2 \geq j_3 \geq j_4 \geq 0$; the integer tests are

$$r_1 + r_2 < 9n, \quad r_1 + r_2 + r_3 + 21 > 11n, \quad r_1 + r_2 + r_3 + 2r_4 \leq 20n. \tag{A.4}$$

These are the original rational cell tests after clearing denominators.

A.2. **Readable rational upper sums.** For each retained cell define the integer

$$b(K) = \left\lceil 10^{18} W(K) \right\rceil = \left\lfloor \frac{10^{18} A_K + B_K - 1}{B_K} \right\rfloor. \tag{A.5}$$

Every cell weight is rational and nonnegative, so $W(K) \leq b(K)/10^{18}$. Tables A.1 and A.2 list sums of these integers, with rows that partition the first-coordinate index j_1. Each entry is obtained by the finite divisions and additions in (A.1)–(A.5).

TABLE A.1. The U_2 upper sum. The final column has denominator 10^{18}.

j_1	Retained cells	$\sum b(K)$
0–127	0	0
128–159	1056	57250926781949766
160–191	2773	153887915136000733
192–223	3376	212249038575035418
224–255	3888	309273586283258073
Total	11093	732661466776243990

The totals imply

$$\begin{aligned} \mathbf{U}_2 &\leq \frac{732661466776243990}{10^{18}} < \frac{732661466777}{10^{12}}, \\ \mathbf{V}_4 &\leq \frac{98012499159664071}{10^{18}} < \frac{98012508379}{10^{12}}. \end{aligned} \tag{A.6}$$

TABLE A.2. The V_4 upper sum. The final column has denominator 10^{18}.

j_1	Retained cells	$\sum b(K)$
0–7	0	0
8–15	60	307379357895441
16–23	7919	38542811400585796
24–31	8263	43932663439422582
32–39	1926	13933990209700055
40–47	118	1295654752060197
48–63	0	0
Total	18286	98012499159664071

The positive differences after multiplication by 10^{18} are 756010 and 9219335929. The final mass calculation uses the original upper bounds on the right; the slightly smaller table bounds are not used.

A.3. **The four losses and the final subtraction.** Table A.3 collects the rational coefficients of $X/8$; the volume calculations for L_4, U_4 are proved in Lemma 4.8.

TABLE A.3. Rational coefficients for the four discarded families.

Family	Calculation	Upper coefficient
U_2	256-grid rectangle sum	$\frac{732661466777}{10^{12}}$
L_4	$C_\omega 10^5 (1/20)^4/576$	$\frac{71}{115200}$
U_4	$C_\omega 10^5 (3/20)^4/1728$	$\frac{213}{12800}$
V_4	64-grid rectangle sum	$\frac{98012508379}{10^{12}}$

Exact subtraction gives

$$1 - \frac{732661466777}{10^{12}} - \frac{71}{115200} - \frac{213}{12800} - \frac{98012508379}{10^{12}} = \frac{342155430899}{2250000000000}.$$

In particular,

$$c_{\mathrm{cert}} - \frac{3}{20} = \frac{4655430899}{2250000000000} > 0.$$

The envelope inequality is also certified rationally:

$$\frac{71}{125} \sum_{j=0}^{4} \frac{(71/125)^j}{j!} = \frac{733930882351}{732421875000} > 1.$$

The accompanying script `cert1120.py` evaluates the original cell sums with exact rational arithmetic. Its output confirms the cell counts, the two upper bounds, the two volume coefficients and the displayed value of c_{cert}. The analytic reduction to these finite sums is proved in Section 4. The script checks only the arithmetic and supplies no asymptotic estimate.

Statements and declarations

Competing interests. The author has no competing interests.

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Data availability. The rational certificate is specified in Appendix A. The accompanying script `cert1120.py` and its output reproduce the finite sums.

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