THEOREM (1 + 1.75) ON THE GOLDBACH CONJECTURE

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Abstract. Every sufficiently large even integer N has a representation $N = p + rq$, where p, q are prime, $r = 1$ or prime, and $r \le q^{a-1} = q^{3/4}$. In the terminology of Li and Liu, this proves Proposition $(1 + a)$ for $a = 1.75 = 7/4$. A weighted sieve combines thirteen signed row estimates with a switched upper bound. The directed comparison gives $D_B > 1.967088903489691335582703173494$ and $c_{K2,G0}I(b) < 1.966859949178424046530804965344$. The exact margin enclosure is $D_B - c_{K2,G0}I(b) > 0.000228954311267289051898208151$.

1. Introduction

We represent large even integers by two primes and one factor. Write $\mathbb{P}$ for the primes. Chen's theorem represents every sufficiently large even integer as a prime plus an integer with at most two prime factors [5]. Paper II supplies the quantitative smaller-factor starting point and the signed sieve sums used here. We change the exterior and switched coefficients and add retained and sixth-sum gains. Table 1.1 lists their separate costs. We use the terminology Proposition $(1 + a)$ introduced by Li and Liu [19, p. 2, (1.6)]. Runbo Li proved lower bounds for $D_{1,2}(N)$ [29]; we do not use that result.

Cai established a small-prime variant of Chen's theorem [4]. Wu developed Chen's double sieve for Goldbach and twin-prime problems [30]. His later paper continues this double-sieve program [31]. Lichtman modified the linear sieve and studied bounds for the twin-prime counting function [20]. He also proved results on large-modulus distribution and Goldbach beyond the square-root barrier [21]. Maynard's first paper proves large-modulus estimates for fixed residue classes [22]. His second gives well-factorable estimates in that setting [23]. His third treats estimates uniform over residue classes [24]. Pascadi studies exponents of distribution for primes and smooth numbers [28]. Li and Liu proved Proposition $(1 + 1.9)$, with a lower bound for $D_{1,1.9}(N)$ [19, Theorem 1.1]. The present proof does not use their estimates.

Theorem 1 (Representation theorem)**.** *Let σ_* be a rational number satisfying* (9.14)*, and let $N_0^*(\sigma_*)$ be the threshold* (2.5)*. Then every even $N \ge N_0^*(\sigma_*)$ admits*

$$N = p + rq, \qquad p, q \in \mathbb{P}, \qquad r \in \{1\} \cup \mathbb{P}, \qquad r \le q^{a-1} = q^{3/4}. \tag{1.1}$$

Section 5 proves the retained comparison (A1) of Table 1.2. Section 6 proves the sixth-sum estimate (A2). Section 7 proves the exterior comparison (A3). Section 8 proves the switched comparison (A4). The constant bounds (A5) follow from the cited Paper I and Paper II estimates, including Paper I, Theorem 2.6, (2.31) [26, Theorem 2.6, (2.31)]. Hence a rational σ_* as in the theorem exists; $N_0^*(\sigma_*)$ may be ineffective.

The proof uses Paper I, [26, Theorems 2.6, 2.8, 3.2 and 4.1], and Paper II, [27, (1.7)–(1.12), (6.4)–(6.4d)]. We restate each input below with its precise scope.

1.1. A positive count.

Put

$$\varepsilon = \frac{1}{10000}, \qquad a = \frac{7}{4}, \qquad b = \frac{3}{7} - \frac{1}{10^6} = \frac{2999993}{7000000}. \tag{1.2}$$

Define

$$\begin{aligned} \mathcal{A}_N &= \{N - p : p \in \mathbb{P},\ p < (1 - \varepsilon)N\}, \\ \mathcal{P}_b &= \mathbb{P} \cup \{rq : r, q \in \mathbb{P},\ r \le q,\ r < N^b\}, \\ G_b(N) &= \#(\mathcal{A}_N \cap \mathcal{P}_b). \end{aligned} \tag{1.3}$$

For a counted semiprime,

$$\begin{aligned} q &> \varepsilon N^{1-b}, \\ q^{3/4} &> \varepsilon^{3/4} N^{b+\kappa_{\mathrm{tr}}} \ge N^b > r \qquad (N \ge N_{\mathrm{tr}}). \end{aligned} \tag{1.4}$$

The transfer parameters are defined in (2.3), and Lemma 9.2 proves (1.4). A counted prime gives $r = 1$. Hence $G_b(N) > 0$ implies (1.1).

Use Paper II's normalization in (2.2) [27, (1.2), §6.1]. The convergent product gives $C_2 > 0$. Thus $W(N) > 0$.

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1.2. **The comparisons.** Use the sums of Paper II [27, (1.7)–(1.11)]. Put

$$\begin{aligned} R &= 3S_1 - 2S_3 - S_4 - S_5 - S_7 + S_8 + S_9 + S_{10} - 2S_{11} - S_{12} - S_{13},\\ \Sigma_W &= R + S_2 - S_6 - S_{\mathrm{sw}},\\ B_b(N) &= \sum_{\substack{r,q\in\mathbb{P}\\ N^b\le r\le\sqrt{N}\\ \varepsilon N<rq<N}} 1_{N-rq\in\mathbb{P}}. \end{aligned} \tag{1.5}$$

Here S_2 is the complete Paper II sum. All original coprimality conditions remain imposed. The last sum counts each eligible ordered prime pair; both orientations are counted only when both satisfy the displayed cutoff, and a square is counted once. Here $R = R_{\mathrm{fix}}$ and $\Sigma_W = \mathcal{W}$ in Section 4.

Write $\overline{J}_6$ for Paper II's sixth upper endpoint [27, (6.1), (6.3a)]. For the optional sharper arithmetic comparison, record

$$\begin{aligned} I(b) &= \log\frac{1-b}{b},\\ c &= \frac{6834282367132846348358930380956825846442683629}{10^{45}}. \end{aligned} \tag{1.6}$$

Both the exterior and switched comparisons use the common coefficient

$$c_{K2,G0} = \frac{6836825163875517756286846321650754372931094689}{10^{45}}. \tag{1.7}$$

Its definition and source comparison are (7.36) and (7.67). Put

$$\begin{aligned} D_{\mathrm{II}} &= \frac{45849916338009170593}{25000000000000000000},\\ \overline{H} &= \frac{389253}{5000000},\\ M_{5,\mathrm{internal}} &= -K_{\mathrm{hi}}\frac{6073714261}{10^9} - L_{G1},\\ g_{\mathrm{RA,lo}} &= \frac{989598353}{10^{10}},\\ \Delta D_{L3} &= \Big(2 - \frac{c_{K2,G0}}{4}\Big)\,\overline{H}. \end{aligned} \tag{1.8}$$

The two mass constants are

$$\begin{aligned} K_{\mathrm{hi}} &= \frac{39894807910705784663904430098309337537484429474781101122 23}{10^{58}},\\ L_{G1} &= -\frac{24395042826519465625420692193994776286887136187440955209509 7}{10043362776618689222137263077132266265763768711142455220633 6}. \end{aligned} \tag{1.9}$$

These are Paper II, (A.10d) and (4.84), respectively [27, (A.10d), (4.84)]. The constant $\overline{H}$ bounds the switched integral. It differs from the packing ceiling 153/256. The exact rational c_{O1} is (8.60), with Table 8.1. It and c belong to the optional certificate comparison only; neither is used to price the main count. The sixth-sum comparison uses the certified floor $g_{\mathrm{RA,lo}}$. The integral gain is denoted $\mathfrak{g}_{\mathrm{RA}}$, with certified enclosure $[0.0989598353353, 0.0989598353355]$ in Section 6. Thus $g_{\mathrm{RA,lo}}$ is rational and $g_{\mathrm{RA,lo}} \le \mathfrak{g}_{\mathrm{RA}}$. It is neither the integral nor its central estimate.

The required comparisons are

$$\begin{aligned} 4G_b(N) &\ge \Sigma_W - 4B_b(N) - W(N)E_{\mathrm{exc}}(N),\\ \frac{R+S_2}{W(N)} &\ge D_{\mathrm{II}} + \overline{J}_6 + 2\overline{H} + \frac{5719}{10^6}\\ &\qquad + M_{5,\mathrm{internal}} - E'(N;\sigma),\\ \frac{S_6}{W(N)} &\le \overline{J}_6 - g_{\mathrm{RA,lo}} + E_{\mathrm{RA}}(N;\sigma),\\ \frac{S_{\mathrm{sw}}}{W(N)} &\le \frac{c_{K2,G0}}{4}\overline{H} + E_{\mathrm{sw}}(N;\sigma). \end{aligned} \tag{1.10}$$

These errors are nonnegative by (5.2), (6.72) and (8.71); the first includes the inherited envelope E_{van}. Use the following error identifications:

$$\begin{aligned} E_{\mathrm{exc}}(N) &= \frac{4E_{\mathrm{pt}}(N)}{W(N)},\\ E_{\mathrm{sw}}(N;\sigma) &= C_{\mathrm{sw}}\sigma + P_{\mathrm{sw}}\sigma^9 + V_{\mathrm{sw}}(N;\sigma). \end{aligned} \tag{1.11}$$

TABLE 1.1. Components of D_B

Component	Value	Definition
D_{II}	1.833996653520367	(1.8)
$5719/10^6$	0.005719	(5.25)
$M_{5,\mathrm{internal}}$	0.005874949945026	(1.8), (1.9)
$g_{\mathrm{RA,lo}}$	0.0989598353	(1.8)
$-1/10000$	-0.0001	(1.14)
ΔD_{L3}	0.022638464724298	(1.8)
D_B	1.967088903489691	(1.14)
$1.96708890348969133558270317349 4 < D_B$		
$D_B < 1.96708890348969133558270317349 5$		

Here E_{pt} is (4.12). The switched quantities are (8.61) and (8.71). Their estimates form input (A4). The retained error includes the conversion errors:

$$E'(N;\sigma) = E_{\mathrm{van}}(N;\sigma) + \sum_{j\in\mathcal{H}} \big(|\Lambda^0_{j,N}| + |\Lambda_{j,N}|\big) + |\Lambda_{G,N}|. \tag{1.12}$$

The row set $\mathcal{H}$ is defined in (5.2). Its two qualified labels denote S_5's lower part and retained S_{10}. The inherited error E_{van} is (5.11); the saved-array contribution $\Lambda^0_{j,N}$ is included once, as in (5.2). The differences Δ_j are (5.22). The conversion arrays are (5.35) and (5.56); the equivalent notation $\Lambda_{j,N} = \Lambda_N(\Delta_j)$ and $\Lambda_{G,N} = \Lambda_N(\Delta_{G_1})$ is fixed in (5.3). The saved-array identifications and conversion bounds are proved in Section 5, as part of (A1). The two finite integral bounds are Propositions B.1 and B.2.

The exterior estimate required is

$$-\frac{4B_b(N)}{W(N)} \ge -c_{K2,G0}I(b) - E_K(N;b,\sigma), \qquad E_K(N;b,\sigma) \ge 0. \tag{1.13}$$

Its convolution input is Paper II, [27, Lemma 5.1, (5.44)–(5.52)].

Define the constant by

$$\boxed{D_B = D_{\mathrm{II}} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} + g_{\mathrm{RA,lo}} - \frac{1}{10000} + \Delta D_{L3}.} \tag{1.14}$$

Table 1.1 records each summand. Its decimals are rounded; the defining formulas are exact.

Suppose the optional six-endpoint estimate is available. Then Proposition 5.10 would add $g_6 = 0.0020549454896034927680\ldots$ of (5.18) to the retained comparison. The assembled retained-count lower bound below would then increase by g_6. Theorem 1 does not use this. Let

$$\mathcal{E}_B = E' + E_{\mathrm{RA}} + E_{\mathrm{sw}} + E_K + E_{\mathrm{exc}}, \qquad \delta_{\mathrm{err}} = \frac{1}{10000}. \tag{1.15}$$

Adding the established comparisons gives

$$\frac{4G_b(N)}{W(N)} \ge D_B + \delta_{\mathrm{err}} - c_{K2,G0}I(b) - \mathcal{E}_B(N;b,\sigma). \tag{1.16}$$

By (A1)–(A5), Proposition 9.1 proves

$$\mathcal{E}_B(N;b,\sigma_*) \le \delta_{\mathrm{err}} \qquad (N \ge N_0^*(\sigma_*),\ N \in 2\mathbb{Z}). \tag{1.17}$$

Table 1.2 lists the five inputs, their exact statements and their contributions to the final constant. The first-line source count in (1.10) is (4.13), valid for $N \ge N_{\mathrm{pt}} = 10^{12}+1$. Lemma 9.2 gives $N_{\mathrm{pt}} < N_{\mathrm{tr}} \le N_0^*(\sigma_*)$.

Table 1.2: Analytic inputs and their contributions

Input	Exact statement and use	Contribution and error
A1	Retained comparison (5.2) with g_6 replaced by 0, Proposition 5.10, at $\sigma = \sigma_*$; used in line 2 of (1.10); proved in Section 5.	$+5719/10^6 + M_{5,\mathrm{internal}}$; error E'.
A2	Proposition 6.12 at $\sigma = \sigma_*$, giving the sixth-sum estimate (6.72) of Proposition 6.13; used in line 3 of (1.10).	$+g_{\mathrm{RA,lo}}$; error E_{RA}.
A3	Proposition 7.4 at $\sigma = \sigma_*$, giving the exterior estimate (7.67) of Proposition 7.5; used in (1.13).	$-c_{K2,G0}I(b)$; error $A_K C_{\mathrm{AA}}\sigma + P_K\sigma^9 + r_K$.

Input	Exact statement and use	Contribution and error
A4	The sieve data and the estimates (8.8), (8.53) and (8.62) of Section 8 for the source (8.3), at $\sigma = \sigma_*$, giving the switched estimate (8.72) and the limit (8.73); used in line 4 of (1.10). This application uses Paper I, Theorem 2.8, at its stated inner spectral stage.	$+\Delta D_{L3}$; error $C_{\mathrm{sw}}\sigma + P_{\mathrm{sw}}\sigma^9 + V_{\mathrm{sw}}$.
A5	The constants C_E in (8.25) and C_{AA} in (7.39) are finite, positive and independent of σ and N, and $C_E \geq C_E^{\mathrm{II}}$.	Fixes the caps in (9.14).

The constants C_E and C_{AA} are distinct. Here C_E is the common maximum of the switched and Paper II sieve constants. The four-row K2 error constant is $C_{\mathrm{AA}} = 4C_{\mathrm{AA37}}$.

The exact constants satisfy

$$\begin{aligned} 0.00022895431126728905189820815\mathbf{1} &< D_B - c_{K2,G0}I(b), \\ D_B - c_{K2,G0}I(b) &< 0.00022895431126728905189820815\mathbf{2}, \\ 1.74997488413789243343320748462\mathbf{2} &< a^*_{\mathrm{buf}}, \\ a^*_{\mathrm{buf}} &< 1.74997488413789243343320748462\mathbf{3} < \frac{7}{4}. \end{aligned} \tag{1.18}$$

The last constant is defined in (9.26). The rational bounds are shown in Section 9. They need the stated analytic comparisons; the arithmetic alone gives no positive-count estimate. The value a^*_{buf} is a scalar diagnostic for this fixed b. It does not prove the theorem for smaller a; that would require comparisons at a different cutoff. The optional sharper scalar comparison (9.29) uses a separate constant $D_{B,c}$; it does not enter the main count.

1.3. **Order of the parameters.** Fix b and all absolute error constants first. Choose one sufficiently small rational σ_*. Then put $\tau_* = (\sigma_*)^9$. Freeze the finite partitions and all coefficient bounds next. Fix every logarithmic precision before the common threshold. Thus the order is

$$\begin{aligned} &\text{absolute constants} \longrightarrow \sigma_* \in \mathbb{Q} \cap (0, 10^{-10}] \longrightarrow \tau_* = (\sigma_*)^9, \\ &\longrightarrow \text{partitions, coefficient bounds and precisions} \longrightarrow N_0^*(\sigma_*) \longrightarrow N. \end{aligned} \tag{1.19}$$

The common threshold is defined in (2.5). It includes every length, structural and error threshold. Every limit is taken at fixed σ_*. These thresholds exist by (A1)–(A5) and Proposition 9.1. The final section combines the comparisons and the error bounds. For the main construction, the comparison estimates used are:

(1) (A1): the actual retained-source comparison in Section 5, including all saved-array conversions. It supplies $5719/10^6 + M_{5,\mathrm{internal}}$ beyond the original retained bound.
(2) (A2): the sixth-sum distribution and coefficient estimates in Section 6, supplying $g_{\mathrm{RA,lo}}$.
(3) (A3): the actual semiprime-source distribution, principal-term and peel estimates in Section 7, giving the exterior cost $c_{K2,G0}I(b)$.
(4) (A4): the switched-source comparison, packing, distribution and peel estimates in Section 8, giving $\Delta D_{L3} = (2 - c_{K2,G0}/4)\overline{H}$.

Each requirement includes its exact nonnegative error estimates and the fixed-σ_* limits allocated in Section 9. The source-counting identity and pointwise bound are supplied in Section 4. The list only records dependencies. The cited sections prove the estimates, with precise scopes and error bounds; the scalar arithmetic here combines them.

The use of AI tools is described at the end.

2. Notation and parameters

All logarithms are natural. Put $L = \log N$. The integer N is even throughout. Table 2.1 fixes the common notation.

Table 2.1: Common notation

Symbol	Meaning
$\mathbb{P}$	The set of primes.
$\varphi(n), d(n)$	Euler's totient and divisor function. The inherited notation $\tau(n) = d(n)$ has an integer argument; the scalar $\tau = \sigma^9$ has none.
$\mu_{\mathrm{Mob}}(n), \mu(n)$	The Möbius function; use μ_* for the sieve cutoff.
$\Omega(n)$	Prime factors counted with multiplicity.
$P^-(n)$	Least prime factor; $P^-(1) = \infty$.
$\mathbf{1}_E$	Indicator of E.
γ, γ_E	Euler's constant, with $\gamma_E = \gamma$.
a, b	Representation exponent and smaller-factor cutoff.
ε	Fixed truncation parameter.
$\alpha, \beta, \mu_*, \lambda, t_0, u$	Fixed sieve exponents in (2.1); cutoff $\mu = \mu_*$.

$C(N), U_C(N), W(N)$	Normalizations in (2.2).
σ, τ, ρ	Partition parameters: $\tau = \sigma^9$, $\rho = 1 + \tau$.
σ_*, τ_*, ρ_*	Their fixed choices for the final argument.
$\kappa_{\mathrm{tr}}, N_{\mathrm{tr}}$	Transfer parameters in (2.3).
δ_{err}	Fixed total error bound, 10^{-4}.
δ_{pk}	The parameter in the factor-partition theorem.
h_G, h, δ_h	Profile displacement $h_G = 9/5000$; $h = h_G$ in (5.1), $\delta_h = h_G/\alpha$.
ϵ_s, δ_s	Local sieve parameters, $\delta_s = \epsilon_s^9$; applications use $\epsilon_s = \sigma$.
η_L	The level margin, 10^{-4}; separate from ε and δ_{err}.
r, q	In (1.1), q is prime and r is either 1 or prime; other uses are locally bound variables.
$c, I(b)$	Optional sharper coefficient and the logarithm in (1.6).
$c_{K2,G0}$	Common K2/L3 coefficient in (1.7).
C_E, C_{AA}	Error constants in (8.25) and (7.38).
$K_{\mathrm{SW}}(A), C_{\mathrm{K2},\sigma}, C_{\mathrm{L3},\sigma}$	The logarithmic Siegel–Walfisz level and the K2 and L3 coefficient bounds; each is local to its stated application.
R, R_{fix}	The signed sum in (1.5); elsewhere R is a local level or remainder.
$\Sigma_W, \mathcal{W}$	Equal signed source sums; see (4.5).
W_{13}, W_{rs}	Arithmetic source weight and spectral operator, respectively. Neither denotes $W(N)$.
$\overline{H}, \overline{H}_L$	Equal switched-integral bounds, 389253/5000000.
$H_{\mathrm{pk,max}}, H_{\max}$	Equal packing ceilings, 153/256.
$D_{\mathrm{II}}, \overline{J}_6$	Original Paper II lower constant and sixth upper endpoint; (1.8), (6.59).
$g_{\mathrm{RA,lo}}$	Certified floor 0.0989598353, used in (1.14).
E', E_{van}	Complete retained error and inherited envelope; (1.12), (5.11).
$\Lambda_{j,N}$, $\Lambda_{G,N}$	Retained conversion differences, defined in (5.35) and (5.56).
$\Delta_G, \Lambda_{G,N}$	Joint-profile gain and conversion difference in (5.56). The spelling $G1$ in subscripts denotes the same group.
$E_{\mathrm{RA}}, E_{\mathrm{sw}}$, E_K, E_{exc}	Nonnegative complete errors in (1.10)–(1.11) and (1.13).
$\mathcal{E}_B$	Their total, including E', in (1.15).
D_B	The constant in (1.14); evaluated in (9.1).
$\mathcal{A}_N, \mathcal{P}_b, G_b, B_b$	Shifted-prime source, admissible factors, positive count and exterior count in (1.3), (1.5).
$S_1, \ldots, S_{13}, S_{\mathrm{sw}}$	Paper II arithmetic sums retained in (1.5); definitions in Section 4.
$\mathfrak{g}_{\mathrm{RA}}, g_{\mathrm{RA,lo}}$	Integral gain and its smaller rational floor; only the floor enters D_B.
$g_6, M_{5,\mathrm{internal}}$, K_{hi}, L_{G1}	Optional six-endpoint gain, shifted-profile gain and mass constants; the main proof sets $g_6 = 0$, see (5.18), (1.8), (1.9).
ΔD_{L3}	Switched gain $(2 - c_{K2,G0}/4)\overline{H}$.
$G_*, G_0, U_0, c_{\mathrm{O1}}$	Common unshifted K2/L3 functional; coefficient cap $G_0 = 6088440363/10^9$; optional sharper upper certificate U_0; its exact coefficient in (8.60).
$\mathcal{H}, \Lambda^0_{j,N}$	Retained row set and saved-array contribution in (5.2).
$\mathcal{E}, E_{\mathrm{pt}}$	Pointwise exceptional set and error in (4.9), (4.12).
$N_0^*, N_{\mathrm{floor}}, T_\bullet$	Common threshold, numerical floor and named threshold requirements in (2.5).
$\zeta, a^*_{\mathrm{buf}}, \mathfrak{E}_*$	Buffer 10^{-6}, scalar endpoint and its exponential in (9.26).
$W_{\mathrm{AA},K}, W_*$	K2 source-weight bound in (7.38) and Paper II bound 10^{14}; these are distinct.
A_K, P_K, r_K	K2 error multiplier, peel multiplier and vanishing remainder in (9.8).
$C_{\mathrm{sw}}, P_{\mathrm{sw}}, V_{\mathrm{sw}}$	Switched error multipliers and positive remainder in (8.61), (8.71).
L, g_{AA}	$\log N$ and Paper II's fixed error allowance in (9.15).

Cusps carry the index $\mathfrak{a}$ in spectral sums. The switched blocks carry the index $\mathfrak{b}$. Partition cells are c or $\mathfrak{c}$. Coefficient sequences are a_n, b_m, with dummy indices interchanged when needed. Spectral moduli and cusp numerators are locally bound variables. They do not redefine the constants c and u. Packing coordinates also have local scope. Bare T and w are local endpoints or weights; the named saved and root-A scales are $T_{\mathrm{save}}, T_{\mathrm{int}}$, and w_{RA}. The exponent $A_0 > 0$ in distribution estimates is a logarithmic precision. The normalization factor is written $2e^{-\gamma}$.

2.1. **Fixed parameters.** Keep a, b, ε from (1.2). The sieve parameters are

$$\begin{aligned} \alpha &= \frac{100}{1149}, \quad \beta = \frac{50}{309}, \quad \mu_* = \frac{25}{128}, \\ \lambda &= \frac{93}{400}, \quad t_0 = \frac{57}{224}, \quad u = \frac{1}{500}. \end{aligned} \tag{2.1}$$

These follow Paper II's notation [27, (1.5)]; ε is ε_A in [27, (1.3)]. Put

$$C_2 = \prod_{\substack{\ell \in \mathbb{P} \\ \ell > 2}} \left(1 - \frac{1}{(\ell-1)^2}\right),$$

$$C(N) = C_2 \prod_{\substack{\ell \in \mathbb{P},\ \ell \mid N \\ \ell > 2}} \frac{\ell - 1}{\ell - 2}, \tag{2.2}$$

$$U_C(N) = \frac{C(N)N}{L^2}, \qquad W(N) = 4(1-\varepsilon)U_C(N).$$

We often suppress the argument N.

Define the transfer parameters by

$$\begin{aligned} \kappa_{\mathrm{tr}} &= a(1-b) - 1 = \frac{7}{4000000}, \\ N_{\mathrm{tr}} &= \varepsilon^{-(a-1)/\kappa_{\mathrm{tr}}} = \varepsilon^{-3000000/7}. \end{aligned} \tag{2.3}$$

Here $a - 1 = 3/4$ is the exponent on the larger prime in the bound for the smaller factor. The transfer argument appears in the final section.

2.2. **Order of choice.** First fix all σ-independent constants in the estimates. These include C_E in (8.25) and C_{AA} in (7.38). Then choose one rational σ_*, with

$$0 < \sigma_* \le 10^{-10}, \qquad \tau_* = (\sigma_*)^9, \qquad \rho_* = 1 + \tau_*. \tag{2.4}$$

The final section gives the remaining upper restrictions. Apply every partition estimate with $\sigma = \sigma_*$. Freeze its finite partitions and coefficient bounds next. Freeze the resulting finite-family bounds and logarithmic precisions too. These choices precede the common threshold.

Use the following named thresholds. Each requires its listed statement for every even N above it, at the fixed σ_*. The cited comparison propositions establish these statements. The numerical floor supplies none of them. The resulting threshold exists but may be ineffective.

(1) T_{src}: the actual-source identity and first line of (1.10), including the pointwise hypotheses in (4.11) and (4.13). Include the explicit pointwise threshold $N_{\mathrm{pt}} = 1 + \lceil \max\{2^{1/\alpha}, \varepsilon^{-3}, \varepsilon^{-1/(1-\lambda-\beta)}\} \rceil$ from Section 4.

(2) T_{ret}: the retained comparison, saved-array identifications and nonnegativity of E_{van} in Section 5.

(3) T_{RA}: the sixth-sum comparison (6.72), with every distribution and coefficient bound specified in Section 6. It also covers the finite source threshold N_6, chosen so that every even $N \ge N_6$ satisfies

$$N^{1/6} > L^{\max(B_L, B_{\mathrm{tail}})}, \quad N^{\upsilon_{\mathrm{RA}} - 1/3} > L^B, \quad \varepsilon N^{1/2} > N^{\mu_*}, \quad NL^{-333} \ge N^{\mu_*}.$$

Thus the operative T_{RA} is the maximum of the sixth-sum comparison threshold and this N_6.

(4) T_K: the actual-source exterior comparison (7.67) and its nonnegative error decomposition, the r_K formula displayed after (7.67).

(5) T_{sw}: the switched comparison (8.72) and the nonnegative decomposition (8.71).

(6) T_{len}: all scale-conversion, zero-mode, factorization, packing, interval-length and logarithmic-precision conditions of the preceding applications, with the fixed finite partitions. In particular, require $N^{\sigma_*} \ge 2$, $N^{60\sigma_* 2^{-126}} \ge 12L^{334}$, (7.61) and (8.46), and include $N_{\mathrm{dom}}, N_{z,0}, N_{\mathrm{LEN,app}}$ from (8.75). The remaining application conditions are exactly those of the cited comparison propositions. The floor implies none of them.

(7) $T_{E'}, T_V, T_{E_{\mathrm{RA}}}, T_r, T_{E_{\mathrm{exc}}}$: respectively the eventual bounds

$$\begin{aligned} E' \le \frac{\delta_{\mathrm{err}}}{20}, \qquad V_{\mathrm{sw}} \le \frac{\delta_{\mathrm{err}}}{20}, \qquad E_{\mathrm{RA}} \le \frac{\delta_{\mathrm{err}}}{10}, \\ r_K \le \frac{\delta_{\mathrm{err}}}{10}, \qquad E_{\mathrm{exc}} \le \frac{\delta_{\mathrm{err}}}{10}. \end{aligned}$$

All arguments are $(N; \sigma_*)$, or $(N; b, \sigma_*)$ for r_K.

These are finitely many requirements after all finite choices. Define the common threshold once by

$$\begin{aligned}\mathcal{T}_* &= \{T_{\mathrm{src}}, T_{\mathrm{ret}}, T_{\mathrm{RA}}, T_K, T_{\mathrm{sw}}, T_{\mathrm{len}},\\ &\qquad T_{E'}, T_V, T_{E_{\mathrm{RA}}}, T_r, T_{E_{\mathrm{exc}}}\},\\ N_0^*(\sigma_*) &= 2\left\lceil \frac{1}{2}\max\bigl(\{N_{\mathrm{tr}}, N_{\mathrm{floor}}\}\cup\mathcal{T}_*\bigr)\right\rceil,\\ N_{\mathrm{floor}} &= 2\left\lceil e^{2^{150}}/2\right\rceil.\end{aligned} \tag{2.5}$$

The floor is a convention added to Paper I's selected-interval length hypothesis [26, §3.2, length hypothesis preceding (3.12)]. It proves none of the listed estimates. The named application estimates establish this threshold. The resulting N_0^* may be ineffective.

This implements the order in (1.19). Afterwards, N is even and $N \ge N_0^*(\sigma_*)$.

Let $\epsilon_{\mathrm{err}}(N;\sigma_*)$ be a real- or complex-valued error function. Write $\epsilon_{\mathrm{err}}(N;\sigma_*) = o_{\sigma_*}(1)$ if and only if

$$\begin{aligned}&\forall \eta > 0\ \exists N_\eta(\sigma_*)\ \forall N \in 2\mathbb{N},\\ &N \ge N_\eta(\sigma_*) \implies |\epsilon_{\mathrm{err}}(N;\sigma_*)| \le \eta.\end{aligned} \tag{2.6}$$

All preceding fixed data are suppressed here. The parameter σ_* remains fixed in this limit. Uniformity in other variables is stated with each estimate.

3. Classical inputs

We use counting-measure ℓ^2 norms. Write $n \sim Y$ for $Y < n \le 2Y$. The symbols $\mu_{\mathrm{Mob}}, \Lambda, \varphi$ have their usual meanings. Write $d(n)$ for the divisor function. The parameters in each fact are local. Put $P^-(1) = \infty$ and $e(t) = \exp(2\pi i t)$. Statements (P1.x) and (P2.x) restate results proved in Papers I and II, respectively, with their original scope. The structured sieve result is Paper I, Theorem 2.6. Its classical source is Iwaniec's Theorem 4 [13, Theorem 4]. The S-form displayed below specializes the $W_R^{\pm}$ estimates of Paper I, Theorem 2.6. It follows from them by $S \le W_R^+$ and $S \ge W_R^-$. Theorem 4 does not state it verbatim. Each new application needs its own coefficient, support and uniformity checks.

Fact 3.1 ((F1) Prime estimates)**.** *For every $A > 0$, one may take $B_\psi(A) = A + 60$ in*

$$\begin{aligned}&\sum_{q\le Q}\max_{(a,q)=1}\left|\sum_{\substack{n\le y\\ n\equiv a\ (q)}}\Lambda(n) - \frac{y}{\varphi(q)}\right| \ll_A y(\log y)^{-A},\\ &y \ge 3, \quad Q \le y^{1/2}(\log y)^{-B_\psi(A)}.\end{aligned} \tag{3.1}$$

For fixed $A, C > 0$, uniformly for $q \le (\log y)^C$,

$$\pi(y;q,a) = \frac{\mathrm{Li}(y)}{\varphi(q)} + O_{A,C}(y(\log y)^{-A}), \quad (a,q) = 1, \quad y \ge 3. \tag{3.2}$$

These are Paper I, (2.20)–(2.21) [26, (2.20)–(2.21), p. 9]*. The endpoint y is fixed. The second constant is ineffective.*

Fact 3.2 ((F1′) Maximal-endpoint prime estimates)**.** *Use the prime Siegel–Walfisz estimate* (3.2)*. For every fixed $S > 0$ and $X \ge 3$, put $Q_X = X^{1/2}(\log X)^{-(S+60)}$. Then*

$$\begin{aligned}&\sum_{q\le Q_X}\max_{(a,q)=1}\sup_{0\le y\le X}\left|\sum_{\substack{n\le y\\ n\equiv a\ (q)}}\Lambda(n) - \frac{y}{\varphi(q)}\right| \ll_S X(\log X)^{-S},\\ &\sum_{q\le Q_X}\max_{(a,q)=1}\sup_{0\le y\le X}\left|\pi(y;q,a) - \frac{\mathrm{Li}(y)}{\varphi(q)}\right| \ll_S X(\log X)^{-S}.\end{aligned} \tag{3.3}$$

Here $\mathrm{Li}(y) = 0$ for $0 \le y < 2$. The level is imposed at X; the endpoints may be chosen separately for each modulus. An interval difference costs at most twice the bound. This is the classical maximal form of Bombieri–Vinogradov [6, Chapter 28]*. Removing prime powers and partial summation pass from the von Mangoldt form to the prime-counting form. The constants may be ineffective through* (3.2)*.*

Fact 3.3 ((F2) Full-support bilinear distribution)**.** *Let a_m, b_n have supports $m \sim M, n \sim Y$. Assume $x = MY \ge 3$, set $L_x = \log x$, and fix $\epsilon, B > 0$. Assume $\epsilon \le \log Y/\log x \le 1 - \epsilon$. For every $A_0 > 0$, assume uniformly in $h, k \ge 1$*

and integers $\ell \neq 0$ with $(\ell, k) = 1$ that

$$\left| \sum_{\substack{n \equiv \ell \ (k) \\ (n,h)=1}} b_n - \frac{1}{\varphi(k)} \sum_{(n,hk)=1} b_n \right| \leq C_{A_0} \|b\|_2 Y^{1/2} d(h)^B (\log(2Y))^{-A_0}. \tag{3.4}$$

Define

$$\Delta_{\mathrm{fs}}(q, a) = \sum_{mn \equiv a \ (q)} a_m b_n - \frac{1}{\varphi(q)} \sum_{(mn,q)=1} a_m b_n.$$

For every $A > 0$, some $B_{\mathrm{BV}} = B_{\mathrm{BV}}(A, B, \epsilon) > 0$ gives

$$\sum_{q \leq Q} \max_{(a,q)=1} |\Delta_{\mathrm{fs}}(q, a)| \ll \|a\|_2 \|b\|_2 x^{1/2} L_x^{-A}, \qquad Q \leq x^{1/2} L_x^{-B_{\mathrm{BV}}}. \tag{3.5}$$

Both sums use the full supports. The constants depend on the family constants C_{A_0}. This follows from the proof of BFI Theorem 0(b) [3, (A1)–(A2), p. 206; Theorem 0(b), (1.5), p. 212; proof, pp. 213–214].

Proof. Character orthogonality and primitive induction give

$$\begin{aligned} \text{left side of (3.5)} \leq \sum_{e \leq Q} \frac{1}{\varphi(e)} \sum_{2 \leq f \leq Q/e} \frac{1}{\varphi(f)} \\ \cdot \sideset{}{^*}\sum_{\chi \ (f)} |A_e(\chi) B_e(\chi)|, \\ A_e(\chi) = \sum_{(m,e)=1} a_m \chi(m). \end{aligned}$$

Define B_e similarly. Split at $F = L_x^K$. Since $M \geq x^\epsilon$, large x gives $F^2 \leq M$. This permits the small-conductor estimate. The hypothesis handles $f \leq F$. The multiplicative large sieve handles $f > F$. Before summing e, the bounds are

$$\ll \|a\|_2 \|b\|_2 x^{1/2} d(e)^B F^{3/2} L_x^{-A_0}, \qquad \ll L_x^2 \|a\|_2 \|b\|_2 (Q + M^{1/2} + Y^{1/2} + x^{1/2}/F).$$

Choose B_2 with $\sum_{e \leq Q} d(e)^B/\varphi(e) \ll L_x^{B_2}$. In the large-conductor bound, summing $1/\varphi(e)$ costs one more logarithm, since $\sum_{e \leq Q} 1/\varphi(e) \ll L_x$. Take $K = A + B_2 + 4$, $A_0 = A + B_2 + 3K/2 + 1$, and $B_{\mathrm{BV}} = K$. Since $M, Y \geq x^\epsilon$, these bounds prove (3.5). The large sieve is BFI (1.6), p. 207, restated in Paper I, (2.19) [3, (1.6), p. 207][26, (2.19), p. 9]. □

We do not use BFI86, Theorem 0(a).

Fact (F2) is the standalone full-support bound. Section 8 checks the coefficients, relative levels and rough factors of the first three switched rows on the actual source.

Fact 3.4 ((F3) The spectral gap)**.** *For every congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$,*

$$\lambda_1(\Gamma) \geq \frac{975}{4096}, \qquad 0 < \lambda = \frac{1}{4} - t^2 < \frac{1}{4}, \quad t \geq 0 \Longrightarrow 0 \leq t \leq \frac{7}{64}, \quad \vartheta = 2t \leq \frac{7}{32}. \tag{3.6}$$

Here $\lambda_1(\Gamma)$ is the smallest nonzero Laplace eigenvalue. The implication applies to every eigenvalue λ of Δ on $L^2(\Gamma\backslash\mathbb{H})$ in the indicated range. The first bound is Kim–Sarnak [17, Appendix 2]*. The implications are elementary.*

Fact 3.5 ((F4) Deligne's bound)**.** *Let $f = \sum_{n \geq 1} \psi_f(n) e(nz)$ be a primitive holomorphic cuspidal newform. Assume $\psi_f(1) = 1$, integer weight $k \geq 2$, level M, and character χ, with $f((az+b)/(cz+d)) = \chi(d)(cz+d)^k f(z)$ on $\Gamma_0(M)$. This agrees with Deligne's convention $\chi(a)^{-1}$. For $p \nmid M$, Deligne proved*

$$|\psi_f(p)| \leq 2p^{(k-1)/2} \tag{3.7}$$

[8, Théorème (8.2), p. 302]. *The same source states that the roots α_p, β_p of $T^2 - \psi_f(p)T + \chi(p)p^{k-1}$ both have modulus $p^{(k-1)/2}$. Since $|\alpha_p|^2 = |\beta_p|^2 = p^{k-1}$, we have $\overline{\alpha_p} = p^{k-1}/\alpha_p$ and likewise for β_p. With $\alpha_p \beta_p = \chi(p)p^{k-1}$, this gives*

$$\chi(p)\overline{\psi_f(p)} = \frac{\alpha_p \beta_p}{p^{k-1}} \left(\frac{p^{k-1}}{\alpha_p} + \frac{p^{k-1}}{\beta_p} \right) = \psi_f(p). \tag{3.8}$$

Choose $\eta_p^2 = \chi(p)$, $|\eta_p| = 1$. Then $b_p = \eta_p^{-1} p^{-(k-1)/2} \psi_f(p)$ is real and $|b_p| \leq 2$. With $T = p^{(k-1)/2} \eta_p Z$, the polynomial becomes

$$T^2 - \psi_f(p)T + \chi(p)p^{k-1} = p^{k-1}\chi(p)(Z^2 - b_p Z + 1).$$

Write $b_p = 2\cos\omega_p$, $0 \leq \omega_p \leq \pi$. The normalized roots are $e^{\pm i\omega_p}$, also in the repeated case $b_p = \pm 2$. An earlier exposition of the modular-form setting is [7].

Fact 3.6 ((F5) Averaged Kloosterman sums)**.** *For $C, M, Y \geq 1$ and $\epsilon > 0$,*

$$\sum_{c \leq C} \left| \sum_{m \leq M} \sum_{n \leq Y} S(m,n;c) \right| \ll_\epsilon (CMY)^\epsilon C(C+MY) \tag{3.9}$$

[9, Theorem 14, equation (8.13), p. 275]. *The sums run over integers $1 \leq c \leq C$, $1 \leq m \leq M$, and $1 \leq n \leq Y$. Here $S(m,n;c) = \sum^*_{u \ (c)} e((mu + n\bar{u})/c)$.*

Use the trivial character on $\Gamma_0(q)\backslash\mathbb{H}$. The sum over f runs through a Petersson-orthonormal basis of $S_k(\Gamma_0(q))$. The sum over j runs through an orthonormal basis of Maass cusp forms, including exceptional forms. The sum over $\mathfrak{a}$ runs through inequivalent cusps with width-one scaling matrices. The measure is $dx\,dy/y^2$. At infinity, the DI coefficients satisfy

$$\begin{aligned} u_j(z) &= \sum_{n \neq 0} \rho_j(n) \sqrt{y} K_{i\kappa_j}(2\pi|n|y) e(nx), \quad \lambda_j = \tfrac{1}{4} + \kappa_j^2, \\ f(z) &= \sum_{n \geq 1} \psi_f(n) e(nz), \\ E_{\mathfrak{a}}(z,s) &= \text{constant terms} + \sqrt{y} \sum_{n \neq 0} \frac{2\pi^s |n|^{s-1/2}}{\Gamma(s)} \varphi_{\mathfrak{a}\infty n}(s) K_{s-1/2}(2\pi|n|y) e(nx). \end{aligned} \tag{3.10}$$

For $0 < \lambda_j < 1/4$, set $\vartheta_j = 2|\Im\kappa_j| = 2\sqrt{1/4 - \lambda_j}$. See Paper I, (2.2)–(2.4) [26, (2.2)–(2.4), p. 5].

Fact 3.7 ((F6) Spectral large sieves)**.** *For $q, K \geq 1$, $Y \geq 1/2$, and a_n supported on $n \sim Y$, put*

$$\begin{aligned} \Sigma_{\mathrm{hol}}(K,a) &= \sum_{\substack{2 \leq k \leq K \\ k \text{ even}}} \frac{(k-1)!}{(4\pi)^{k-1}} \sum_f \left| \sum_{n \sim Y} a_n n^{-(k-1)/2} \psi_f(n) \right|^2, \\ \Sigma_{\mathrm{Maa}}(K,a) &= \sum_{|\kappa_j| \leq K} \frac{1}{\cosh(\pi\kappa_j)} \left| \sum_{n \sim Y} a_n \rho_j(n) \right|^2, \\ \Sigma_{\mathrm{Eis}}(K,a) &= \sum_{\mathfrak{a}} \int_{-K}^{K} \left| \sum_{n \sim Y} a_n n^{it} \varphi_{\mathfrak{a}\infty n}(1/2+it) \right|^2 dt. \end{aligned}$$

The large-sieve estimates are

$$\begin{aligned} &\Sigma_{\mathrm{hol}}(K,a), \Sigma_{\mathrm{Maa}}(K,a), \Sigma_{\mathrm{Eis}}(K,a) \ll_\epsilon (K^2 + Y^{1+\epsilon}/q)\|a\|_2^2, \\ &\sum_{j \text{ exceptional}} V^{\vartheta_j} \left| \sum_{n \sim Y} a_n \rho_j(n) \right|^2 \ll_\epsilon (1 + \sqrt{YV/q})(1 + \sqrt{Y^{1+\epsilon}/q})\|a\|_2^2, \quad V \geq 1. \end{aligned} \tag{3.11}$$

These are the specialized forms in Paper I, (2.5)–(2.8) [9, Theorem 2, (1.28)–(1.30), p. 230; Theorem 5, (1.38), p. 232] [26, (2.5)–(2.8), pp. 6–7].

For $\phi \in C_c^\infty((0,\infty))$, define

$$\begin{aligned} \widehat{\phi}(t) &= \frac{\pi}{\sinh \pi t} \int_0^\infty \frac{J_{2it}(x) - J_{-2it}(x)}{2i} \phi(x) \frac{dx}{x}, \\ \check{\phi}(t) &= \frac{4}{\pi} \cosh \pi t \int_0^\infty K_{2it}(x) \phi(x) \frac{dx}{x}, \qquad \widetilde{\phi}(l) = \int_0^\infty J_l(x) \phi(x) \frac{dx}{x}. \end{aligned} \tag{3.12}$$

The value at $t = 0$ is the removable limit.

Fact 3.8 ((F7) Corrected same-cusp trace formula)**.** *For $q, m, n \geq 1$ and the test class above,*

$$\sum_{h \geq 1} \frac{S(m,n;qh)}{qh} \phi\left(\frac{4\pi\sqrt{mn}}{qh} \right) = \frac{H_\phi(m,n)}{\pi} - M_\phi(m,n) - \frac{E_\phi(m,n)}{\pi}, \tag{3.13}$$

where

$$H_\phi(m,n)=\sum_{\substack{k\ge 2\\ k\ \text{even}}}\frac{i^k(k-1)!\widetilde{\phi}(k-1)}{(4\pi\sqrt{mn})^{k-1}}\sum_f\overline{\psi_f(m)}\psi_f(n),$$

$$M_\phi(m,n)=\sum_j\frac{\widehat{\phi}(\kappa_j)}{\cosh\pi\kappa_j}\overline{\rho_j(m)}\rho_j(n),$$

$$E_\phi(m,n)=\sum_{\mathfrak{a}}\int_{\mathbb{R}}(n/m)^{it}\overline{\varphi_{\mathfrak{a}\infty m}(1/2+it)}\varphi_{\mathfrak{a}\infty n}(1/2+it)\widehat{\phi}(t)\,dt.$$

No extra diagonal term occurs, also for $m=n$. This is the corrected same-cusp specialization proved in Paper I, (2.7.T15)–(2.7.T18) [26, (2.7.T15)–(2.7.T18)]. *Compare* [10, Lemma 4.5, (4.8)–(4.17)], *in the DI convention* [9, (1.15)–(1.17)].

Fact 3.9 ((F8) Mixed-cusp formulas)**.** *For $q=rs$, $(r,s)=1$, use the trivial character and*

$$\sigma_{1/s}=\begin{pmatrix}\sqrt{r} & 0\\ s\sqrt{r} & 1/\sqrt{r}\end{pmatrix},\quad \mathcal{C}(\infty,1/s)=\{cs\sqrt{r}:c\ge 1,\ (c,r)=1\}.$$

DI82 (1.6), p. 224, gives the arithmetic identity [9, (1.6), p. 224]

$$S_{\infty,1/s}(m,n;cs\sqrt{r})=e(n\bar{s}/r)S(m\bar{r},n;cs),\qquad m,n\in\mathbb{N}.$$

Here $s\bar{s}\equiv 1 \pmod r$ and $r\bar{r}\equiv 1 \pmod{cs}$. The accompanying trace formulas, proved in Paper I, are

$$\mathcal{K}_+=H_{\infty,1/s}/\pi-M_{\infty,1/s}-E_{\infty,1/s}/\pi,\qquad \mathcal{K}_-=(\pi/2)M^-_{\infty,1/s}+(1/2)E^-_{\infty,1/s}.$$

Here $\mathcal{K}_\pm=\sum_{\gamma\in\mathcal{C}}\gamma^{-1}S_{\infty,1/s}(m,\pm n;\gamma)\phi(4\pi\sqrt{mn}/\gamma)$. The plus terms have the definitions in (F7), with their second coefficient at $1/s$. For M^- and E^-, replace $\widehat{\phi}$ by $\check{\phi}$ and n by $-n$ only in the second Fourier coefficient. The cusps are singular, and the scaling matrices have width one. The classical width of $1/s$ is r. For $r>1$ they are inequivalent. The normalization is that of Paper I, (2.13)–(2.14). Compare [10, Lemmas 4.3, 4.5, (4.7)–(4.19)] [26, (2.13)–(2.14), p. 8].

Fact 3.10 ((F9) Bessel representations)**.** *For $x>0$, $n\in\mathbb{Z}$, and $|\Re\nu|<1$,*

$$\begin{aligned}J_n(x)&=\pi^{-1}\int_0^\pi\cos(x\sin u-nu)\,du\\&=i^{-n}\pi^{-1}\int_0^\pi e^{ix\cos u}\cos(nu)\,du,\\J_\nu(x)&=(2/\pi)\int_0^\infty\sin(x\cosh u-\pi\nu/2)\cosh(\nu u)\,du,\\Y_\nu(x)&=-(2/\pi)\int_0^\infty\cos(x\cosh u-\pi\nu/2)\cosh(\nu u)\,du,\\K_\nu(x)&=\sec(\pi\nu/2)\int_0^\infty\cos(x\sinh u)\cosh(\nu u)\,du.\end{aligned}$$

The infinite integrals are improper. These are DLMF (10.9.2), (10.9.8), and (10.32.7) [25, (10.9.2), (10.9.8), (10.32.7)]. *Paper I uses them on printed p. 20, (2.7.T31)–(2.7.T32).*

Fact 3.11 ((F10) Kontorovich–Lebedev inversion)**.** *For g on $(0,\infty)$, the inversion pair is*

$$f(y)=\frac{2y\sinh\pi y}{\pi^2}\int_0^\infty g(x)K_{iy}(x)\frac{dx}{x},\qquad g(x)=\int_0^\infty f(y)K_{iy}(x)\,dy.$$

It holds under either of two conditions. (a) $g/x\in C^1$, with xg and $x(g/x)'$ absolutely integrable. (b) Local piecewise continuity and bounded variation, with

$$\int_0^{1/2}\frac{|g(x)|}{x}\log(1/x)\,dx<\infty,\qquad \int_{1/2}^\infty\frac{|g(x)|}{\sqrt{x}}\,dx<\infty.$$

Under condition (b), the inverse equality is asserted at continuity points of g. In our applications, the inverse equality is used only at continuity points of g. See DLMF (10.43.30)–(10.43.32) [25, (10.43.30)–(10.43.32)]. *These hypotheses hold for $g=\phi/x$, $\phi\in C_c^\infty((0,\infty))$. They also hold for $g=xK_{2it}(x)$ when $|\Im t|\le 7/64$. Near zero, $K_{2it}(x)\ll_t x^{-2|\Im t|}(1+|\log x|)$, so the first integral in condition (b) converges absolutely. At infinity, $K_{2it}(x)\ll_t x^{-1/2}e^{-x}$, so the second converges too. Both functions are smooth on compact subintervals.*

Fact 3.12 ((F11) Hecke operators on the full space)**.** *For trivial character and $(k,q)=1$, Paper I proves*

$$\|T_k\|_{\mathrm{Maass},q} \le d(k)k^{7/64}, \qquad \|T_k\|_{\mathrm{hol},q} \le d(k).$$

Their proof uses the orthogonal cuspidal representation decomposition of Jacquet–Langlands, Chapter III, §16 [16, Chapter III, §16]*. It also uses the holomorphic oldform classes of Atkin–Lehner* [1, Lemmas 6, 13–15; Theorems 3(i), 5]*. Atkin–Lehner's weight k means classical weight $2k$. For an unramified normalized newform, the algebra needed is*

$$\lambda_f(p^r) = \sum_{j=0}^{r} \alpha_p^{r-j}\beta_p^j, \qquad \lambda_f(mn) = \lambda_f(m)\lambda_f(n) \quad ((m,n)=1).$$

The Maass Satake bound is $|\alpha_p|, |\beta_p| \le p^{7/64}$ [17, Appendix 2]*. The holomorphic bound is (F4). Full-space diagonalization and unitary transport give the displayed norms, as in Paper I, (2.7.T37)–(2.7.T40)* [26, (2.7.T37)–(2.7.T40), pp. 21–22].

Fact 3.13 ((F12) Eisenstein inputs)**.** *Paper I uses three classical Eisenstein properties. The Eisenstein family has a meromorphic continuation, which allows compact Fourier-coefficient extraction. It has an arithmetic Fourier normalization. There is a complete irredundant cusp set. For the cusp u/w, let d_{den} be the lower-row denominator variable in $cz + d_{\mathrm{den}}$ and write the printed variable as $d = -d_{\mathrm{den}}$. The printed denominator is therefore $cz - d$, and the Fourier phase changes with this sign choice. Thus replacing d by $-d$ changes the congruence and the phase together, not one sign in isolation. The normalization includes the orbit conditions*

$$(c,d)=1, \quad (c,q)=w, \quad cd \equiv uw \pmod{(w^2,q)}.$$

With $c = w\gamma$, the last condition is $\gamma d \equiv u \pmod{(w,q/w)}$. The cusp set has the form

$$u/w, \quad w \mid q, \quad u \pmod{(w,q/w)}, \quad (u,(w,q/w))=1,$$

with admissible lifts chosen coprime to q. The sources are Iwaniec, Chapter 6 and Theorem 3.4; DI82 (1.17), Lemmas 2.3 and 3.6; and Kiral–Young, Proposition 3.1 and Corollary 3.2 [15, Chapter 6; Theorem 3.4] [9, (1.17), p. 227; Lemma 2.3, p. 240; Lemma 3.6, p. 247] [18, Proposition 3.1, Corollary 3.2]*. These are the conventions used in Paper I, (2.7.T20)–(2.7.T24) and (2.7.T41)–(2.7.T43)* [26, (2.7.T20)–(2.7.T24), (2.7.T41)–(2.7.T43), pp. 18–19, 22–23].

Fact 3.14 ((F13) Further spectral estimates)**.** *For exceptional eigenfunctions, the first line below follows from (F3), since $975/4096 > 768/4096 = 3/16$ and $7/32 < 1/2$. Paper I derives the interval estimate from (F5), (F6), (F7), (P1.1) and this first line. The last line is its supported imaginary-transform estimate:*

$$\begin{gathered}
\lambda_j \ge 3/16, \qquad 0 < \vartheta_j \le 1/2, \\
\sum_{2\le q\le Q} \sideset{}{^{(q)}}\sum_{j \text{ exceptional}} V^{\vartheta_j} \left|\sum_{n\in I} \rho_j(n)\right|^2 \ll_\epsilon (QY)^\epsilon (Q + Y + \sqrt{YV})Y, \\
Q, Y, V \ge 1, \qquad I \subset (Y, 2Y] \cap \mathbb{Z} \quad \text{an integer interval}, \\
|\widehat{\phi}(i\nu)| + |\check{\phi}(i\nu)| \ll \mathfrak{B}(1 + |\log Z|)(1 + Z^{-2\nu})/(1+Z), \quad 0 < \nu \le 1/4.
\end{gathered} \tag{3.14}$$

The last line uses the supported test class of (P1.1). The interval argument uses DI82's proof of Theorem 7, (8.17), with the corrected same-cusp normalization (F7) [9, proof of Theorem 7, (8.17)] [26, (2.9), (2.7.T32), pp. 6, 20]*. The weaker gap $3/16$ is also DI82, Theorem 4, p. 232, which we do not use separately. The reconstruction of the last estimate uses (F9).*

Fact 3.15 ((F14) Transform identities and interchanges)**.** *Paper I proves the following identities and accompanying interchange bounds in (2.7.T26)–(2.7.T32). Use the transforms* (3.12) *without rescaling. For $\phi \in C_c^\infty((0,\infty))$, put*

$$\begin{gathered}
\mathcal{R}_\phi(v) = \int_0^\infty K_{2iv}(x)\phi(x)x^{-2}\,dx, \qquad k_\phi(v) = \frac{4}{\pi^2} v \sinh(2\pi v)\mathcal{R}_\phi(v), \\
\mathcal{H}(t,v) = \frac{\cosh \pi t}{\cosh \pi(t-v)\cosh \pi(t+v)}, \\
\int_{\mathbb{R}} k_\phi(v)K_{2iv}(x)\,dv = \phi(x)/x, \\
\int_{\mathbb{R}} v\sinh(2\pi v)\mathcal{H}(t,v)K_{2iv}(x)\,dv = \cosh(\pi t)xK_{2it}(x).
\end{gathered} \tag{3.15}$$

By (F10), the KL hypotheses hold for both applications, to $\phi(x)/x$ and to $xK_{2it}(x)$. For the latter, the product-integral condition is $|\Re(2it)| + |\Re(2iv)| < 1$, with $v \in \mathbb{R}$. The permitted t are real or exceptional, $|\Im t| \le 7/64$. The differentiations and contour, limit, sum, and integral interchanges in (2.7.T26)–(2.7.T32) hold in these ranges. This includes absolute summability after the spectral estimates. For each fixed smooth test family, the decay bound is

$$\mathcal{R}_\phi(v) \ll_{\phi,j} e^{-\pi|v|}(1+|v|)^{1-2j} \qquad (j \ge 0).$$

Let $\operatorname{supp}\phi \subset [Z/21, 9Z]$. *The integrations by parts and differentiated estimates retain the five-derivative seminorm* $\mathfrak{B}$ *of (P1.1), uniformly in every auxiliary parameter. For the following bounds* $U > 0$, *and the oscillatory kernels are the normalized combinations in* (3.12)*; these estimates are not assertions about an unnormalized* J_{2it} *tail. The hyperbolic factor in the raw Bessel representation is cancelled only after this normalization. For truncated normalized oscillatory representations, the tails are bounded by*

$$O\Big(\frac{1+|t|}{x\sinh U}\Big) \quad (t\in\mathbb{R}), \qquad O\Big(\frac{(1+\nu)e^{2\nu U}}{x\sinh U}\Big) \quad (t=i\nu,\ 0<\nu\le 1/4).$$

These tails permit the stated limiting interchanges. The derivative and decay constants are uniform in the families consumed by Theorem 2.8, with the losses prescribed there. The first inversion identity follows from (F10). The product integral, second kernel identity, and uniform differentiation and interchange steps are proved in the same passage [26, (2.7.T26)–(2.7.T32), pp. 19–20]. *A reconstruction from the individual Bessel representations uses (F9).*

Fact 3.16 ((P1.1) Supported real transforms)**.**
For $Z > 0$, *let* $\operatorname{supp}\phi \subset [Z/21, 9Z]$ *and set* $\mathfrak{B} = \max_{0\le j\le 5}\sup_{x>0} x^j|\phi^{(j)}(x)|$. *Then*

$$\begin{aligned} |\widehat{\phi}(t)| + |\check{\phi}(t)| &\ll \frac{\mathfrak{B}(1+|\log Z|)}{1+Z}\min\{1,((1+Z)/|t|)^3\}, \quad t\in\mathbb{R},\\ |\widetilde{\phi}(l)| &\ll \frac{\mathfrak{B}(1+|\log Z|)}{1+Z}\min\{1,((1+Z)/l)^3\}, \quad l\in\mathbb{N}. \end{aligned} \tag{3.16}$$

The constants are absolute. The minimum at $t = 0$ *is* 1. *This is the real-parameter part of Paper I, (2.7.T31)* [26, (2.12), (2.7.T31), pp. 8, 20]. *The real-transform bound uses exactly the Bessel representations in (F9). It does not cover the imaginary-parameter or interchange steps.*

Write γ_E for Euler's constant. The continuous linear-sieve functions are normalized by

$$\begin{aligned} sF(s) &= 2e^{\gamma_E} \quad (0<s\le 3), \qquad sf(s) = 0 \quad (0<s\le 2),\\ (sF(s))' &= f(s-1) \quad (s>3), \qquad (sf(s))' = F(s-1) \quad (s>2). \end{aligned} \tag{3.17}$$

Thus $F(1) = 2e^{\gamma_E}$ and $f(s) = 0$ for $0 < s \le 2$. This is Paper I, (2.23), p. 10 [26, (2.23), p. 10].

Fact 3.17 ((P1.2) Structured linear sieve)**.** *Let* $\mathcal{A}$ *be a finite integer sequence with nonnegative multiplicities, let* $\mathcal{P}$ *be a set of primes, and let* $\mathcal{X} > 0$. *For a multiplicative function* ω, *set*

$$\begin{aligned} P(z) &= \prod_{\substack{p<z\\ p\in\mathcal{P}}} p, \qquad S(\mathcal{A},\mathcal{P},z) = \sum_{a\in\mathcal{A}} 1_{(a,P(z))=1},\\ |\mathcal{A}_d| &= \frac{\omega(d)}{d}\mathcal{X} + r(\mathcal{A},d), \qquad V(z) = \prod_{\substack{p<z\\ p\in\mathcal{P}}}\Big(1-\frac{\omega(p)}{p}\Big). \end{aligned} \tag{3.18}$$

Assume $0 < \omega(p) < p$ *on* $\mathcal{P}$, $\omega(p) = 0$ *off* $\mathcal{P}$, *and* $\omega(p^a) \ge 0$. *Suppose fixed* $K \ge 2$ *and* $L_\omega > 1$ *satisfy, for every* $z > w \ge 2$,

$$\begin{aligned} \prod_{\substack{w\le p<z\\ p\in\mathcal{P}}}\Big(1-\frac{\omega(p)}{p}\Big)^{-1} &\le \frac{\log z}{\log w}\Big(1+\frac{K}{\log w}\Big),\\ \sum_{\substack{w\le p<z\\ p\in\mathcal{P}}}\sum_{a\ge 2}\frac{\omega(p^a)}{p^a} &\le \frac{L_\omega}{\log(3w)}. \end{aligned} \tag{3.19}$$

Fix $D \ge 2$, $0 < \epsilon_s < 1/3$ *and* $\delta_s = \epsilon_s^9$. *Define*

$$\mathcal{G}(D) = \{D^{\epsilon_s^2(1+\delta_s)^n} : n \ge 0\}. \tag{3.20}$$

Let $\mathcal{H}(D)$ *consist of the finite tuples* $\mathbf{D} = (D_1,\dots,D_k)$ *from this grid such that* $D_k \le \cdots \le D_1 < D^{1/2}$. *Set*

$$\begin{aligned} \mathcal{D}^+(D) &= \{\mathbf{D}\in\mathcal{H}(D) : D_1\cdots D_{2l}D_{2l+1}^3 < D \quad (0\le l\le (k-1)/2)\},\\ \mathcal{D}^-(D) &= \{\mathbf{D}\in\mathcal{H}(D) : D_1\cdots D_{2l-1}D_{2l}^3 < D \quad (1\le l\le k/2)\}. \end{aligned} \tag{3.21}$$

Each of these sets has fewer than $\exp(8\epsilon_s^{-3})$ *elements. There are coefficients* ϕ_d^ν *and* $\lambda_d^\nu(\mathbf{D})$, $\nu\in\{+,-\}$, *with* $|\phi_d^\nu|, |\lambda_d^\nu(\mathbf{D})| \le 1$, *independent of* $\mathcal{A},\mathcal{X},\omega$, *for which*

$$\begin{aligned} S(\mathcal{A},\mathcal{P},z) &\le \mathcal{X}V(z)\{F(s)+E_s\} + R_1^+ + R^+ && (2\le z\le D),\\ S(\mathcal{A},\mathcal{P},z) &\ge \mathcal{X}V(z)\{f(s)-E_s\} - R_1^- - R^- && (2\le z\le D^{1/2}),\\ s &= \frac{\log D}{\log z}, \qquad E_s \ll \epsilon_s + \epsilon_s^{-8}e^{K+L_\omega}(\log D)^{-1/3}. \end{aligned} \tag{3.22}$$

The implied constant in the last line is absolute. The signed remainders are

$$R_1^\nu = \sum_{\substack{d<D^{\epsilon_s}\\ d\mid P(D^{\epsilon_s^2})}} \phi_d^\nu r(\mathcal{A},d) \tag{3.23}$$

and

$$R^\nu = \sum_{\mathbf{D}\in\mathcal{D}^\nu(D)} \sum_{\substack{d<D^{\epsilon_s}\\ d\mid P(D^{\epsilon_s^2})}} \lambda_d^\nu(\mathbf{D}) \sum_{\substack{D_i\le p_i<D_i^{1+\delta_s}\\ p_i\mid P(z)\\ p_i\neq p_j\ ((i,j)\in I(\mathbf{D}))}} r(\mathcal{A},dp_1\cdots p_k). \tag{3.24}$$

Here $I(\mathbf{D})$ imposes distinctness between the equal-label positions in Iwaniec's construction. Unequal grid intervals are disjoint. The coefficients may depend on $\mathcal{P},D,z,\epsilon_s,\mathbf{D}$ and these distinctness conditions. We do not claim that coefficients at different cutoffs agree. See Paper I, Theorem 2.6, (2.22), (2.24), (2.27)–(2.33), pp. 10–12 [26, Theorem 2.6, (2.22), (2.24), (2.27)–(2.33), pp. 10–12]. *Its classical source is Iwaniec, Theorem 4* [13, Theorem 4].

Below, the prime set is $\mathcal{P}_N=\{p : p\nmid 2N\}$. Take the nominal density $\omega(p)=1$ on $\mathcal{P}_N$, $\omega(p)=0$ elsewhere and $\omega(p^a)=0$ for $a\ge 2$. Then the second sum in (3.19) is zero. Mertens' product estimate gives one fixed absolute $K_{\mathrm{M}}\ge 2$ with

$$\prod_{w\le p<z}(1-1/p)^{-1}\le \frac{\log z}{\log w}\left(1+\frac{K_{\mathrm{M}}}{\log w}\right)\qquad (z>w\ge 2). \tag{3.25}$$

The first product in (3.19) is no larger than this product. Hence $K=K_{\mathrm{M}}$, $L_\omega=2$ are valid for this nominal density. The factor in (3.22) is then $e^{K_{\mathrm{M}}+2}$. See [26, pp. 10–11]. Each actual sequence still needs its own decomposition (3.18) and error estimate.

For every relative level $L_c\ge 2$ we use, exactly as in Paper I, (2.33a)–(2.33b), p. 12 [26, (2.33a)–(2.33b), p. 12],

$$\begin{gathered} 0<\sigma\le 10^{-10},\qquad \tau=\sigma^9,\qquad \epsilon_s=\sigma,\quad \delta_s=\tau,\quad D=L_c,\\ \mathcal{G}(L_c)=\{L_c^{\sigma^2(1+\tau)^n}: n\ge 0\},\qquad d<L_c^{\sigma},\quad d\mid P(L_c^{\sigma^2}).\end{gathered} \tag{3.26}$$

Thus the prime intervals, the smooth-divisor support and the level all refer to the same L_c. First fix $\sigma>0$. Then let N tend to infinity.

Fact 3.18 ((P1.3) Finite factorization)**.** *Let $X\ge 2$, $K\in\mathbb{N}$, and $z\ge 1$. Assume $z^K>X$. For every integer $1\le n\le X$,*

$$\Lambda(n)=\sum_{j=1}^{K}(-1)^{j-1}\binom{K}{j}\sum_{\substack{m_1\cdots m_j n_1\cdots n_j=n\\ m_i\le z}}\mu_{\mathrm{Mob}}(m_1)\cdots\mu_{\mathrm{Mob}}(m_j)\log n_1. \tag{3.27}$$

This is Paper I, Lemma 2.7, (2.36), p. 13 [26, Lemma 2.7, (2.36), p. 13].

Proof. For $\Re s>1$, set $M_z(s)=\sum_{m\le z}\mu_{\mathrm{Mob}}(m)m^{-s}$ and $Y_z(s)=\zeta(s)M_z(s)$. The polynomial identity $1-(1-Y_z)^K=\sum_{j=1}^K(-1)^{j-1}\binom{K}{j}Y_z^j$ gives

$$-\frac{\zeta'}{\zeta}=\sum_{j=1}^{K}(-1)^{j-1}\binom{K}{j}(-\zeta')\zeta^{j-1}M_z^j+\left(-\frac{\zeta'}{\zeta}\right)(1-Y_z)^K.$$

Every nonzero coefficient of Y_z-1 has index greater than z. The last term therefore has no coefficient at $n\le X<z^K$; coefficient extraction proves (3.27). □

For $n\le 2X_m$, choose $K=3$, $z=2X_m^{1/3}$. Then $z^3=8X_m>2X_m$. Any interval partition is imposed after (3.27). The product condition and all representation multiplicities remain in the resulting coefficients. See [26, (2.36)–(2.37), p. 13].

For the packing theorem, put $\mu_*=25/128$ and define the packing ceiling by

$$H_{\mathrm{pk,max}}=\frac{153}{256}.$$

Use this constant in the packing hypotheses below. Put

$$\begin{gathered} v_{\mathrm{pk}}(x)=\min\left\{\frac{1+x}{2},\frac{32-34x}{39},\frac{57}{64}-\frac{3}{2}x\right\}\quad (x\ge 0),\\ \psi_{\mathrm{pk}}(x,y)=x1_{x\ge y}\quad (x\in\mathbb{R},\ y\ge 0),\\ w_{\mathrm{pk}}(x_1,x_2,x_3)=\psi_{\mathrm{pk}}\left(\min\left\{\frac{132-68x_3}{214},\frac{153-96x_3}{256},1-2x_2\right\},\frac{1+x_1}{2}\right).\end{gathered} \tag{3.28}$$

For $x_1 \ge x_2 \ge x_3 \ge 0$, define

$$\begin{aligned} H(x_1,x_2,x_3) = \max\{&v_{\mathrm{pk}}(x_1), v_{\mathrm{pk}}(x_2), v_{\mathrm{pk}}(x_1+x_2), v_{\mathrm{pk}}(x_1+x_2+x_3),\\ &w_{\mathrm{pk}}(x_1,x_2,x_3), w_{\mathrm{pk}}(x_1,x_3,x_2),\\ &\psi_{\mathrm{pk}}(v_{\mathrm{pk}}(x_1+x_3), x_1+2x_2+x_3),\\ &\psi_{\mathrm{pk}}(v_{\mathrm{pk}}(x_2+x_3), 2x_1+x_2+x_3)\}, \end{aligned} \tag{3.29}$$

$$\widetilde{H}(x_1,x_2,x_3) = \begin{cases} H(x_1,x_2,x_3), & x_1 \le \mu_*,\\ v_{\mathrm{pk}}(x_1), & x_1 > \mu_*. \end{cases}$$

These are the definitions preceding Paper I, Theorem 4.1, p. 53 [26, p. 53].

Fact 3.19 ((P1.4) Three-factor packing)**.** *Let $0 < \delta_{\mathrm{pk}} \le 3\cdot 10^{-9}$, $1/2 \le h \le H_{\mathrm{pk,max}}$, $d_* = h - 2\delta_{\mathrm{pk}}$, and $\delta_{\mathrm{pk}} \le g \le 1/3 - \delta_{\mathrm{pk}}/2$. Let $e_1 \ge \cdots \ge e_m \ge 0$ be a finite labelled list. Equal values keep distinct labels. The empty list is allowed. Extend the list by zeros, writing $\bar{e}_i = 0$ for $i > m$. Suppose*

$$\sum_{i<j} e_i + 2e_j \le h - 2\delta_{\mathrm{pk}} \quad (1 \le j \le m), \qquad h \le \widetilde{H}(\bar{e}_1, \bar{e}_2, \bar{e}_3). \tag{3.30}$$

The rule of Paper I, Section 4.1, partitions the labels as $J_1 \sqcup J_2 \sqcup J_3$. With $\Sigma_s = \sum_{i\in J_s} e_i$ and $\theta_{\mathrm{sp}} = 7/32$,

$$\begin{aligned} \Sigma_1 &\le g,\\ 2g + \Sigma_2 + 2\Sigma_3 &\le 1 - 4\delta_{\mathrm{pk}},\\ \theta_{\mathrm{sp}} + g + \Sigma_1 + 5\Sigma_2 + 2\Sigma_3 &\le 2 - 4\delta_{\mathrm{pk}},\\ (1+2\theta_{\mathrm{sp}})g + \Sigma_1 + (5-2\theta_{\mathrm{sp}})\Sigma_2 + (2+2\theta_{\mathrm{sp}})\Sigma_3 &\le 2 - 4(1+\theta_{\mathrm{sp}})\delta_{\mathrm{pk}}. \end{aligned} \tag{3.31}$$

The partition depends only on these data. This is Paper I, Theorem 4.1, (4.1)–(4.3), pp. 53–54 [26, Theorem 4.1, (4.1)–(4.3), pp. 53–54].

Fact 3.20 ((P1.5) Continuous Hecke comparison)**.** *Assume the three Eisenstein statements in (F12). Let $w \mid q$, put $w' = (w, q/w)$, and let χ be a Dirichlet character modulo w'. Then Paper I, (2.7.T41)–(2.7.T43), gives*

$$\varphi_{u/w,\infty,n}(z) = (w'/(wq))^z \sum_{\substack{\gamma\ge 1\\(\gamma, q/w)=1}} \gamma^{-2z} \sum_{\substack{d \ (\gamma w)\\(d,\gamma w)=1\\ \gamma d \equiv u \ (w')}} e(-nd/(\gamma w)),$$

$$\mathcal{V}_\chi(n;z) = \sideset{}{^*}\sum_{u \ (w')} \chi(u)\varphi_{u/w,\infty,n}(z), \tag{3.32}$$

$$\mathcal{V}_\chi(kn';z) = \overline{\chi(k)} \left(\sum_{d\mid k} \chi(d)^2 d^{1-2z}\right) \mathcal{V}_\chi(n';z), \quad (k,q) = (k,n') = 1.$$

The first identity initially holds for $\Re z > 1$. The last extends meromorphically. For finitely supported $v_{n'}$ in this coprime range,

$$\begin{aligned} \sum_{\mathfrak{a}} \left| \sum_{n'} v_{n'} |kn'|^{it} \varphi_{\mathfrak{a}\infty, kn'}(1/2+it) \right|^2 \\ \le d(k)^2 \sum_{\mathfrak{a}} \left| \sum_{n'} v_{n'} |n'|^{it} \varphi_{\mathfrak{a}\infty, n'}(1/2+it) \right|^2. \end{aligned} \tag{3.33}$$

It holds almost everywhere in real t, for either index sign [26, (2.7.T41)–(2.7.T43), pp. 22–23].

Proof. In (3.32), write the modulus $c_1 = w\gamma = c_1' c_1''$, where $(c_1', q) = 1$ and every prime of c_1'' divides q. CRT factors the inner character sum as $\chi(c_1') G_\chi(-n; c_1'') c_{c_1'}(n)$. Here G_χ is the corresponding finite Gauss sum and $c_{c_1'}$ is the Ramanujan sum. The outer character supplies another $\chi(c_1')$. The coprime factor is

$$L^{(q)}(2z, \chi^2)^{-1} \sum_{\substack{d \mid |n|\\(d,q)=1}} \chi^2(d) d^{1-2z},$$

where $L^{(q)}$ omits Euler factors at primes dividing q. For $(k,q) = (k,n') = 1$, its divisor sum factors at $n = kn'$. Changing the Gauss-sum variable supplies $\overline{\chi(k)}$. Absolute convergence for $\Re z > 1$, followed by meromorphic

continuation, proves the last identity in (3.32). At $z = 1/2 + it$, the character multiplier is

$$\overline{\chi(k)} \sum_{d|k} \chi(d)^2 (k/d^2)^{it}.$$

Its modulus is at most $d(k)$. Orthogonality over the complete cusp set proves the inequality. □

Fact 3.21 ((P1.6) Exceptional parameters on the full space)**.** *Apply (F3) to $\Gamma_0(q)$. Every exceptional eigenfunction in the full $\Gamma_0(q)$ space then has*

$$\vartheta_j = 2\sqrt{1/4 - \lambda_j} \le 7/32.$$

The bound depends only on the eigenvalue, so it covers oldforms. These are the parameters in Paper I, (2.7.T32), (2.7.T52), and (2.7.T56) [26, (2.7.T32), (2.7.T52), (2.7.T56), pp. 20, 24–25].

Fact 3.22 ((P1.7) The spectral estimate)**.** *Paper I, Theorem 2.8, proves the estimate below. Its proof uses the large sieves (F6) and the trace formulas (F7)–(F8). It also uses the transform bounds (P1.1), (F13)–(F14), the full-space Hecke bounds (F11), and the continuous and exceptional comparisons (P1.5)–(P1.6). The normalized unitary cusp transport, proved in Paper I, (2.7.T33)–(2.7.T40), is as follows. Choose $rw - sy = 1$ and use*

$$\begin{gathered}
W_{rs} = r^{-1/2} \begin{pmatrix} r & y \\ rs & rw \end{pmatrix}, \qquad W_{rs}\Gamma_0(rs)W_{rs}^{-1} = \Gamma_0(rs), \\
\rho_{F,1/s}(n) = e(n\bar{s}/r)\rho_{F\circ W_{rs},\infty}(n), \qquad \psi_{F,1/s}(n) = e(n\bar{s}/r)\psi_{F|W_{rs},\infty}(n), \\
\varphi_{\mathfrak{a},1/s,n}(z) = e(n\bar{s}/r)\varphi_{W_{rs}^{-1}\mathfrak{a},\infty,n}(z), \\
T_k F(z) = k^{-1/2} \sum_{hd=k} \sum_{j \ (d)} F((hz+j)/d), \qquad c_{T_kF}(n) = \sum_{h|(k,n)} c_F(kn/h^2).
\end{gathered} \tag{3.34}$$

The last line is the Maass normalization, with $(k, rs) = 1$. For holomorphic forms use the normalized coefficients of (F11). These formulas fix the normalization of the following theorem.

Let $C, D, Y, R, S, A_T \ge 1$ and $0 < \epsilon_0 < 1/100$. Let $a_T \in \mathbb{Z}$ satisfy $0 < |a_T| \le A_T$. Let $b_{n,r,s}$ be supported on

$$0 < n \le Y, \qquad r \sim R, \qquad s \sim S, \qquad (a_T, rs) = 1. \tag{3.35}$$

Let g be smooth, supported in $(C, 2C] \times (D, 2D] \times \mathbb{R}^3_{>0}$, with

$$|\partial^\nu g(c, d, n, r, s)| \le G_\nu (c^{-\nu_1} d^{-\nu_2} n^{-\nu_3} r^{-\nu_4} s^{-\nu_5})^{1-\epsilon_0} \quad (\nu \in \mathbb{Z}^5_{\ge 0}). \tag{3.36}$$

Set $\mathcal{X} = 2 + A_T CDYRS$ and $\theta_{\mathrm{sp}} = 7/32$. For every $\epsilon > 0$,

$$\left| \sum_{(r,s)=1} \sum_{0<n\le Y} b_{n,r,s} \sum_{\substack{c,d\ge 1 \\ (rd,sc)=1}} g(c, d, n, r, s) e\left(\frac{a_T n \overline{rd}}{sc}\right) \right| \ll_{\epsilon,\epsilon_0,\{G_\nu\}} \mathcal{X}^{\epsilon+18\epsilon_0} J_*(a_T), \tag{3.37}$$

where the inverse is modulo sc, and

$$\begin{aligned}
J_*(a_T)^2 = {} & \sum_{\substack{n''|a_T^\infty \\ n''\le 2Y}} A_T^{\theta_{\mathrm{sp}}} (n'')^{\theta_{\mathrm{sp}}} \\
& \times \left[CS\left(\frac{Y}{n''} + RS\right)(C + DR) + |a_T| YRS \right] \|\widetilde{b}(n'')\|_2^2 \\
& + \Big[CS(CD\sqrt{R})^{2\theta_{\mathrm{sp}}} (Y + RS)^{1-\theta_{\mathrm{sp}}} \\
& \quad \times (C + DR)^{1-2\theta_{\mathrm{sp}}} + D^2 YR \Big] \|b\|_2^2, \\
\|b\|_2^2 = \sum_{n,r,s} |b_{n,r,s}|^2, \qquad & \|\widetilde{b}(n'')\|_2^2 = \sum_{\substack{n',r,s \\ (n',a_T)=1}} |b_{n'n'',r,s}|^2.
\end{aligned} \tag{3.38}$$

Here $n'' \mid a_T^\infty$ means that every prime factor of n'' divides a_T. It includes $n'' = 1$; when $a_T = \pm 1$, only $n'' = 1$ occurs. The bound is uniform in $(a_T, A_T, C, D, Y, R, S, b)$. These are Paper I, Theorem 2.8, (2.38)–(2.43) [26, Theorem 2.8, (2.38)–(2.43), pp. 14–15].

Section 8 checks the masked coefficients, intervals and support of the fourth switched row.

4. Finite source identities and decomposition

Throughout this section, N is an even integer with $N \geq 2$. Further size hypotheses appear at each display or just before it. Use $L = \log N$ from Section 2. We prove the finite identities of this section below. They count each occurrence with its multiplicity. The later analytic estimates are separate hypotheses. Use $\mathcal{A}_N, G_b(N), \varepsilon, a, b$ from (1.2)–(1.3) and $\alpha, \beta, \mu_*, \lambda, t_0, u$ from (2.1). Take the roughness cutoff $z_s = N^\alpha$ from the Root–A scale definition in (6.1). Throughout, n is a positive source integer. When $n \in \mathcal{A}_N$, $n = N - p$ for a prime p. In a source sum, e is the integer being counted. The letters p, p_i, a_i, r denote prime variables, q is a positive modulus, and m is the factor or cofactor specified by its sum. Paper-II counts use $n = dm$; the marked count of Section 6 uses $m = (N-e)/p$. These roles are local to the displayed source. Here $\Omega(n)$ counts prime factors with multiplicity. For squarefree d with $(d, N) = 1$, define

$$S^{\mathrm{II}}(d; y) = \#\{n = dm \in \mathcal{A}_N : P^-(m) \geq y,\ (m, Nd) = 1\}. \qquad (2 \mid N,\ N \geq 2). \tag{4.1}$$

Set this count to zero for other d. For admissible d, it is Paper II's (1.7) count in our notation [27, (1.7)]. The zero value for other d is our convention. Put $x_i = \log p_i / \log N$. The tenth row uses

$$\begin{aligned} C_{10} &= \frac{57}{64} - \frac{1}{500} - \frac{11}{5}\alpha, \\ B_{10}(x) &= \begin{cases} \beta, & \beta \leq x < \mu_*, \\ \min\{\beta, \max\{\alpha, C_{10} - 5x/2\}\}, & \mu_* \leq x < \lambda. \end{cases} \end{aligned} \qquad (2 \mid N,\ N \geq 2). \tag{4.2}$$

Table 4.1 lists all thirteen sums with every divisor and cutoff of [27, (1.6)–(1.7), row table]. All p_i in the table are prime. Each admissible divisor contributes separately.

Table 4.1: The Paper II source sums. Each row sums $S^{\mathrm{II}}(d; y)$.

Row	Coefficient	Divisor and domain	Cutoff y
S_1	3	$d = 1$	N^α
S_2	1	$d = 1$	N^β
S_3	-2	$d = p_1,\ \alpha \leq x_1 < \mu_*$	N^α
S_4	-1	$d = p_1,\ \mu_* \leq x_1 < 1/4$	N^α
S_5	-1	$d = p_1,\ 1/4 \leq x_1 < t_0$	N^α
S_6	-1	$d = p_1,\ t_0 \leq x_1 < 1/3$	N^α
S_7	-1	$d = p_1,\ \mu_* \leq x_1 < \lambda$	N^α
S_8	1	$d = p_1p_2$, $\alpha \leq x_2 < x_1 < \beta$	N^α
S_9	1	$d = p_1p_2$, $\alpha \leq x_2 < \beta \leq x_1 < \mu_*$	N^α
S_{10}	1	$d = p_1p_2,\ \mu_* \leq x_1 < \lambda$, $\alpha \leq x_2 < B_{10}(x_1)$	N^α
S_{11}	-2	$d = p_1p_2,\ \lambda \leq x_1 < 1/3$, $x_1 < x_2 < (1 - x_1)/2$	p_2
S_{12}	-1	$d = p_1p_2$, $\alpha \leq x_1 < 1/3 \leq x_2 < (1 - x_1)/2$	p_2
S_{13}	-1	$d = p_1p_2$, $\beta \leq x_1 < \lambda \leq x_2 < (1 - x_1)/2$	$\sqrt{N/(p_1p_2)}$

Let $\mathcal{D}_i$ be the displayed divisor domain and $y_i(d)$ its cutoff. Define the row occurrences by

$$\vartheta_i(n) = \sum_{\substack{d \in \mathcal{D}_i \\ d \mid n \\ d \text{ squarefree},\ (d,N)=1}} 1_{P^-(n/d) \geq y_i(d)} 1_{(n/d, Nd)=1}. \qquad (2 \mid N,\ N \geq 2). \tag{4.3}$$

Thus $S_i = \sum_{d \in \mathcal{D}_i} S^{\mathrm{II}}(d; y_i(d))$. Keep every cofactor and endpoint restriction. Define

$$\begin{aligned} S_i &= \sum_{n \in \mathcal{A}_N} \vartheta_i(n) \qquad (1 \leq i \leq 13), \\ W_{13}(n) &= 3\vartheta_1(n) + \vartheta_2(n) - 2\vartheta_3(n) - \vartheta_4(n) - \vartheta_5(n) - \vartheta_6(n) - \vartheta_7(n) \\ &\quad + \vartheta_8(n) + \vartheta_9(n) + \vartheta_{10}(n) - 2\vartheta_{11}(n) - \vartheta_{12}(n) - \vartheta_{13}(n), \qquad (2 \mid N,\ N \geq 2). \\ s_{\mathrm{sw}}(n) &= 1_{\Omega(n) \geq 3} \max\{W_{13}(n), 0\}, \\ S_{\mathrm{sw}} &= \sum_{n \in \mathcal{A}_N} s_{\mathrm{sw}}(n), \qquad w(n) = W_{13}(n) - s_{\mathrm{sw}}(n), \qquad \mathcal{W} = \sum_{n \in \mathcal{A}_N} w(n). \end{aligned} \tag{4.4}$$

This is the complete switched source of Paper II, (1.9) [27, (1.9)]. Every divisor prime and cofactor cutoff is at least N^{α}. For S_{13} the cutoff exceeds $N^{(1-\lambda)/4} > N^{\alpha}$. Thus every $\vartheta_i(n)$ vanishes when $P^-(n) < N^{\alpha}$, so the extra roughness indicator of Paper II is redundant. Use $P^-(1) = \infty$ as in Table 2.1. Finite linearity gives

$$\begin{aligned}\mathcal{W} &= 3S_1 + S_2 - 2S_3 - S_4 - S_5 - S_6 - S_7 \\ &\quad + S_8 + S_9 + S_{10} - 2S_{11} - S_{12} - S_{13} - S_{\mathrm{sw}}. \end{aligned} \qquad (2 \mid N,\ N \ge 2). \tag{4.5}$$

4.1. **The representation count.** Put

$$\begin{aligned} g_b(e) &= \#\{(p_1,p_2) \in \mathbb{P}^2 : N^b \le p_1 \le \sqrt{N},\ \varepsilon N < p_1p_2 = e < N\}, \\ B_b &= \sum_e g_b(e) 1_{N-e\in\mathbb{P}}, \qquad P_2 = \#\{n \in \mathcal{A}_N : \Omega(n) \le 2\}. \end{aligned} \qquad (2 \mid N,\ N \ge 2). \tag{4.6}$$

When e has distinct prime factors and both choices satisfy the first-factor cutoff, both orientations are counted. For $e = p^2$, there is only the ordered pair (p,p). Hence

$$0 \le g_b(e) \le 2, \qquad P_2 - G_b(N) \le B_b \qquad (\varepsilon N > 1). \qquad (2 \mid N,\ N \ge 2). \tag{4.7}$$

For the second inequality, write $n = rs$, $r \le s$, with both factors prime. If n is excluded from G_b, then

$$N^b \le r \le \sqrt{n} < \sqrt{N}, \qquad g_b(n) \ge 1. \qquad (2 \mid N,\ N \ge 2,\ \varepsilon N > 1). \tag{4.8}$$

Prime squares are counted with their actual multiplicity.

Define

$$\mathcal{E} = \{n \in \mathcal{A}_N : P^-(n) \ge z_s,\ n \text{ is not squarefree or } (n,N) > 1\}. \qquad (2 \mid N,\ N \ge 2). \tag{4.9}$$

For $z_s = N^{\alpha} \ge 2$,

$$\begin{aligned} \#\{n \in \mathcal{E} : n \text{ is not squarefree}\} &\le N \sum_{k \ge \lceil z_s \rceil} k^{-2} \le \frac{2N}{z_s}, \\ \#\{n \le N : P^-(n) \ge z_s,\ (n,N) > 1\} &\le \sum_{\substack{\ell\in\mathbb{P},\ \ell\mid N \\ \ell \ge z_s}} \frac{N}{\ell} \le \frac{11N}{z_s}, \qquad (2 \mid N,\ N \ge 2^{1/\alpha}). \\ |\mathcal{E}| &\le 13N^{1-\alpha}. \end{aligned} \tag{4.10}$$

The prime divisors in the middle sum are distinct. Their number is at most $\lfloor 1/\alpha \rfloor = 11$.

Set $N_{\mathrm{pt}} = 1 + \lceil \max\{2^{1/\alpha}, \varepsilon^{-3}, \varepsilon^{-1/(1-\lambda-\beta)}\} \rceil$.

Lemma 4.1 (Pointwise source weight bound)**.** *For all $N \ge N_{\mathrm{pt}}$, the weight bounds are*

$$\begin{aligned} w(n) = 0 \quad (P^-(n) < z_s), \qquad w(n) \le 4\,1_{\Omega(n)\le 2} \quad (n \in \mathcal{A}_N \setminus \mathcal{E}), \\ \max\{0, w(n) - 4\,1_{\Omega(n)\le 2}\} \le 169 \quad (n \in \mathcal{E}), \qquad E_{\mathrm{pt}} = 0. \end{aligned} \qquad (2 \mid N,\ N \ge N_{\mathrm{pt}}). \tag{4.11}$$

Proof. Start from the roughness observation above. If $\Omega(n) \ge 3$, then $w(n) = \min\{W_{13}(n), 0\} \le 0$. If $(n,N) > 1$, the coprimality conditions force every row occurrence to vanish. For $\Omega(n) \le 2$ and $n \in \mathcal{E}$ with $(n,N) = 1$, only $n = p^2$ occurs; only S_1, S_2 contribute, giving $w(n) = 4$. For $n \notin \mathcal{E}$ with $\Omega(n) \le 2$, a prime has weight 4. For $n = rs$, with primes $r < s$, the bands $[\alpha,\beta)$, $[\beta,\mu_*)$, $[\mu_*,\lambda)$, $[\lambda,1/3)$ and $[1/3,1/2)$ for $\log_N r$ give weights $1,2,2,3,4$, respectively, directly from the row table. This agrees with Paper II, (1.11) [27, (1.10)–(1.11)]. Here N_{pt} is chosen so that $\varepsilon N > N^{2/3}$ and $\varepsilon N > N^{\lambda+\beta}$. Then the larger prime exceeds $N^{1/3}$. The two-prime divisor rows vanish, since their products are below $N^{\lambda+\beta}$ for S_8–S_{10} and below $N^{2/3}$ for S_{11}–S_{13}. These observations prove all three bounds, including the stronger exceptional bound with right-hand side zero. □

Put

$$E_{\mathrm{pt}} = \frac{1}{4} \sum_{n\in\mathcal{E}} \max\{0, w(n) - 4\,1_{\Omega(n)\le 2}\}. \qquad (2 \mid N,\ N \ge 2). \tag{4.12}$$

The pointwise lemma gives $E_{\mathrm{pt}} = 0$ for $N \ge N_{\mathrm{pt}}$, within its stated scope. We still record cruder bounds. Summation gives

$$\begin{aligned} 0 \le E_{\mathrm{pt}} \le \frac{169}{4}|\mathcal{E}| \le \frac{2197}{4} N^{1-\alpha}, \\ \mathcal{W} \le 4P_2 + 4E_{\mathrm{pt}}, \qquad 4G_b(N) \ge \mathcal{W} - 4B_b - 4E_{\mathrm{pt}}. \end{aligned} \qquad (2 \mid N,\ N \ge N_{\mathrm{pt}}). \tag{4.13}$$

The transfer to $r \le q^{3/4}$ is (1.4).

4.2. **Signed arithmetic sums.** Put

$$\begin{aligned} R_{\mathrm{fix}} &= 3S_1 - 2S_3 - S_4 - S_5 - S_7 + S_8 + S_9 + S_{10} \\ &\quad - 2S_{11} - S_{12} - S_{13}. \end{aligned} \tag{4.14}$$

$(2 \mid N,\ N \ge 2).$

Then (4.5) gives

$$\mathcal{W} - 4B_b = R_{\mathrm{fix}} + S_2 - S_6 - S_{\mathrm{sw}} - 4B_b. \tag{4.15}$$

$(2 \mid N,\ N \ge 2).$

Define

$$\begin{aligned} G_{\mathrm{rows}} &= 2S_{11} + S_{12} + S_{13}, \\ \mathcal{C}_{\mathrm{ret}} &= 3S_1 + S_2 - S_4 - S_5 - S_7 + S_{10} - G_{\mathrm{rows}}, \\ \mathcal{Q} &= -2S_3 + S_8 + S_9. \end{aligned} \tag{4.16}$$

$(2 \mid N,\ N \ge 2).$

Coefficient comparison gives

$$\begin{aligned} \mathcal{C}_{\mathrm{ret}} + \mathcal{Q} &= R_{\mathrm{fix}} + S_2, \\ \mathcal{W} - 4B_b &= \mathcal{C}_{\mathrm{ret}} + \mathcal{Q} - S_6 - S_{\mathrm{sw}} - 4B_b. \end{aligned} \tag{4.17}$$

$(2 \mid N,\ N \ge 2).$

Substitution proves both equalities coefficient by coefficient. The coefficients $3S_1$ and $+S_{10}$ are fixed. The integer supports may overlap. These coefficient identities do not depend on later estimates. Take the signed normalized chains $Q_{j,k}$, their index set $\mathcal{I}$, and the stage index $k \in \{0, \dots, 10\}$ from Section 5. See its "original signed chains" paragraph and (5.4). Each $Q_{j,k}$ already includes its signed normalized coefficient, so no extra multiplicity factor enters. Finite telescoping gives

$$\sum_{j \in \mathcal{I}} Q_{j,0} = \sum_{j \in \mathcal{I}} Q_{j,10} + \sum_{j \in \mathcal{I}} \sum_{k=1}^{10} (Q_{j,k-1} - Q_{j,k}). \tag{4.18}$$

$(2 \mid N,\ N \ge 2).$

This is the normalized-chain identity cited above. The arithmetic coefficient identities above keep their unnormalized definitions. Paper II uses joint endpoints for $S_1, S_2\text{-}T_8/4, G_{\mathrm{rows}}$ [27, (6.4)–(6.4d), pp. 70–72]. Here T_8 is Paper II's stage-eight endpoint in the S_2 chain. The notation $S_2\text{-}T_8/4$ is a joint-chain label; it defines no new arithmetic sum. Their joint endpoint is kept before the differences are split.

4.3. **A finite upper decomposition.** Each original representation occurrence uses the shifted-prime set of (1.3):

$$\mathcal{A}_N = \{N - p : p \in \mathbb{P},\ p < (1 - \varepsilon)N\}. \tag{4.19}$$

$(2 \mid N,\ N \ge 2).$

The two arithmetic sources stay separate. Take A_b, E_{top} from (7.2) and S_b from the Section 7 definition (7.13); A from (8.3); $\mathcal{P}_N$ from (8.1); and $b_A, P_N, S_v, S_A, E_{\mathrm{src}}$ from (8.6). The ordered prime coordinates a_i and their prefix domains are those of (8.9) and the paragraph before it. Define

$$\begin{aligned} h_b(n) &= A_b(N - n), \qquad \mathscr{L}_b = \{p \in \mathbb{P} : p \nmid 2N\}, \qquad E_b = E_{\mathrm{top}}, \\ h_{\mathrm{sw}}(n) &= A(N - n), \qquad \mathscr{L}_{\mathrm{sw}} = \mathcal{P}_N, \qquad E_{\mathrm{sw}}^{\mathrm{src}} = E_{\mathrm{src}}, \end{aligned} \tag{4.20}$$

$(2 \mid N,\ N \ge 2).$

Here A_b, E_{top} are the Section 7 source objects in (7.2). The objects $A, \mathcal{P}_N, E_{\mathrm{src}}$ are the Section 8 source objects in (8.3), (8.1), and (8.6). Both sequences are nonnegative. No extra $1_{(e,2N)=1}$ factor enters. The script-font symbols $\mathscr{L}_b$, $\mathscr{L}_{\mathrm{sw}}$, $\mathscr{L}_X$, and $\mathscr{U}_X$ denote the variable sieve families and their sifted sums. The fixed prime support is written $\mathcal{P}_N$. For $X \in \{b, \mathrm{sw}\}$, write $h_X, \mathscr{L}_X, E_X$ for the corresponding line of (4.20), and put

$$\begin{aligned} P_X(y) &= \prod_{\substack{\ell < y \\ \ell \in \mathscr{L}_X}} \ell, \\ \mathscr{U}_X(d; y) &= \sum_{d \mid n} h_X(n) 1_{(n/d, P_X(y)) = 1}. \end{aligned} \tag{4.21}$$

$(2 \mid N,\ N \ge 2).$

Take $I_L, I_H, J, J_{\mathrm{tail}}, J_t, z_s, \eta_{\mathrm{m}}, v_{\mathrm{RA}}$ from the Root–A definition (6.1). Take $c_{R,K}$ from (6.4), and $R_L, R_{\mathrm{tail}}, B_L, B_{\mathrm{tail}}, B, c_{\mathrm{lo}}, \Gamma$ from (6.6) and its preceding choice of fixed logarithmic exponents. Take D_{m} from (6.52). Write Q_{RA} for the level called Q in (6.6). This alias is distinct from every signed chain $Q_{j,k}$. We do not redefine these objects here. Both source comparisons require the finite threshold N_6 of Section 2. It enters the operative T_{RA}, the maximum of the sixth-sum comparison threshold and N_6. We do not redefine it here.

Use the common choice $\sigma = \sigma_*$ from (2.4). It lies in the Section 7 range of (7.1), and the switched comparison uses it. Take τ, ρ from Section 7 (7.1); Section 8 uses the same values. We make no new choice or definition of σ, τ, ρ here. Partition the logarithmic prime ranges into half-open intervals $[t_i, \rho t_i)$, with $t_i = u\rho^{k_i}$, $k_i \ge 0$. Clip the intervals at μ_*. Each prime has one cell. For $X = b$, use the retained cells of the Section 7 (7.19) retained-cell definition and the cutoff in (7.18). Write them as $\mathcal{R}_b, z_{b,c}, \chi_{b,c}$. For $X = \mathrm{sw}$, use the Section 8 definitions (8.12) and (8.10). Write them as $\mathcal{R}_{\mathrm{sw}}, z_{\mathrm{sw},c}, \chi_{\mathrm{sw},c}$. These definitions come from the two source sections. They form no common envelope in this section. We use no equality between the retained collections of the three sections. The closed-cell infima of Sections 7 and 8 and their endpoint rules apply only within their own source. For $X = b$, the packing function is H, as specified

in (7.19). For $X = \mathrm{sw}$, it is $\widetilde{H}$, as specified in (8.10) and its preceding definition of Θ_c. The two functions are defined in Paper II, equation (2.17d), and the paragraph after it. We have $\widetilde{H} = H$ only on the stated largest-coordinate range, and use no equality outside it. For either source write $z_{X,c} = N^{t_3}$, with t_3 the lower endpoint of the third coordinate of its cell c. Here $z_{\mathrm{sw},c}$ is the object called z_c in Section 8; we use only the source-indexed name. The Section 7 cutoff is defined in (7.18), with the lower-endpoint convention stated before (7.31).

The source index here is always $X \in \{b, \mathrm{sw}\}$. For the b-source the underlying prime support is $\mathcal{P}_N$, so $P_b(y) = P_N(y)$. Both sieve products use primes below y not dividing $2N$. For the b-source, the substitution $e = N - n$ identifies the generic sieve with S_b of Section 7 on that source:

$$\mathscr{U}_b(d;y) = \sum_{d\mid n} A_b(N-n)1_{(n/d,P_b(y))=1} = \sum_{d\mid N-e} A_b(e)1_{((N-e)/d,P_b(y))=1} = S_b(d;y). \qquad (2 \mid N,\ N \ge 2). \tag{4.22}$$

For $X = \mathrm{sw}$, identify the generic cofactor sieve with the Section 8 sieve on precisely the following rows:

- $d = 1$ with cutoff $y = N^{\mu_*}$, the source-comparison row $\mathscr{U}_{\mathrm{sw}}(1; N^{\mu_*}) = S_A$ from Section 8 (8.6);
- $v = a_1 \cdots a_j$, $0 \le j \le 2$, with cutoff $y = N^u$;
- $v = a_1a_2a_3$, with dynamic cutoff $y = a_3$, or retained-cell cutoff $y = z_{\mathrm{sw},c} = N^{t_3}$;
- $vr = a_1a_2a_3r$ on a retained cell, with $z_{\mathrm{sw},c} \le r < a_3$, at cutoff $y = r$ or $y = z_{\mathrm{sw},c}$.

Here the empty prefix is $v = 1$, and all primes belong to $\mathcal{P}_N$ and have the ordering in (8.9) and (8.11). Let d be the displayed prefix v or product vr. Every prime divisor of d is at least y, so $(d, P_{\mathrm{sw}}(y)) = 1$. The definitions give $P_{\mathrm{sw}}(y) = P_N(y)$. Write $S_d(y)$ for the $S_v(y)$ of Section 8 from (8.6) with $v = d$. For $d \mid n$, the Section 8 definition (8.6) gives

$$\mathscr{U}_{\mathrm{sw}}(d;y) = \sum_{d\mid n} h_{\mathrm{sw}}(n)1_{(n/d,P_N(y))=1} = \sum_{d\mid n} b_A(n)1_{(n,P_{\mathrm{sw}}(y))=1} = S_d(y). \qquad (2 \mid N,\ N \ge 2). \tag{4.23}$$

This identity holds on these rows only, not for arbitrary d, y. These E-terms are unnormalized source-restoration costs. The two source upper comparisons are stated separately: Here B'_b is the prime-occurrence part of B_b, and E_{top} is the removed-top cost.

Lemma 4.2 (Upper-source comparisons)**.**

$$B_b \le E_b + \mathscr{U}_b(1; N^{\mu_*}), \qquad S_{\mathrm{sw}} \le E^{\mathrm{src}}_{\mathrm{sw}} + \mathscr{U}_{\mathrm{sw}}(1; N^{\mu_*}) \quad \textit{for even } N \ge N_6. \tag{4.24}$$

Proof. The first inequality follows from Section 7 (7.16) and $E_b = E_{\mathrm{top}}$; the second is the Section 8 source comparison (8.7). We do not identify their hypotheses or their restoration costs. □

We invoke the source comparison once, with the named hypotheses displayed above.

For real cutoffs $1 \le y \le w_{\mathrm{cut}}$, the least allowed prime gives

$$\mathscr{U}_X(d; w_{\mathrm{cut}}) = \mathscr{U}_X(d;y) - \sum_{\substack{y\le r<w_{\mathrm{cut}}\\ r\in\mathscr{L}_X}} \mathscr{U}_X(dr;r). \qquad (2 \mid N,\ N \ge 2). \tag{4.25}$$

Write $n = dm$. If m survives at y but fails at w_{cut}, choose its least allowed prime in $[y, w_{\mathrm{cut}})$. Exactly one summand records that prime. Repeated copies remain in the cofactor. Thus the identity also counts prime powers correctly.

Three applications give

$$\begin{aligned} \mathscr{U}_X(1; N^{\mu_*}) &= T_{X,0} - T_{X,1} + T_{X,2} - T_{X,3}, \\ T_{X,j} &= \sum_{\substack{N^u\le a_j<\cdots<a_1<N^{\mu_*}\\ a_i\in\mathscr{L}_X}} \mathscr{U}_X(a_1\cdots a_j; N^u) \quad (0 \le j \le 2), \\ T_{X,3} &= \sum_{\substack{N^u\le a_3<a_2<a_1<N^{\mu_*}\\ a_i\in\mathscr{L}_X}} \mathscr{U}_X(a_1a_2a_3; a_3). \end{aligned} \qquad (2 \mid N,\ N \ge 2). \tag{4.26}$$

For $j = 0$, the sum is $\mathscr{U}_X(1; N^u)$. The primes are ordered; no factorial occurs. Every multiplicity remains in h_X.

For $d = a_1a_2a_3$ in a retained cell of source X, put

$$\begin{aligned} -\mathscr{U}_X(d;a_3) &= -\mathscr{U}_X(d; z_{X,c}) + \sum_{\substack{z_{X,c}\le r<a_3\\ r\in\mathscr{L}_X}} \mathscr{U}_X(dr;r) \\ &\le -\mathscr{U}_X(d; z_{X,c}) + \sum_{\substack{z_{X,c}\le r<a_3\\ r\in\mathscr{L}_X}} \mathscr{U}_X(dr; z_{X,c}). \end{aligned} \qquad (2 \mid N,\ N \ge 2). \tag{4.27}$$

The equality is (4.25). Nonnegativity proves the inequality. Actual cell membership gives $z_{X,c} \le a_3$. The closed infimum creates no additional occurrences.

Let $\sum_{\mathrm{ret},X}$ restrict the ordered triples to $\mathcal{R}_X$, equivalently to $\chi_{X,c}=1$. For either source define the peel terms

$$\begin{aligned} R_{X,3} &= \sum_{\mathrm{ret},X} \mathscr{U}_X(a_1a_2a_3; z_{X,c}),\\ P_{X,3} &= \sum_{\mathrm{ret},X} \sum_{\substack{z_{X,c}\le r<a_3\\ r\in\mathscr{L}_X}} \mathscr{U}_X(a_1a_2a_3r; z_{X,c}). \end{aligned} \qquad (2\mid N,\ N\ge 2). \tag{4.28}$$

For each source $X\in\{b,\mathrm{sw}\}$, the non-retained depth-three terms are nonnegative and occur with a minus sign, so we drop them. The retained terms use the dynamic inequality above:

$$-T_{X,3}\le -R_{X,3}+P_{X,3}. \qquad (2\mid N,\ N\ge 2). \tag{4.29}$$

For even $N\ge N_6$, define the two upper expressions and their slacks, one per source. The next comparison proves their nonnegativity:

$$\begin{aligned} U_b &:= T_{b,0}-T_{b,1}+T_{b,2}-R_{b,3}+P_{b,3}, & Z_b &:= E_b+U_b-B_b\ge 0,\\ U_{\mathrm{sw}} &:= T_{\mathrm{sw},0}-T_{\mathrm{sw},1}+T_{\mathrm{sw},2}-R_{\mathrm{sw},3}+P_{\mathrm{sw},3}, & Z_{\mathrm{sw}} &:= E^{\mathrm{src}}_{\mathrm{sw}}+U_{\mathrm{sw}}-S_{\mathrm{sw}}\ge 0. \end{aligned} \qquad (2\mid N,\ N\ge N_6). \tag{4.30}$$

Hence

$$\mathscr{U}_X(1;N^{\mu_*})\le \begin{cases} U_b, & X=b,\\ U_{\mathrm{sw}}, & X=\mathrm{sw}. \end{cases} \qquad (2\mid N,\ N\ge N_6). \tag{4.31}$$

These inequalities assume the two source comparisons in (4.24). They give no estimate for either restoration cost.

4.4. **The conditional lower expression.** Take A_{mix} from the Root–A definition (6.8), and Z_6 with its comparison from Section 6 (6.9). We redefine neither object. This sum uses the ordinary upper Rosser weights $\mu_d^+(D)$ with the odd-index cubic gates of (6.3). Their upper-sieve property is the Iwaniec upper-bound lemma [14, Lemma 1, (3.2), (3.6)] [13, Lemmas 2–3, p. 313]. This section introduces no new weight. The Section 6 comparison gives

$$S_6\le A_{\mathrm{mix}}, \qquad Z_6\ge 0 \qquad (2\mid N,\ N\ge N_6). \tag{4.32}$$

For a mark p, put $m=(N-e)/p$ and $P_{\mathrm{RA}}=\prod_{\ell<z_s,\ \ell\nmid 2N}\ell$. For $D>1$, the cited Iwaniec upper-bound lemmas give, for the upper Rosser divisor sum, [14, Lemma 1, (3.2), (3.6)] [13, Lemmas 2–3, p. 313],

$$\sum_{d\mid(m,P_{\mathrm{RA}})}\mu_d^+(D)\ge 1_{(m,P_{\mathrm{RA}})=1}\ge 1_{P^-(m)\ge z_s} \qquad (2\mid N,\ N\ge 2,\ D>1).$$

Dropping $(m,Np)=1$ enlarges the original nonnegative occurrence count. For even $N\ge N_6$, each of $R_L,R_{\mathrm{tail}},Q_{\mathrm{RA}}$ exceeds $N^{1/3}$. Hence take $D=R_L/p,R_{\mathrm{tail}}/p,Q_{\mathrm{RA}}/p$ on, respectively, $I_L\times J$, $I_H\times J_{\mathrm{tail}}$, $I_H\times J_t$. These pieces partition the original source: $e=N/4$ belongs to I_L, and the shared mark endpoint belongs to J_{tail}. Since $p\ge N^{t_0}>z_s$, the mark is unique in $q=pd$. The condition $d\mid m$ is exactly $e\equiv N\pmod{pd}$. The original gates give $c_{R,K}(pd)=\mu_d^+(R/p)$. Summation gives precisely $S_6\le A_{\mathrm{mix}}$ and $Z_6\ge 0$, with all mark multiplicities. A product-cell partition only splits these exact sums. It must keep the original $pd<R$, the cubic gates and every clipped endpoint. In particular c_{lo} and Γ keep the gates at Q_{RA}. Subdivide each of the three marked pieces in (6.7) into cells in its two displayed coordinates. Let $\delta=L^{-D_{\mathrm{m}}}$, and use half-open multiplicative intervals of relative width at most $1+\delta$ in each coordinate. Insert all row cuts, the points $N/4$, $1/3-\eta_{\mathrm{m}}$, $1/3$, and the two endpoints of I_H as cell boundaries. Denote the resulting finite, disjoint cell families by $\Theta_L^\sharp,\Theta_{\mathrm{tail}}^\sharp,\Theta_t^\sharp$. Use the separate piece index $Y\in\{L,\mathrm{tail},t\}$ and the mapping

$$(I_Y,K_Y,R_Y)=\begin{cases}(I_L,J,R_L), & Y=L,\\ (I_H,J_{\mathrm{tail}},R_{\mathrm{tail}}), & Y=\mathrm{tail},\\ (I_H,J_t,Q_{\mathrm{RA}}), & Y=t.\end{cases} \qquad (2\mid N,\ N\ge N_6). \tag{4.33}$$

Thus each $C\in\Theta_Y^\sharp$ satisfies $C\subseteq(\mathbb{P}\cap I_Y)\times(\mathbb{P}\cap N^{K_Y})$ and $(p,N)=1$, with $0<(N-e)/p$ on every included occurrence. Their unions are exactly the corresponding coprime marked source pieces $I_L\times J$, $I_H\times J_{\mathrm{tail}}$, and $I_H\times J_t$. A cell touching a boundary is kept as a boundary cell. For $Y\in\{L,\mathrm{tail},t\}$ and $C\in\Theta_Y^\sharp$, define

$$\begin{aligned} S_{6,Y,C} &:= \sum_{(e,p)\in C}\mathbf{1}_{p\in\mathbb{P}}\mathbf{1}_{(p,N)=1}\mathbf{1}_{p\mid N-e}\mathbf{1}_{0<(N-e)/p}\mathbf{1}_{e\in\mathbb{P}}\mathbf{1}_{P^-((N-e)/p)\ge z_s}\mathbf{1}_{((N-e)/p,Np)=1},\\ A_{Y,C} &:= \sum_{(e,p)\in C}\mathbf{1}_{p\in\mathbb{P}}\mathbf{1}_{(p,N)=1}\mathbf{1}_{p\mid N-e}\mathbf{1}_{0<(N-e)/p}\mathbf{1}_{e\in\mathbb{P}}\sum_{\substack{d\mid(N-e)/p\\ d\mid P_{\mathrm{RA}}}}c_{R_Y,K_Y}(pd)\mathbf{1}_{e\equiv N\ (pd)}, \end{aligned} \qquad (2\mid N,\ N\ge 2). \tag{4.34}$$

Here R_Y is R_L, R_{tail}, or Q_{RA}. The corresponding coefficient gates stay in the inner sum. The half-open families make every marked occurrence belong to exactly one C, including $e = N/4$ and the shared mark endpoint. The one-term Rosser inequality at each occurrence gives $S_{6,Y,C} \le A_{Y,C}$, cell by cell. Reindexing the finite sums gives

$$S_6 = \sum_Y \sum_{C \in \Theta_Y^\sharp} S_{6,Y,C}, \qquad \sum_Y \sum_{C \in \Theta_Y^\sharp} A_{Y,C} = A_{\mathrm{mix}}. \qquad (2 \mid N,\ N \ge N_6). \tag{4.35}$$

The finite comparison uses these cellwise Rosser inequalities. Thus the displayed comparison is the sum of finitely many explicit cell inequalities. No product cell, endpoint, or boundary strip is omitted. The later complete-cell estimates and boundary-strip bound are quantitative inputs only and do not alter this finite identity. The three slacks below are cited or defined by the source comparisons above, before this substitution. Substitution in (4.15) gives

$$\begin{aligned} \mathcal{W} - 4B_b - 4E_{\mathrm{pt}} = R_{\mathrm{fix}} + S_2 - A_{\mathrm{mix}} - U_{\mathrm{sw}} - 4U_b \\ - E_{\mathrm{sw}}^{\mathrm{src}} - 4E_b - 4E_{\mathrm{pt}} + Z_6 + Z_{\mathrm{sw}} + 4Z_b. \end{aligned} \qquad (2 \mid N,\ N \ge N_6). \tag{4.36}$$

This equality follows from the source identity and the cited and local definitions. The Section 6 comparison (6.9) gives $Z_6 \ge 0$. The comparisons in (4.30) give $Z_{\mathrm{sw}}, Z_b \ge 0$. For even $N \ge \max\{N_6, N_{\mathrm{pt}}\}$, (4.13) and nonnegativity of these slacks yield

$$\begin{aligned} 4G_b(N) \ge R_{\mathrm{fix}} + S_2 - A_{\mathrm{mix}} \\ - (T_{\mathrm{sw},0} - T_{\mathrm{sw},1} + T_{\mathrm{sw},2} - R_{\mathrm{sw},3} + P_{\mathrm{sw},3}) \\ - 4(T_{b,0} - T_{b,1} + T_{b,2} - R_{b,3} + P_{b,3}) \\ - E_{\mathrm{sw}}^{\mathrm{src}} - 4E_b - 4E_{\mathrm{pt}}. \end{aligned} \qquad (2 \mid N,\ N \ge \max\{N_6, N_{\mathrm{pt}}\}). \tag{4.37}$$

The two expanded signs are

$$\begin{aligned} -U_{\mathrm{sw}} &= -T_{\mathrm{sw},0} + T_{\mathrm{sw},1} - T_{\mathrm{sw},2} + R_{\mathrm{sw},3} - P_{\mathrm{sw},3}, \\ -4U_b &= -4T_{b,0} + 4T_{b,1} - 4T_{b,2} + 4R_{b,3} - 4P_{b,3}. \end{aligned} \qquad (2 \mid N,\ N \ge N_6). \tag{4.38}$$

Use one bound for the complete S_{sw}. Paper II gives its coefficient-eight estimate [27, (1.9), (5.29)–(5.43)]; the Buchstab upper bound concerns the same occurrence. Sections 7 and 8 estimate these two five-term upper expressions (four principal rows and their peel term) separately. The displayed expansion fixes each peel term's sign. The finite switched-source domination is already in the local Z_{sw} identity above. The remaining distribution and quantitative estimates for this occurrence are separate hypotheses of the later bill. They lie outside the finite decomposition of this section.

4.5. **The lower expression with the original S_2.** Keep the original S_2 throughout. Use $\overline{H}$, $M_{5,\mathrm{internal}}$ and ΔD_{L3} from (1.8). Here $\overline{H}$ is the switched integral bound. The distinct constant 153/256 bounds the packing envelope. The complete normalized switched error is $E_{\mathrm{sw}}(N;\sigma)$. The source-restoration error $E_{\mathrm{sw}}^{\mathrm{src}}$ is unnormalized. The main route takes $g_6 := 0$ and excludes the optional six-endpoint improvement. Fix this common parameter and an even $N \ge \max\{N_6, N_{\mathrm{pt}}\}$ throughout the following conditional lemma. Take $W, E_{\mathrm{van}}, \Lambda^0_{j,N}, \Lambda_N, \Delta_j, \Delta_{G1}$ from (2.2), (1.12), and Section 5, and $g_{\mathrm{RA,lo}}, E_{\mathrm{RA}}$ from (6.72). Use the complete switched and exterior errors from (8.72) and (7.67). The fixed constants are defined in (1.8) and (1.7).

Lemma 4.3 (Addition of the four comparisons)**.** *Assume* (4.11) *and the following four inequalities. Write*

$$\begin{aligned} \frac{R_{\mathrm{fix}} + S_2}{W(N)} &\ge D_{\mathrm{II}} + \overline{J}_6 + 2\overline{H} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} - E'(N;\sigma), \\ E'(N;\sigma) &= E_{\mathrm{van}}(N;\sigma) + \sum_{j \in \mathcal{H}} \big(|\Lambda^0_{j,N}| + |\Lambda_N(\Delta_j)|\big) + |\Lambda_N(\Delta_{G1})|. \end{aligned} \qquad (2 \mid N,\ N \ge \max\{N_6, N_{\mathrm{pt}}\} \text{ (assumed)}). \tag{4.39}$$

Here D_{II} denotes the original Paper II floor. The main route does not use the six endpoint replacements. Here $\mathcal{H} = \{4, 5, 7, 10\}$, with 5 denoting the lower part of S_5 and 10 its retained S_{10} part. Section 5 gives the hypotheses for this estimate.

The other three comparisons have the forms

$$\begin{aligned} \frac{S_6}{W(N)} &\le \overline{J}_6 - g_{\mathrm{RA,lo}} + E_{\mathrm{RA}}(N;\sigma), \\ \frac{S_{\mathrm{sw}}}{W(N)} &\le 2\overline{H} - \Delta D_{L3} + E_{\mathrm{sw}}(N;\sigma), \qquad (2 \mid N,\ N \ge \max\{N_6, N_{\mathrm{pt}}\} \text{ (assumed)}). \\ \frac{4B_b}{W(N)} &\le c_{K2,G0} I(b) + E_K(N;b,\sigma). \end{aligned} \tag{4.40}$$

All error terms are nonnegative.

Then the following normalized lower bound holds:

(4.41)
$$\frac{4G_b(N)}{W(N)} \ge D_{\mathrm{II}} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} + g_{\mathrm{RA,lo}} + \Delta D_{L3} - c_{K2,G0} I(b) - \mathcal{E}_B(N;b,\sigma), \qquad (2 \mid N,\ N \ge \max\{N_6, N_{\mathrm{pt}}\}).$$
$$\mathcal{E}_B(N;b,\sigma) = E'(N;\sigma) + E_{\mathrm{RA}}(N;\sigma) + E_{\mathrm{sw}}(N;\sigma) + E_K(N;b,\sigma) + \frac{4E_{\mathrm{pt}}}{W(N)}.$$

Proof. Substitute the four assumed estimates into (4.15) and use (4.13). The terms $\overline{J}_6$ and $2\overline{H}$ cancel exactly. Each complete upper source is subtracted once. In particular, no additional switched majorant is added.

Using the same retained constant as in Section 9, put

(4.42)
$$D_B = D_{\mathrm{II}} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} + g_{\mathrm{RA,lo}} - \frac{1}{10000} + \Delta D_{L3},$$
$$\mathcal{E}_B(N;b,\sigma) \le \frac{1}{10000} \quad\Longrightarrow\quad \frac{4G_b(N)}{W(N)} \ge D_B - c_{K2,G0} I(b). \qquad (2 \mid N,\ N \ge \max\{N_6, N_{\mathrm{pt}}\}).$$

□

5. The retained side

Keep every original source of Paper II, including the complete sum S_2. Fix $0 < \sigma \le 10^{-10}$ satisfying Paper II, (1.27). Put

$$L = \log N, \qquad \tau = \sigma^9, \qquad \rho = 1 + \tau, \qquad h = h_G = \frac{9}{5000}, \qquad \eta_L = \frac{1}{10000}. \tag{5.1}$$

The choice of σ precedes the limit. Use W from (2.2). Write R_{fix} for (4.14).

The comparison required in this section is

(5.2)
$$\frac{R_{\mathrm{fix}} + S_2}{W} \ge D_{\mathrm{II}} + g_6 + \overline{J}_6 + 2\overline{H} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} - E'(N;\sigma),$$
$$E'(N;\sigma) = E_{\mathrm{van}}(N;\sigma) + \sum_{j\in\mathcal{H}} \big(|\Lambda^0_{j,N}| + |\Lambda_{j,N}|\big) + |\Lambda_{G,N}|, \qquad \mathcal{H} = \{4,5,7,10\}.$$

In $\mathcal{H}$, indices 5 and 10 denote the lower S_5 and retained S_{10} parts. Every displayed error is nonnegative. The notation agrees with (1.12):

$$\Lambda_{j,N} = \Lambda_N(\Delta_j), \qquad \Lambda_{G,N} = \Lambda_N(\Delta_{G1}). \tag{5.3}$$

The required limit is $E'(N;\sigma) \to 0$. Proposition 5.10 proves (5.2). Its inputs are listed in Remark 5.9. The six-endpoint lemma is optional: without it, the same proof gives (5.2) with g_6 replaced by 0. We display the unchanged comparisons first. Then we specify the five incremental and four saved-array conversion terms.

5.1. **The original signed chains.** Let $\mathcal{I} = \mathcal{I}_{\mathrm{row}} \sqcup \{\mathrm{T5}\}$ index the signed chains of $R_{\mathrm{fix}} + S_2$. Here $\mathcal{I}_{\mathrm{row}}$ is the set of Paper II's nineteen row chains. The chain T5 is Paper II's outer chain for the upper part of S_5 [27, (2.28), (5.10), (6.1) and §6.1, condition 1]. For $j \in \mathcal{I}$, let $Q_{j,k}$ include its normalized coefficient. Paper II uses ten stages [27, (6.4)–(6.4d)]:

$$Q_{j,0} = Q_{j,10} + \sum_{k=1}^{10} (Q_{j,k-1} - Q_{j,k}). \tag{5.4}$$

The initial expression is the actual arithmetic source. The last expression is a directed rational endpoint. For $j = \mathrm{T5}$ the stages are those of Lemma 5.1: $Q_{\mathrm{T5},k}$ is defined there for $k \le 3$, and $Q_{\mathrm{T5},k} = Q_{\mathrm{T5},3}$ for $3 \le k \le 10$. For the three joint groups, use

(5.5)
$$\mathcal{O} \in \{S_1, S_2\text{-}T_8/4, G_1\}, \qquad Q_{\mathcal{O},9} = \sum_{j\in\mathcal{O}} Q_{j,9},$$
$$\sum_{j\in\mathcal{O}} Q_{j,0} = E_{\mathcal{O}} + (Q_{\mathcal{O},9} - E_{\mathcal{O}}) + \sum_{j\in\mathcal{O}} \sum_{k=1}^{9} (Q_{j,k-1} - Q_{j,k}),$$
$$Q_{\mathcal{O},9} - E_{\mathcal{O}} \ge 0.$$

The joint signs are kept before rounding. The G_1 weights are $(2,1,1)$.

Signed sources and parent assignment. Here "parent" records the assignment of an arithmetic occurrence and its source-restoration share. It does not assert that each Buchstab child is numerically bounded by the unsplit parent. Let $T_{j,r}$ be the nonnegative plain-source terms of Paper II, (2.19) and (3.2). Let Δ_j^{src} be its signed source correction. Then

$$3S_1 = 3(T_{1,0} - T_{1,1} + T_{1,2} - T_{1,3}) + 3\Delta_1^{\mathrm{src}}, \qquad S_2 = T_{2,0} - T_{2,1} + T_{2,2} - T_{2,3} + \Delta_2^{\mathrm{src}}.$$

For every third-depth S1 occurrence, $E_3 \le p_3$ gives $-3S_{\mathrm{pl}}(b,p_3) \ge -3S_{\mathrm{pl}}(b,E_3)$. For S2 we make this replacement only on Paper II's selected whole-K blocks $\mathcal{F}$, including PF and $\mathcal{F} \cap R1$. There $-S_{\mathrm{pl}}(b,p_3) \ge -S_{\mathrm{pl}}(b,E_3)$. On the strict complement $\mathcal{B}$ keep the actual p_3 and Paper II's strict-BV comparison. The last child of S2 keeps its negative sign on all three pieces PF, $\mathcal{F} \cap R1$, and $\mathcal{B}$. The strict BV piece stays in S2.

For the fixed-cutoff rows use their literal divisor domains. Their signed sources are $-2S_3$, $-S_4$, $-S_{5,\mathrm{low}}$, $-S_7$, $+S_8$, and $+S_9$, respectively. On each source cell, an upper sieve bound $S \le U$ gives $cS \ge cU$ when $c < 0$. A lower sieve bound $L \le S$ gives $cS \ge cL$ when $c > 0$. These comparisons keep the source corrections and the row's allocated errors. Paper II, (2.28), gives the exact assignment identity

$$-S_5 = -S_{5,\mathrm{low}} - S_{5,\mathrm{top}}, \quad S_{5,\mathrm{low}} = \sum_{1/4 \le \log_N p < t_*} S(p, N^\alpha), \quad S_{5,\mathrm{top}} = \sum_{t_* \le \log_N p < \tau_S} S(p, N^\alpha).$$

The endpoint t_* belongs to the upper/BV part. The split precedes the structured sieve or Heath–Brown expansion and keeps parent labels. S5-low and T5 both have parent S5. S5-low gets no restoration address of its own. For S10 the source partition $S_{10} = S_{10,\mathrm{Ret}} + S_{10,\mathrm{Drop}}$ is exact, with $S_{10,\mathrm{Ret}}, S_{10,\mathrm{Drop}} \ge 0$. Hence $+S_{10} \ge +S_{10,\mathrm{Ret}}$ before its lower sieve bound.

The joint source keeps the representation weights $(2,1,1)$. Paper II, (4.14), gives, with its stated exceptional-source and small-prime allowance $E_G \ge 0$,

$$G_1 \le 2S(\mathcal{B}_{11}, N^{1/2}) + S(\mathcal{B}_{12}, N^{1/2}) + S(\mathcal{B}_{13}, N^{1/2}) + E_G.$$

Make its cut-measure replacement with the original signed correction before applying (4.18). The four signed descendants of $-G_1$ then have coefficients $(-1,+1,-1,+1)$ on the single weighted source. Their last positive term is bounded from below by its retained cells. The change from actual a_3 to label base E keeps the signed peel correction of Paper II, (4.45)–(4.57). The four rows share one joint source parent and one restoration allocation. Thus S1, S2, S4, S5, S7, S10 and G1 are the seven parents of the sixteen listed retained rows. S3, S8 and S9 keep their original single-comparison parents, and T5 remains the upper part of S5. These are sign-correct source reductions for the complete parent chains. They do not discharge the distinct S6, bad-source or switched-source comparisons in Section 4.

The nineteen original row chains of $\mathcal{I}_{\mathrm{row}}$ are the sixteen rows of Appendix B.7 and Paper II's original S_3, S_8, S_9 comparisons. The strict BV piece of S_2 lies inside the joint S_2 group. The source $-S_5$ of R_{fix} is split at N^{t_*}, $t_* = 7109/28000$, before any sieve step. Its lower part is the row S_5-low of $\mathcal{I}_{\mathrm{row}}$. Its upper part is the chain T5. The chains of S_6 and S_{sw} are excluded. Thus the endpoint sum over $\mathcal{I}$ is

$$\sum_{j \in \mathcal{I}} Q_{j,10} = B_{\mathrm{print}} - \delta_{G1} + \overline{J}_6 + 2\overline{H}, \qquad \delta_{G1} = 4.98 \cdot 10^{-54}. \tag{5.6}$$

This follows from Paper II's printed thirteen-term sum [27, §6.4, immediately before (6.6)]. Its joint endpoint is $Q_{G1,10} = L_{G1} - \delta_{G1}$; the printed sum uses L_{G1}. The term $-\overline{J}_5$ of (5.7) is $Q_{\mathrm{T5},10}$, by Lemma 5.1. Apart from $-\overline{J}_6 - 2\overline{H}$, every other term of (5.7) is an endpoint. It ends a chain of $\mathcal{I}_{\mathrm{row}}$ or a joint group. This is a finite coefficient identity. The printed sum is

$$\begin{aligned} B_{\mathrm{print}} = {} & \underline{D}_1 + \underline{d}_2 - \frac{U_{3,A}}{2} - \frac{U_4 + U_{5,\mathrm{low}} + U_7}{4} \\ & + \frac{L_{8,A} + L_{9,A} + L_{10}}{4} + L_{G1} - \overline{J}_5 - \overline{J}_6 - 2\overline{H} \\ > {} & 1.834037421096413313459579145421940449323041697278696810730 64. \end{aligned} \tag{5.7}$$

The letters denote the rational endpoints printed in Paper II, Appendix A.2–A.6 and (A.7), (A.10), (A.13)–(A.14). The constant $\overline{J}_6$ keeps its rational definition:

$$\overline{J}_6 = 1.31485819591573215335132472767822853107587704688862376195 85. \tag{5.8}$$

The upper part of S_5 has the following outer chain. Put $t_* = 7109/28000$, $\tau_S = 57/224$, and let

$$\overline{J}_5 = \frac{1}{4}\big(c_5 + 8.84 \cdot 10^{-57}\big), \qquad c_5 = 0.03658166423843258896399704402764682632822246022922592990, \tag{5.9}$$

which is Paper II's directed upper endpoint for $J_5(0) = \int_{t_*}^{\tau_S} \frac{dt}{t(1/2-t)}$ [27, (5.16), (5.20), (A.14)–(A.14a)].

Lemma 5.1 (The outer chain of the upper part of S_5)**.** *Define*

$$Q_{\mathrm{T5},0} = -\frac{1}{W} \sum_{t_* \le \log_N p_1 < \tau_S} S(p_1, N^\alpha), \quad Q_{\mathrm{T5},1} = -\frac{T_5}{W},$$
$$Q_{\mathrm{T5},2} = -J_5(0), \qquad Q_{\mathrm{T5},3} = -\overline{J}_5,$$

where the first sum is the second term of Paper II, (2.28), and T_5 is Paper II, (5.10). Then

$$Q_{\mathrm{T5},0} - Q_{\mathrm{T5},1} \ge -E_{\mathrm{T5},1}(N), \qquad Q_{\mathrm{T5},1} - Q_{\mathrm{T5},2} \ge -E_{\mathrm{T5},2}(N), \qquad Q_{\mathrm{T5},2} - Q_{\mathrm{T5},3} \ge 0,$$

with $E_{\mathrm{T5},1}, E_{\mathrm{T5},2} \ge 0$ tending to 0 at fixed σ. Both are entries of E_{van}. In particular $Q_{\mathrm{T5},0} \ge -\overline{J}_5 - E_{\mathrm{T5},1}(N) - E_{\mathrm{T5},2}(N)$.

Proof. Paper II splits $-S_5$ at N^{t_*} before the structured sieve or HB. The endpoint t_* belongs to the upper part [27, (2.28) and the three lines after it]. The coefficient is -1.

Stage one. For a prefix $p_1 \in \mathcal{P}_N$, dropping the condition $(m, Nb) = 1$ gives $S(p_1, N^\alpha) \le S_{p_1}(N^\alpha)$ in the notation of Paper II, (5.9) [27, the sentence after (5.9)]. Prefixes $p_1 \mid N$ are removed at their first failure. This is the error-table entry "Exceptional source factors", of size $O_\sigma(L^3/N + N^{-\alpha/2+o(1)})$ [27, §6.3]. A prime equal to N^{t_*} or N^{τ_S} divides N [27, after (5.20)]. With the coefficient -1 this gives the first inequality.

Stage two. Paper II applies the upper sieve (5.4) to each $\mathcal{A}_b$, $b = p_1$. Its parameters are $X_{\mathcal{A}} = \mathrm{Li}(X)/\varphi(b)$, level $Q = N^{1/2}L^{-B}$ ($\kappa = 0$), $D_b = Q/b$ and $z_b = N^\alpha$ [27, (5.12)]. For $t = \log_N p_1 \in [t_*, \tau_S)$ one has $a_0(t) = 1/2 - t \ge 55/224$, $t_* - \alpha = 5368241/32172000 > 0$, and

$$\frac{a_0}{\alpha} \in \Big[\frac{12639}{4480}, \frac{7917759}{2800000}\Big] \subset (2, 3)$$

[27, table after (5.12), row T5]. Hence eventually $s_b = (a_0 - \delta_N)/\alpha \in (2, 3)$, where $F(s) = 2e^\gamma/s$ [27, between (5.4) and (5.5)]. By (5.14) the remainders are $o(U_C)$. The main terms are (5.15), and (5.16) gives $T_5 \le 4(1-\varepsilon)U_C J_5(0) + o(U_C)$ with the vanishing logarithmic loss (5.21) [27, (5.14)–(5.16), (5.21) and the sentence "S5 keeps its vanishing logarithmic loss"]. Since $W = 4(1-\varepsilon)U_C$, this is the second inequality. Its error is the error-table entry "S5 BV remainder" plus the prime-measure and Euler conversion of that row [27, §6.3].

Stage three. By Paper II, (A.14)–(A.14a), $4J_5(0)$ lies in the ball of centre c_5 and radius $8.84 \cdot 10^{-57}$, so $J_5(0) \le \overline{J}_5$. Paper II also gives the independent rational enclosure $P_5 < 4J_5(0) < P_5 + R_5$ inside the same ball [27, after (A.14a)].

The chain uses $\kappa = 0$, so the increment $J_5(\kappa) - J_5(0)$ does not occur. Paper II's allowance $E_{5,\kappa} = 4 \cdot 10^{-8}$ is a summand of $g_{\mathrm{BV}} = 0.00000575$ in (5.10) and stays there once. Paper II, Appendix A, records the hypothetical increment and its once-booked decomposition. They are in the paragraph opening *For comparison, if S5-top used* $\kappa = 10^{-6}$ and in the BV-allowance line right after it [27, Appendix A, comparison paragraph and following decomposition]. Equation (A.14a) gives only the nominal value at $\kappa = 0$. □

The chain T5 does not meet the changes of this section. Its three stages use no sieve level ℓ_c and no saved profile. They use no Class II, packing or Class I term. Paper II's prepared Class II, packing and Class I contributions of this row are zero [27, §2.4, paragraph "S5-top and S6 use BV separately"]. The lift below changes the kernels on I_4, $I_5 = [1/4, t_*)$, I_7 and $\mathcal{D}_{10}$ only. The interval I_5 is disjoint from $[t_*, \tau_S)$. The joint shift changes the G_1 cone only. Hence Lemma 5.1 holds unchanged after both changes.

Keep the whole fixed deduction of Paper II:

$$\begin{aligned} F_{\mathrm{net}} &= 0.000006484294227956901 + 0.00000575 + 0.00002853328 \\ &\quad + 4.98 \cdot 10^{-54} + 2 \cdot 10^{-24} + 10^{-390} \\ &< 0.000040767574227956903. \end{aligned} \tag{5.10}$$

These are Paper II's six deductions [27, (6.6)]. The first remains paid although its source is excluded. The joint G_1 reconciliation also remains paid. Every adverse row difference has an absolute bound. Keep their whole vanishing envelope, E_{van}. The row estimates required here are

$$\sum_{j \in \mathcal{I}} \sum_{k=1}^{10} (Q_{j,k-1} - Q_{j,k}) \ge -\big(F_{\mathrm{net}} - \delta_{G1}\big) - E_{\mathrm{van}}(N; \sigma), \qquad E_{\mathrm{van}}(N; \sigma) \longrightarrow 0. \tag{5.11}$$

The δ_{G1} part is already in the endpoint sum, so the remaining chain deduction is $F_{\mathrm{net}} - \delta_{G1}$. Their total is exactly F_{net}. The fixed AA37 share is paid under the admissible condition $0 < \sigma \le 4g_{\mathrm{AA}}/(C_E W_*)$ of Paper II, (1.27) [27]. See Paper II, §6.5. This condition is part of the fixed-parameter choice made before the limit. The analytic inputs are Paper II, (6.4b)–(6.5), and its direct and G_1 estimates [27, (4.66)–(4.69), (6.4b)–(6.5)]. Their logarithmic parameters are

$$K_0 = 8042, \qquad A' = 8047, \qquad A_{\mathrm{work}} = 8051, \qquad K_0 + 2 - A' = -3. \tag{5.12}$$

These are Paper II, (4.66), for the original coefficient families. This section leaves them unchanged.

Lemma 5.2 (Inherited chain errors)**.** *The bound* (5.11) *holds, and* $E_{\mathrm{van}}(N;\sigma) \to 0$ *as* $N \to \infty$ *at fixed* σ*. Each row or separately listed stage difference used below has an allocated nonnegative share in Paper II's error table. Denote it by* $\mathcal{E}_j$*. Paper II, §6, charges these shares once. An entry may be grouped across rows, so the lemma claims nothing for arbitrary subfamilies and does not claim pairwise disjoint entries. The only omission bounds used are those for the whole chains or stage differences named in the proof below.*

Proof. The signed-assembly theorem of Paper II proves $D_{\mathrm{src}} \geq B_{\mathrm{main}} - F_{\mathrm{pers}} - E_{\mathrm{van}}$ with $E_{\mathrm{van}} \to 0$ for the source (6.1) [27, §6.1, (6.1)–(6.2), (6.4)–(6.4d)]. Its proof bounds every adverse stage difference by an allocated share of Paper II's error table, §6.3. Each entry is a nonnegative bound or a proved zero; some entries are grouped across rows. Paper II assigns each tuple to its first failed condition and charges each difference or part once below [27, §6.3]. The fixed entries are the six deductions of (5.10), that is $F_{\mathrm{pers}} = F_{\mathrm{net}}$ [27, (6.6)]. The others tend to zero at fixed σ, and their sum is E_{van}. The rows of $\mathcal{I}$ are rows of (6.1). The S_6 level increment is still deducted when that chain is omitted. The δ_{G1} part is already in (5.6). This gives (5.11). For T5 the differences and entries are those of Lemma 5.1. Separately listed entries cover the omissions used in this chapter. The whole S_6 and S_{sw} chains use the S6 BV remainder, S6 level increment and $E_{S_{\mathrm{sw}}}$ entries. The whole G_1 chain uses the §4.6 table and (4.85) of Paper II [27, §4.6, (4.85)]. The stage-nine strip for the retained part of S_{10} uses the S10 selector entry. These bounds apply to the named whole chains or stage difference after allocating their shares. No arbitrary omitted subfamily is asserted. For the direct rows, the stage-nine entry "Profile comparison" is zero adverse. The stage-ten entry "Directed enclosure" is zero additional [27, §6.3]. For the retained part of S_{10}, the strip between the retained cells and $\mathcal{D}_{10}$ is the entry "S10 selector", 7.160τ. The term $g_\tau = 2 \cdot 10^{-24}$ replaces it [27, §6.3, rows "S10 selector" and g_τ].

The entries rest on the following inputs, all used by Paper II on these same sources. The linear-sieve main terms and remainders are Paper I, Theorem 2.6, (2.30)–(2.33b). This is the structured form of Iwaniec's Theorem 4 [13, Theorem 4, p. 311], applied with $D = L_c$, $\epsilon_s = \sigma$ and $\delta_s = \tau$ [27, §6.2]. Before the exterior sign, applied once, each block remainder splits as

$$\frac{C_c}{M_c}\langle \gamma_c, \Delta_P\rangle + NC_{\mathrm{sf},c} - \langle \Xi_{U,c}, \Delta_P\rangle + \langle \Xi_{A,c}, \Delta_0 - \Delta_P\rangle,$$

with $\Xi_{A,c} = O_c\mathbf{1}_{q \text{ not squarefree}}$ [27, (6.4a)–(6.4b), (4.23)]. The term NC_{sf} vanishes on the direct rows, where $\Delta_0 = \Delta_P$. On G_1 it is $O_\sigma(N^{-\alpha}L^5)$ by Paper II, (4.64)–(4.65). The terms Ξ_U and Ξ_A are bounded jointly, once, by $232\Lambda_\sigma C_{\mathrm{ret},\sigma} N^{-\ell_0\sigma^2}(1+L)^4$, with $\ell_0 = 264897/2060000$ [27, (B.1)–(B.7)]. Here $C_{\mathrm{ret},\sigma}$ is the retained-side combinatorial constant; it depends on the fixed σ and not on N. The persistent part $C_E\sigma W_{\mathrm{AA}}/4$ of Paper II, (B.8), lies in g_{AA}; only its secondary part belongs to E_{van}. The Class II estimate (6.5) of Paper II applies Paper I, Theorem 3.2, (3.13), with the precisions (5.12). The spectral input of that theorem is Paper I, Theorem 2.8; see Fact 3.22. In the large-norm case of Paper I, (3.48), its mean-square input is Lemma 6.5(ii), in place of [3, Theorem 0(a)]. Lemma 6.5(ii) applies under the Paper I hypotheses and selected parameter box stated there. It uses neither the optional six-endpoint lemma nor any broader BFI86 scope. Its premises are checked for these sources as follows. In Paper I, conditions (3.1), (3.2), (3.6), (3.8) with the length condition, and (3.11) display its hypotheses (H1)–(H5). Paper II verifies them for these same sources, coefficients, boxes and scales [27, §6.2, hypothesis table, row "C1–C5, LEN"]. Paper I proves condition (3.25) for every selected factor of this form. The proof uses the prime Siegel–Walfisz theorem (2.21) [26, (3.25) and its proof]. Conditions (3.26)–(3.27) are the definition of λ_r and the Cauchy–Schwarz step of that proof. They hold under (H1)–(H5), with $|\lambda_r| \leq \tau(r)$ from the roughness in (H2) [26, (3.26)–(3.27)]. Stages one to eight are unchanged, so these are Paper II's own checks. The prime-measure and Euler conversions are the prime number theorem, Mertens' theorem and Paper II, (4.76)–(4.78). The profile lift and the shifted joint profile below act at stages nine and ten only. They change no source, cell, level, mask, box, norm or precision at stages one to eight. Hence E_{van} is Paper II's own envelope for these rows. □

Paper II's rational floor satisfies

$$B_{\mathrm{print}} - F_{\mathrm{net}} > D_{\mathrm{II}} = \frac{45849916338009170593}{25000000000000000000}; \tag{5.13}$$

see [27, (6.3a), (6.6)]. The selected S2 endpoint used in this floor is d_{T8}. Paper II's tail reserve is

$$\delta_{T8} = \frac{1}{20000}, \qquad d_{T8} = \frac{P_- - (2/e^\gamma)V_{\mathrm{full}} - (2/e^\gamma)J_+ + C_-}{4} - \delta_{T8}. \tag{5.14}$$

Thus δ_{T8} is deducted once inside d_{T8}, as in Paper II, (A.7)–(A.8). It is not a second deduction from D_{II}. The separate fixed term 275009680τ remains outside d_{T8} in the S2 estimate [27, (3.22)–(3.23), (A.7)]. Combining the preceding estimates gives

$$\frac{R_{\mathrm{fix}} + S_2}{W} \geq D_{\mathrm{II}} + \overline{J}_6 + 2\overline{H} - E_{\mathrm{van}}(N;\sigma). \tag{5.15}$$

This deduction uses the individual signed chains. Its analytic input is (5.11).

5.2. **Six improved endpoints.** Let

$$\mathcal{J} = \{1, 2, 4, 5, 7, 10\}. \tag{5.16}$$

For $j \in \mathcal{J}$, let P_j be its Paper II profile integral. The indices 5 and 10 keep the preceding conventions. Let O_j and N_j be directed lower endpoints. The required numerical comparisons are

$$P_j \geq N_j \geq O_j, \qquad P_j - O_j = (N_j - O_j) + (P_j - N_j). \tag{5.17}$$

The old and improved directed endpoints are the following exact terminating decimals. They are given in the normalized D contributions used by the signed sum. For S4, S5-low and S7 the sign $-1/4$ is included; for S10 the sign $+1/4$ is included.

Endpoint source	Directed contribution in D units
S1, [27, (A.4), (A.5a)]	$O_1 = 9.67653874894124412908238749318694090949698021569432709634$
	$N_1 = 9.67655703542073109779637615721289987202051812605345855204 2$
S2, [27, (A.8), (A.8a)]	$O_2 = 1.619581356989083525591334043321057292200767140816 5$
	$N_2 = 1.62063469556829$
S4, [27, Appendix A.2–A.6]	$O_4 = -0.8342438370724665$
	$N_4 = -0.834181700688906460517498143106062 5$
S5-low, [27, Appendix A.2–A.6]	$O_5 = -0.05972210829776175$
	$N_5 = -0.059718855067707665767321408650351 75$
S7, [27, Appendix A.2–A.6]	$O_7 = -0.574850630119163$
	$N_7 = -0.574815106890481420929612425345365$
S10-Ret, [27, Appendix A.2–A.6]	$O_{10} = 0.214711863944485$
	$N_{10} = 0.215594271533099346859788739403732 55$

The rational endpoint differences sum to

$$\begin{aligned} g_6 &= \sum_{j \in \mathcal{J}} (N_j - O_j) \\ &= 0.00205494548960349276801138300685497032277076954263145570 2. \end{aligned} \tag{5.18}$$

Exact rational addition of the six entries above gives the displayed sum, and each entry satisfies $N_j \geq O_j$. The values O_j are Paper II's directed endpoints, and $P_j \geq O_j$ is proved there [27, (A.4), (A.5a), (A.8), (A.8a), Appendix A.2–A.6]. The values N_j are lower bounds read from Paper II's printed independent enclosures. The following optional lemma states them.

Lemma 5.3 (Six directed endpoints from Paper II; optional)**.** *For each $j \in \mathcal{J} = \{1, 2, 4, 5, 7, 10\}$, the Paper II profile integral P_j, with its signed normalized coefficient, satisfies $P_j \geq N_j$, with N_j as tabulated above.*

Proof. For $j = 1$, Paper II, (A.5a), gives the printed 3/4 ball; its lower endpoint is N_1. For $j = 2$, Paper II, (A.8a), gives the printed interval whose lower endpoint is N_2. Paper II states that this interval covers the continuous functional and the infinite tail. The d_{T8} deduction in (6.6a) is released once in $N_2 - O_2$. The term F_{net} in (6.6b) contains no δ_{T8} [27, (6.6a)–(6.6b)]. For $j = 4, 5, 7$, the Appendix A.3 balls include the adverse $\tau = 10^{-90}$ shifts. Since the profile function is nonincreasing, the shifted raw integrals are at least the τ-free ones. Their upper endpoints with the negative coefficient therefore give $P_j \geq N_j$. For $j = 10$, the Appendix A.3 S_{10}-Ret ball encloses the τ-free actual-coordinate profile on $\mathcal{D}_{10}$. Its lower endpoint gives $P_{10} \geq N_{10}$. The separate $D_*\tau$ strip is paid in the S_{10} selector estimate. These are the printed enclosures in Paper II, (A.5a), (A.8a), and Appendix A.3. Its numerical-certificate section records the adverse-shift and actual-coordinate conventions [27, (A.5a), (A.8a), Appendix A.3]. The remaining directed endpoints O_j are unchanged. □

With O_j in place of N_j, (5.17) is Paper II's own endpoint comparison and g_6 is replaced by 0. Only the endpoint changes in (5.4). All earlier signed differences remain unchanged. Thus (5.17) adds g_6 once. The S_5 upper part remains separate. The joint G_1 endpoint is absent from $\mathcal{J}$.

5.3. **Four stronger profile comparisons.** Use the sieve functions of Paper I [26, (2.23), immediately before Theorem 2.6]. Put

$$\widehat{F} = \frac{F}{2e^{\gamma}}, \qquad \widehat{f} = \frac{f}{2e^{\gamma}}, \qquad K = \frac{57}{64} - u. \tag{5.19}$$

Define

$$\begin{aligned} s(x) &= \frac{K - 5x/2}{\alpha}, \qquad q(x, y) = \frac{K - 5x/2 - y}{\alpha}, \\ B_0(x) &= \min\{\beta, \max\{\alpha, K - 11\alpha/5 - 5x/2\}\}. \end{aligned} \tag{5.20}$$

The domains are

$$\begin{aligned} I_4 = [\mu_*, 1/4), \quad I_5 = [1/4, 7109/28000), \quad I_7 = [\mu_*, \lambda), \\ \mathcal{D}_{10} = \{(x, y) : \mu_* \leq x < \lambda,\ \alpha \leq y < B_0(x)\}. \end{aligned} \tag{5.21}$$

These are Paper II's domains [27, (2.27), (2.29), (2.31), (2.38)–(2.40)]. Set

$$\begin{aligned}\Delta_j &= \int_{I_j} \frac{\widehat{F}(s(x)) - \widehat{F}(s(x)+h/\alpha)}{\alpha x}\,dx && (j=4,5,7),\\ \Delta_{10} &= \iint_{\mathcal{D}_{10}} \frac{\widehat{f}(q(x,y)+h/\alpha) - \widehat{f}(q(x,y))}{\alpha xy}\,dx\,dy.\end{aligned} \tag{5.22}$$

The three upper rows carry a negative sign. The lower S_{10} row carries a positive sign. Their original coefficients are included in these gains, so there is no further factor 1/4.

Paper II's actual levels give

$$\begin{aligned}\ell_c &\ge \alpha s(x) + h + \frac{1}{10000} - 3\tau && (j=4,5,7),\\ \ell_c &\ge \alpha q(x,y) + h + \frac{1}{10000} - \left(\frac{5\lambda}{2}+\beta\right)\tau && (j=10).\end{aligned} \tag{5.23}$$

See [27, §2.4, proof of (2.26)–(2.31); §2.5, proof of (2.40)]. The exact split behind the displayed reserve is

$$u - \eta_L = \frac{19}{10000} = \frac{9}{5000} + \frac{1}{10000} = h_G + \frac{1}{10000}. \tag{5.24}$$

After the h_G lift, Paper II's S4, S5-low (including its B branch) and S7 comparisons leave $1/10000 - 3\tau$. Its S10-Ret comparison leaves $1/10000 - \tau(5\lambda/2+\beta)$. Both are positive for $\tau \le 10^{-90}$. The G1-3 shift leaves $g_s = 1/10000 - (225/256)\tau > 2\mu_*\tau$ and the μ_* crossing leaves $1/10000 - (75/256)\tau > 0$, by [27, (4.72)]. These are pointwise profile-numerator inequalities. They do not change the actual level ℓ_c, the level D_c, or the masks, boxes, Class-II and AA37 inputs. The other level conditions of Paper II remain pointwise and unmodified. They include the G1 guard, S1–S3, S2/S3/S8/S9, and the whole-K margin. See Paper II, (2.21), (2.40), (4.72), and (4.19). Decreasing F gives the first three comparisons. Increasing f gives the fourth comparison. These operations leave the actual levels unchanged. The H6 lift and the M5 shift act only at stages 9–10, after Class II at stage 7. They change no Class-II, AA37, box, mask or precision input: $\ell_c = \Theta_c - \eta_L - \sum_i t_i$ contains no saved profile, and (5.12) remains unchanged.

Proposition B.1 gives finite integral enclosures. With the directed calculation of the computational appendix, they prove the following. Its exact rational interval is $[0.00572004709432798871422299118637922914938188759004277312087148967296502249 9 \pm 7.09\cdot 10^{-76}]$. This is a ball for the sum of the four rowwise lower bounds, not for $\sum_j \Delta_j$. We use the terminating credit $572004/10^8$. It lies below that exact lower endpoint and leaves the existing 10^{-20} allowance intact. This allowance is conservative; it is not an additional gain in the final constant.

$$\frac{572004}{10^8} < \Delta_4 + \Delta_5 + \Delta_7 + \Delta_{10} < \frac{606486}{10^8}, \qquad \frac{572004}{10^8} - 10^{-20} > \frac{5719}{10^6}. \tag{5.25}$$

The unchanged S_{10} strip also satisfies

$$\frac{10987/10000}{4\alpha^2}\left(\frac{5\lambda}{2}+\beta\right)\frac{\lambda-\mu_*}{\mu_*} = \frac{21137254516197687}{4120000000000000} < 5.137 < 7.160. \tag{5.26}$$

The lifted S_{10} argument satisfies $q + h/\alpha \le 11519839/3200000 + h/\alpha < 4$. Thus $f \le f(4) = 2e^\gamma \log 3/4$ throughout the strip, as used in this density bound. These are finite bounds, including their stated domains. The retained cells also leave a reserve. Put

$$\begin{aligned}R_{10} &= \left[\frac{1}{5},\frac{21}{100}\right]\times\left[\frac{9}{100},\frac{1}{10}\right], \qquad \delta_{10} = \frac{1}{20000\alpha},\\ c_{10} &= \frac{1}{\alpha}\cdot\frac{7\delta_{10}}{480}\cdot\frac{1}{210} = \frac{146689}{320000000000}, \qquad c_{10} - \frac{21137254516197687}{4120000000000000}\,\tau > 10^{-7}.\end{aligned} \tag{5.27}$$

Lemma 5.5 uses (5.26) and (5.27), so its comparison over the whole of $\mathcal{D}_{10}$ needs no deduction. Paper II's S_{10} selector entry 7.160τ is replaced in Paper II by $g_\tau = 2\cdot 10^{-24}$, a summand of (5.10). The fixed g_τ amount is deducted once. The retained reserve above eliminates the S_{10} selector deduction in this comparison. The inherited G_1 peel and its label replacement are jointly covered by that same summand.

We next define the finite profile arrays. Let $X_0 = [u,1]$. On this compact interval, use the zero extension of the profile outside its displayed domain, and put

$$m_N = \sum_{\substack{p\in\mathbb{P},\ p>2,\ p\nmid N\\ N^u\le p\le N}} \frac{\delta_{\log_N p}}{p-1}, \qquad m(dt) = \frac{dt}{t}. \tag{5.28}$$

The symbol δ_x denotes unit mass at x. Define

$$V_N(z) = \prod_{\substack{p<z,\ p\in\mathbb{P}\\ p\nmid N}} \left(1-\frac{1}{p-1}\right), \tag{5.29}$$

and

$$\begin{aligned} X_D &= \sum_{v\in\mathbb{P}} \mathbf{1}_{N/L^{30}<v<(1-\varepsilon)N}, \\ r_{D,N} &= \frac{2e^{\gamma}\alpha X_D V_N(N^{\alpha})}{W}. \end{aligned} \tag{5.30}$$

For $j = 4, 5, 7$, set

$$\begin{aligned} k_j^0(x) &= -\frac{\mathbf{1}_{I_j}(x)}{\alpha}\widehat{F}(s(x)), \\ k_j^h(x) &= -\frac{\mathbf{1}_{I_j}(x)}{\alpha}\widehat{F}(s(x)+h/\alpha). \end{aligned} \tag{5.31}$$

For $j = 10$, set

$$\begin{aligned} k_{10}^0(x,y) &= \frac{\mathbf{1}_{\mathcal{D}_{10}}}{\alpha}\widehat{f}(q(x,y)), \\ k_{10}^h(x,y) &= \frac{\mathbf{1}_{\mathcal{D}_{10}}}{\alpha}\widehat{f}(q(x,y)+h/\alpha). \end{aligned} \tag{5.32}$$

Put $e_j = 1$ for $j = 4, 5, 7$, and $e_{10} = 2$. The original and lifted arrays are

$$\begin{aligned} P_j &= \int k_j^0\, d(m^{e_j}), \qquad S_{j,N}^0 = r_{D,N}\int k_j^0\, d(m_N^{e_j}), \\ S_{j,N}^h &= r_{D,N}\int k_j^h\, d(m_N^{e_j}), \quad B_{j,N} = S_{j,N}^h - S_{j,N}^0. \end{aligned} \tag{5.33}$$

The powers denote product measures. These formulas use P_j from (5.17). Write $d_j = k_j^h - k_j^0$. For $j = 4, 5, 7$, and then $j = 10$,

$$\begin{aligned} d_j(x) &= \frac{\mathbf{1}_{I_j}(x)}{\alpha}[\widehat{F}(s(x)) - \widehat{F}(s(x)+h/\alpha)], \\ d_{10}(x,y) &= \frac{\mathbf{1}_{\mathcal{D}_{10}}}{\alpha}[\widehat{f}(q(x,y)+h/\alpha) - \widehat{f}(q(x,y))]. \end{aligned} \tag{5.34}$$

Thus $\Delta_j = \int d_j\, d(m^{e_j})$. Define the two conversion errors:

$$\Lambda_{j,N}^0 = S_{j,N}^0 - P_j, \qquad \Lambda_{j,N} = B_{j,N} - \Delta_j \qquad (j \in \mathcal{H}). \tag{5.35}$$

Lemma 5.4 (Finite profile shift)**.** *For the arrays above,*

$$S_{j,N}^h = P_j + \Lambda_{j,N}^0 + \Delta_j + \Lambda_{j,N} \geq P_j + \Delta_j - |\Lambda_{j,N}^0| - |\Lambda_{j,N}|. \tag{5.36}$$

Proof. Substitute the two definitions in (5.35). □

Both absolute conversion terms appear as separate summands of E'. We do not assume that Paper II's original envelope contains the saved-array error. Its stage 8 converts the actual-level array before stage 9 domination [27, §6.1, stage table]. The arithmetic application uses the preceding level domination.

Restoration. The four rows of $\mathcal{H}$ are Paper II's rows S_4, the lower part of S_5, S_7 and the retained part of S_{10}. Their stage-two source-restoration shares are assigned once to the source parents S_4, S_5, S_7 and S_{10}. The literal S_5 split at 7109/28000 is made before the structured sieve or Heath–Brown expansion, as above. Both parts retain the parent S_5 for this restoration accounting [27, (2.28), §6.1, stage table and the list of S10, S5 and S4/S7 endpoints]. There is no separate restoration for the lower part of S_5. The upper part of S_5 is the outer chain T5 of Lemma 5.1, which the lift does not change. Each tuple of an actual source is assigned to one entry by Paper II's first-failure rule [27, (6.4d) and the two sentences after it]. This section changes no source and no stage from one to eight, so the assignment is Paper II's. Appendix B.7 lists the finite correspondence between these rows and Paper II's error entries. It is an index of that assignment, and each entry occurs once.

Lemma 5.5 (Lifted direct comparison)**.** *Let $j \in \mathcal{H}$, and let $Q_{j,8}$ be Paper II's stage-eight quantity for this row. It is the continuum mass of the actual-level main term after the prime-measure and Euler conversion. There each actual cell carries its actual relative level ℓ_c. Put $Q_{j,9}^h = P_j + \Delta_j = \int k_j^h\, d(m^{e_j})$. Then $Q_{j,8} \geq Q_{j,9}^h$.*

Proof. Stages one to eight are Paper II's. Paper II passes from $Q_{j,8}$ to $Q_{j,9} = P_j$ by pointwise domination of the actual-level integrand by the specified profile. The cells and the continuum measure are the same [27, §6.1, stage table; §2.4, after (2.31); §2.5, after (2.40)]. We repeat that domination with the lifted profile. For each B-labelled parent contributing to S_4, S_7 or S_5-low, we first rederive the buffer used below. Write

$$m(t) = \min\left\{\frac{1+t}{2}, \frac{57}{64} - \frac{3\rho t}{2}\right\}, \qquad t_{\mathrm{dd}} = \frac{25}{32(1+3\rho)}.$$

Paper II gives $\Theta_c \geq m(t_1)$. The branch comparison is the exact identity

$$\frac{1+t_1}{2} - \left(\frac{57}{64} - \frac{3\rho t_1}{2}\right) = \frac{1+3\rho}{2}(t_1 - t_{\mathrm{dd}}).$$

Thus the decreasing branch is active exactly when $t_1 > t_{\mathrm{dd}}$. Equality is assigned to A, where both bounds agree. This active branch test is applied to every B-labelled parent. The branch is never inferred from the largest completed coordinate alone. Paper II keeps the inherited label through clipping. In S4 and S7 the B parents are separate intersections of their respective row domains with $t_1 > t_{\mathrm{dd}}$. No common parent interval is asserted. S5-low has $1/4 \leq r < t_* = 7109/28000$ and $t_1 \leq r < \rho t_1$, so $t_1 > 1/(4\rho) > t_{\mathrm{dd}}$ because $\rho < 8$. It is therefore entirely B. Let a parent in S4 or S7 be A-labelled. Then $t_1 \leq t_{\mathrm{dd}}$, and Paper II's first branch in (2.17f) [27, (2.17f)] gives $\Theta_c \geq (1+t_1)/2$. Take an actual exponent $r \in [t_1, \rho t_1)$. The S4 and S7 row domains are clipped by $r \geq \mu_*$ [27, §2.1, fixed-cutoff rows], and the retained bin gives $r \geq t_1$. The residual is therefore

$$2(t_1 - \mu_*) + \frac{5}{2}(r - t_1) = 2(r - \mu_*) + \frac{1}{2}(r - t_1) \geq 0.$$

$$\ell_c - \left(\frac{57}{64} - u - \frac{5r}{2}\right) \geq u - \eta_L - \frac{25}{64} + 2t_1 + \frac{5}{2}(r - t_1) = \frac{19}{10000} + 2(t_1 - \mu_*) + \frac{5}{2}(r - t_1) \geq \frac{19}{10000}.$$

Here $\mu_* = 25/128$ is the retained cutoff parameter; the symbol $\mu(n)$ elsewhere denotes the Möbius function. Thus the A pieces also have the required $19/10000 - 3\tau$ buffer. This is a row-local calculation. It does not merge the distinct S4 and S7 parent domains. For any B parent, Paper II's floor gives

$$\Theta_c \geq \frac{57}{64} - \frac{3}{2}\rho t_1, \qquad \ell_c - \left(\frac{57}{64} - u - \frac{5r}{2}\right) \geq (u - \eta_L) + \frac{5}{2}(r - t_1) - \frac{3}{2}\tau t_1 \geq \frac{19}{10000} - 3\tau. \tag{5.37}$$

The first bound is Paper II's linear branch of the full-anatomy floor [27, (2.17d)–(2.17f)]. Using $\ell_c = \Theta_c - \eta_L - t_1$, $r \geq t_1$, $u - \eta_L = 19/10000$, and $t_1 < t_* < 1$ proves the second line by direct subtraction. The B pieces of S4 and S7 use the same row-local calculation. Their parent intervals stay distinct.

For $j = 4, 5, 7$ the row coefficient is negative. At a point x of an actual cell, the actual sieve argument is ℓ_c/α and, by (5.23),

$$\frac{\ell_c}{\alpha} \geq s(x) + \frac{h}{\alpha} + \frac{10^{-4} - 3\tau}{\alpha} > s(x) + \frac{h}{\alpha}.$$

These are Paper II's margins $19/10000 - 3\tau$ for S_4, S_7 and the B branch of the lower part of S_5. Here h has moved into the profile [27, §2.4, display before "The S4/S7 A/B levels give the same buffer"]. Since F is decreasing, $-\widehat{F}(\ell_c/\alpha) \geq -\widehat{F}(s(x) + h/\alpha)$. For $j = 10$ the coefficient is positive, and on the retained cells

$$\frac{\ell_c}{\alpha} \geq q(x,y) + \frac{h}{\alpha} + \frac{10^{-4} - (5\lambda/2 + \beta)\tau}{\alpha} > q(x,y) + \frac{h}{\alpha},$$

which is Paper II's $\alpha(s_{\mathrm{live}} - s_{\mathrm{saved}}) \geq (u - \eta_L) - \tau((5/2)\lambda + \beta)$ with h moved into the profile [27, §2.5, proof after (2.40)]. Since f is nondecreasing, $\widehat{f}(\ell_c/\alpha) \geq \widehat{f}(q(x,y) + h/\alpha)$. Paper II uses the same monotonicity of F and f for these comparisons [27, §2.4]. For $j = 4, 5, 7$ the cells cover I_j, and integrating the pointwise inequality against m gives $Q_{j,8} \geq \int k_j^h \, dm = P_j + \Delta_j$.

For $j = 10$ the retained cells need not cover $\mathcal{D}_{10}$. Let $R_\tau \subset \mathcal{D}_{10}$ be their union, put $a_{10} = 5\lambda/2 + \beta$, and write $T_h(E) = \int_E k_{10}^h \, d(m^2)$. Then

$$Q_{10,8} = \frac{1}{\alpha} \sum_{c \text{ retained}} \int_c \widehat{f}(\ell_c/\alpha) \, d(m^2).$$

Strip. Let $(x,y) \in \mathcal{D}_{10} \setminus R_\tau$ lie in a cell with labels $t_1 \leq x$, $t_2 \leq y$. The cell is dropped, so $y \geq t_2 > \mathrm{sel}(t_1) \geq B_0(t_1) - a_{10}\tau \geq B_0(x) - a_{10}\tau$. This uses the two deficit bounds of Paper II, §2.5, and the fact that B_0 is nonincreasing [27, §2.5, proof of (2.40)]. Also $y < B_0(x)$. On $\mathcal{D}_{10}$ one has $11/5 \leq q$ and $q + h/\alpha < 4$, so $\widehat{f}(q + h/\alpha) \leq \widehat{f}(4) = \log 3/4$. With $y \geq \alpha$ and $\int_{\mu_*}^{\lambda} dx/x \leq (\lambda - \mu_*)/\mu_*$, this gives $T_h(\mathcal{D}_{10} \setminus R_\tau) \leq D_* \tau$, where D_* is the constant in (5.26).

Reserve. On a retained cell the margin in (5.23) satisfies $10^{-4} - a_{10}\tau > 1/20000$, so $\ell_c/\alpha \geq q + h/\alpha + \delta_{10}$. Here $2 < q + h/\alpha < q + h/\alpha + \delta_{10} < 4$. For $2 < s < 4$, $\widehat{f}(s) = \log(s-1)/s$ and

$$\widehat{f}'(s) = \frac{s/(s-1) - \log(s-1)}{s^2} > \frac{4/3 - 11/10}{16} = \frac{7}{480},$$

since $\log 3 < 11/10$. As f is nondecreasing, $\widehat{f}(\ell_c/\alpha) - \widehat{f}(q + h/\alpha) \geq 7\delta_{10}/480$ at every point of R_τ.

A fixed retained rectangle. Let $(x, y) \in R_{10}$, in a cell with labels $t_1 \in (x/\rho, x]$, $t_2 \in (y/\rho, y]$. Then $\mu_* < t_1 < \lambda$ and $\alpha < t_2 < \beta$. On $[\mu_*, 21/100]$ we have $B_0 = \beta$, because $57/64 - u - 11\alpha/5 - (5/2)(21/100) > \beta$. Hence $\mathrm{sel}(t_1) \geq \beta - a_{10}\tau > 1/10 \geq t_2$, and the cell is retained: $R_{10} \subset R_\tau$. Also $m^2(R_{10}) = \log \frac{21}{20} \log \frac{10}{9} \geq \frac{1}{21} \cdot \frac{1}{10} = \frac{1}{210}$.

Combining the three steps with the pointwise inequality above,

$$Q_{10,8} \geq T_h(R_\tau) + c_{10} \geq T_h(\mathcal{D}_{10}) - D_*\tau + c_{10} > T_h(\mathcal{D}_{10}) = P_{10} + \Delta_{10},$$

by (5.27). No deduction is made. The reserve is the part of Paper II's level margin left after the lift h. It exists on cells of S_{10} only and is used only here. □

The finite arrays $S^0_{j,N}, S^h_{j,N}$ are not used in this inequality. The errors $|\Lambda^0_{j,N}|$ and $|\Lambda_{j,N}|$ are nonnegative and are kept in E' so that E' agrees with (1.12). They tend to zero by Lemma 5.8.

5.4. **The internal joint shift.** The joint occurrence is $2S_{11} + S_{12} + S_{13}$. Its six-branch function is Paper II, (4.70) [27, (4.70)–(4.72)]:

$$\begin{aligned} v_1(t) &= \min\left\{\frac{1+t}{2}, \frac{32-34t}{39}, \frac{57}{64} - \frac{3t}{2}\right\}, \\ v(t) &= v_1(t) - u, \qquad \psi(x,y) = x\mathbf{1}_{x\geq y}, \\ \Theta_*(a,b,c) &= \max\{v(a), v(b), v(a+b), v(a+b+c), \\ &\qquad \psi(v(a+c), a+2b+c), \psi(v(b+c), 2a+b+c)\}. \end{aligned} \tag{5.38}$$

Subtract u before testing each guard. Set

$$\mathcal{C} = \{(t_1,t_2,t_3) : u \leq t_3 < t_2 < t_1 < \mu_*\}, \qquad a_3(\mathbf{t}) = \frac{\Theta_*(\mathbf{t}) - t_1 - t_2 - t_3}{t_3}. \tag{5.39}$$

Shift the output of Θ_* by h. Keep its guards and actual source unchanged:

$$\begin{aligned} a_3^s(\mathbf{t}) &= a_3(\mathbf{t}) + \frac{h}{t_3}, \\ J &= \iiint_{\mathcal{C}} \frac{1 - f(a_3(\mathbf{t}))}{t_1 t_2 t_3^2}\, d\mathbf{t}, \\ J_s &= \iiint_{\mathcal{C}} \frac{1 - f(a_3^s(\mathbf{t}))}{t_1 t_2 t_3^2}\, d\mathbf{t}. \end{aligned} \tag{5.40}$$

Use $f(s) = 0$ for $s < 2$. Keep its true values at every larger argument. The new margins are

$$g_s = \frac{1}{10000} - \frac{225}{256}\tau > 2\mu_*\tau, \qquad g_{s,\mu_*} = \frac{1}{10000} - \frac{75}{256}\tau > 0. \tag{5.41}$$

The second margin concerns a parent interval crossing μ_*. Both follow by subtracting h in Paper II, (4.72); see Lemma 5.6. They hold throughout $0 < \tau \leq 10^{-90}$. If $f(a_3^s) > 0$, the retained-cell condition follows:

$$\ell_c \geq \Theta_* + h - t_1 - t_2 - t_3 + g_s > 2t_3 + 2\mu_*\tau \geq 2\rho t_3. \tag{5.42}$$

Thus the canonical restriction still loses zero. At fixed numerator, use the increase of $sf(s)$. This is the same comparison as Paper II, (4.72).

Lemma 5.6 (Shifted retention)**.** *The margins* (5.41) *hold for* $0 < \tau \leq 10^{-90}$. *On the original joint source, every positive shifted lower contribution lies in an original retained whole cell of Paper II, (4.19). On such a cell the actual lower main term dominates the shifted canonical one.*

Proof. Use an original G1-3 cell with inherited labels $t_1 \geq t_2 \geq t_3 > 0$, and order its distinct actual prefix primes as $a_1 > a_2 > a_3$. Put $x_i = \log_N a_i$. Unequal parent bins enforce the actual order, while within an equal-label group we order the primes. Hence $t_i \leq x_i < \rho t_i$, including clipped bins. No strict inequality between the labels is required.

Let Θ_c be the original full-anatomy infimum and $\ell_c = \Theta_c - \eta_L - \sum_i t_i$. Put $B_s(x) = \Theta_*(x) + h - \sum_i x_i$. The function v_1 is 3/2-Lipschitz. On the closed ordered parent a coordinate changes by at most $\mu_*\tau$. A sum of at most three coordinates changes by at most $3\mu_*\tau$. If a saved guarded branch is active at x, its buffered guard has nonnegative gap, including equality. Removing its buffer adds u. Throughout the parent its unbuffered gap is therefore at least $u - (9/2)\mu_*\tau - 4\mu_*\tau > 0$. Thus it stays available in the full anatomy. Take an unguarded or active guarded saved branch. The corresponding full branch is at least its value at x plus $u - (9/2)\mu_*\tau$. Taking their maximum and then the parent infimum preserves this lower bound. Ties between maximizing branches cause no loss. Therefore

$$\ell_c \geq B_s(x) + g_s + \sum_i (x_i - t_i), \qquad g_s = u - \eta_L - h - (9/2)\mu_*\tau = 10^{-4} - (225/256)\tau.$$

For a parent crossing μ_*, the largest exponent is within $\mu_*\tau$ of μ_*, and $v_1(\mu_*) = 153/256$. Every saved branch is at most $153/256 - u$, so the crossing comparison gives instead $10^{-4} - (75/256)\tau \geq g_s$. Clipping only shrinks the

infimum domain. These arguments also apply when two or all three labels coincide. For $0 < \tau \le 10^{-90}$ we have $g_s > 2\mu_*\tau > 0$.

If $f(B_s(x)/x_3) > 0$, then $B_s(x) > 2x_3$ and hence

$$\ell_c > 2x_3 + 2\mu_*\tau \ge 2t_3 + 2\tau t_3 = 2\rho t_3.$$

Thus the whole original cell is retained by Paper II, (4.19). On positive support $\ell_c > B_s(x) > 0$ and $0 < t_3 \le x_3$ imply

$$\frac{\ell_c}{t_3} \ge \frac{B_s(x)}{x_3}, \qquad f(\ell_c/t_3) \ge f(B_s(x)/x_3), \qquad \frac{f(\ell_c/t_3)}{t_3} \ge \frac{f(B_s(x)/x_3)}{x_3}.$$

These use only nonnegativity and monotonicity of f. Equal labels invalidate none of the weak inequalities. Also $E = N^{t_3} \le a_3$, so $V_N(E) \ge V_N(a_3)$. At zero shifted contribution the required comparison follows from nonnegativity without any assertion that $B_s(x) > 0$.

Keep the original orbit normalization. If the equal-label groups are G_h, put $M_c = \prod_h |G_h|! \in \{1, 2, 6\}$. For a full labelled realization $\mathbf{a}$, write $\mathbf{a}^{\downarrow}$ for its decreasing rearrangement within the equal-label groups. The shifted comparison above is evaluated at $\mathbf{x}^{\downarrow}$ and is invariant under these group permutations. For every nonnegative, invariant main-mass weight w_c, therefore,

$$\frac{1}{M_c}\sum_{\text{full}} K_P(\mathbf{a})w_c(\mathbf{a})V_N(E)f(\ell_c/t_3) \ge \frac{1}{M_c}\sum_{\text{full}} K_P(\mathbf{a})w_c(\mathbf{a})V_N(a_3^{\downarrow})f(B_s(\mathbf{x}^{\downarrow})/x_3^{\downarrow}).$$

Each distinct ordered triple has exactly M_c labelled realizations. The prefix mask K_P and its correction are kept exactly as in Paper II, (4.22)–(4.23). No repeated-prefix realization is reclassified. The tail mask and all remainder coefficients are likewise unchanged. This proves the finite lower-main-term comparison with the cell mass divided by M_c. The corresponding continuum comparison uses the preceding inequality with the $1/t_3$ and $1/x_3$ factors. Equal-label bins are kept. Their strictly ordered actual triples lie in the original cone. The orbit identity converts the full-block sum to that ordered sum exactly. Only actual coordinate diagonals have zero continuum measure. The proof imposes no $N \to \infty$ disappearance of equal-bin regions. □

Put $H_{\log} = \log(\mu_*/u)$. Paper II's remaining terms satisfy

$$\begin{aligned} 0 \le R_G \le R_{\text{up}} &= 500(F(30) - 1) + 500H_{\log}(1 - f(30)) \\ &\quad + 250H_{\log}^2(F(30) - 1). \end{aligned} \tag{5.43}$$

Its constant terms cancel exactly [27, (4.73)–(4.75)]:

$$500\left(1 - H_{\log} + \frac{H_{\log}^2}{2}\right) - \iiint_{\mathcal{C}} \frac{d\mathbf{t}}{t_1t_2t_3^2} = \frac{1}{\mu_*}. \tag{5.44}$$

The shifted upper comparison is therefore

$$G_1^s = \frac{1}{\mu_*} + R_G + J_s, \qquad \Phi_{G_1}^s = -K_G G_1^s, \qquad K_G = \frac{2I_{11} + I_{12} + I_{13}}{2e^{\gamma}}. \tag{5.45}$$

The source-mass integrals are [27, (4.77)]

$$\begin{aligned} I_{11} &= \int_{\lambda}^{1/3} \frac{\log((1-2t)/t)}{t(1-t)}\,dt \\ &\quad + \int_{\lambda}^{1/4} \frac{dt}{t}\int_{t}^{(1-t)/3} \frac{\log((1-t-2v)/v)}{v(1-t-v)}\,dv, \\ I_{12} &= \int_{\alpha}^{1/3} \frac{\log(2-3t)}{t(1-t)}\,dt, \\ I_{13} &= \int_{\beta}^{\lambda} \frac{\log((1-t-\lambda)/\lambda)}{t(1-t)}\,dt. \end{aligned} \tag{5.46}$$

We have $I_{11}, I_{12}, I_{13} \ge 0$ and $I_{12} > 0$. Thus $K_G > 0$. The finite shifted enclosure gives

$$0 \le G_1^s \le \frac{6073714261}{10^9} \quad (\text{shifted mesh } n_{\text{mesh}} = 512). \tag{5.47}$$

Paper II's mass enclosure gives $0 \le K_G \le K_{\text{hi}}$ [27, (A.10d)]. The exact constants are

$$K_{\text{hi}} = \frac{39894807910705784663904430098309337537484429474781101122 23}{10^{58}}. \tag{5.48}$$

Paper II, (4.84), gives

$$L_{G1} = -\frac{24395042826519465625420692193994776286887136187440955209 5097}{10043362776618689222137263077132266265763768711142455220 6336}. \tag{5.49}$$

Define

$$M_{5,\mathrm{internal}} = -K_{\mathrm{hi}}\frac{6073714261}{10^9} - L_{G1}. \tag{5.50}$$

Their exact substitution gives

$$\begin{aligned} &0.0058749499450263575220 < M_{5,\mathrm{internal}},\\ &M_{5,\mathrm{internal}} < 0.0058749499450263575221. \end{aligned} \tag{5.51}$$

The finite enclosure and identity are proved. They concern the shifted profile integral. The required actual-source inequality is

$$-\frac{2S_{11}+S_{12}+S_{13}}{W} \ge L_{G1} + M_{5,\mathrm{internal}} - E^D_{G1} - |\Lambda_{G,N}|. \tag{5.52}$$

Proof of (5.52). Paper II proves

$$\frac{2S_{11}+S_{12}+S_{13}}{W} \le K_G G_1 + E^D_{G1}, \qquad G_1 = \frac{1}{\mu_*} + R_G + J,$$

from (4.73)–(4.77) [27, (4.75), (4.78)]. Run the same argument with J replaced by J_s. The first three residual rows and the bound $R_G \le R_{\mathrm{up}}$ do not change, and neither does the cancellation (5.44). In the fourth row, Lemma 5.6 replaces Paper II's comparison after (4.72). The canonical restriction loses zero, and the actual lower main term dominates the shifted one. The source and its weights $(2,1,1)$ do not change, and neither do the peel (4.45)–(4.57) and the mass conversion (4.76)–(4.78). Hence

$$\frac{2S_{11}+S_{12}+S_{13}}{W} \le K_G G_1^s + E^D_{G1}$$

with the same E^D_{G1}. By Proposition B.2, $0 \le G_1^s \le 6073714261/10^9$, which is (5.47), with the residual bound supplied by (5.43). By Paper II, (A.10d), $0 \le K_G \le K_{\mathrm{hi}}$. Both factors are nonnegative, so by (5.50)

$$-K_G G_1^s \ge -K_{\mathrm{hi}}\frac{6073714261}{10^9} = L_{G1} + M_{5,\mathrm{internal}}.$$

This gives (5.52) even without $|\Lambda_{G,N}|$. That term is nonnegative and is kept so that the error agrees with (1.12). □

The existing E^D_{G1} is a disjoint component of the original $F_{\mathrm{net}} + E_{\mathrm{van}}$ allowance. Its fixed peel 145664566.4567τ is covered by g_τ, and its fixed $C_E\sigma$ share by g_{AA}. Its vanishing part belongs to E_{van}. When replacing the joint chain, carry these same components once. The reconciliation in (5.10) remains paid. These components are the G_1 comparator term NC_{sf} of Paper II, (4.64)–(4.65), and the G_1 share of the joint bound. That bound treats Ξ_U and Ξ_A in Paper II, (B.1)–(B.7). Both are attached to the original source measure and not to the profile. Hence the shift neither creates nor removes them.

5.5. **Conversion of the finite arrays.** We give the remaining definitions before stating the lemma. All representation variables in the following table are prime. Their product is e, with $\varepsilon N < e < N$. Let $R_{11,3}(e)$ and $R_{11,4}(e)$ count the first two forms. Let $R_{12}(e)$ and $R_{13}(e)$ count the last two forms. Put $R_{11} = R_{11,3} + R_{11,4}$. These are Paper II's representation weights [27, (4.4), source table preceding (4.4)].

TABLE 5.1. The internal representation counts

Count	Product	Restrictions
$R_{11,3}$	$p_1p_2p_3$	$N^\lambda \le p_1 < N^{1/3}$, $p_1 < p_2$, $p_1p_2^2 < N$, $p_3 \ge p_2$.
$R_{11,4}$	$p_1p_2p_3p_4$	The same conditions on p_1, p_2; $p_2 \le p_3 \le p_4$.
R_{12}	$p_1p_2p_3$	$N^\alpha \le p_1 < N^{1/3} \le p_2$, $p_1p_2^2 < N$, $p_3 \ge p_2$.
R_{13}	$p_1p_2p_3$	$N^\beta \le p_1 < N^\lambda \le p_2$, $p_1p_2^2 < N$, $p_1p_2p_3^2 \ge N$.

For the last form the endpoint orders are distinct. Every permitted representation keeps its multiplicity. Define the upper cut and masses by

$$\begin{aligned} N_{\mathrm{cut}} &= N(1 - L^{-333}),\\ X_j^{\mathrm{cut}} &= \sum_e R_j(e)\mathbf{1}_{\varepsilon N < e \le N_{\mathrm{cut}}} \qquad (j = 11, 12, 13),\\ X_G &= 2X_{11}^{\mathrm{cut}} + X_{12}^{\mathrm{cut}} + X_{13}^{\mathrm{cut}},\\ r_{G,N}(t) &= \frac{tX_G V_N(N^t)}{WK_G}. \end{aligned} \tag{5.53}$$

The cut precedes the sieve decomposition [27, (4.16)–(4.17)]. The source-mass input used below is

$$\begin{aligned} X_D &= (1-\varepsilon+o(1))\,\frac{N}{L}, \\ X_G &= ((1-\varepsilon)(2I_{11}+I_{12}+I_{13})+o(1))\,\frac{N}{L}. \end{aligned} \tag{5.54}$$

Lemma 5.7 (Source masses)**.** *The formulas* (5.54) *hold.*

Proof. The sum X_D counts the primes in $(N/L^{30},(1-\varepsilon)N)$, so the prime number theorem gives $X_D=(1-\varepsilon+o(1))N/L$. For the representation weights of Table 5.1, Paper II proves $X_j=((1-\varepsilon)I_j+o(1))N/L$ for the uncut masses, $j=11,12,13$ [27, (4.76)–(4.77)]. The cut strip $N_{\mathrm{cut}}<e<N$ carries joint mass at most $100(NL^{-333}+1)$ [27, (4.16)–(4.17)], which is $o(N/L)$. Summing with the weights $(2,1,1)$ gives the formula for X_G. □

On X_0^3, define the zero-extended kernel

$$d_G(\mathbf{t}) = \mathbf{1}_{\mathcal{C}}(\mathbf{t})K_G\frac{f(a_3^s(\mathbf{t}))-f(a_3(\mathbf{t}))}{t_3}. \tag{5.55}$$

Its arrays and integral are

$$\begin{aligned} B_{G,N} &= \iiint r_{G,N}(t_3)d_G(\mathbf{t})\,d(m_N^3)(\mathbf{t}), \\ \Delta_G &= \iiint d_G\,d(m^3) = K_G(J-J_s), \\ \Lambda_{G,N} &= B_{G,N}-\Delta_G. \end{aligned} \tag{5.56}$$

Equivalently,

$$\begin{aligned} B_{G,N} = \frac{X_G}{W} \sum_{\substack{p_3<p_2<p_1,\ p_i\in\mathbb{P},\ p_i\nmid N \\ u\le\log_N p_3<\log_N p_2<\log_N p_1<\mu_*}} & \frac{V_N(p_3)[f(a_3^s(\mathbf{t}))-f(a_3(\mathbf{t}))]}{(p_1-1)(p_2-1)(p_3-1)}, \\ & t_i=\log_N p_i. \end{aligned} \tag{5.57}$$

These definitions specify all five additional conversion errors.

Lemma 5.8 (Conversion of the five profile differences)**.** *Fix $0<\sigma\le 10^{-10}$. For m_N,m in* (5.28)*, the finite arrays* (5.33) *and* (5.56) *satisfy*

$$\Lambda_{j,N}\longrightarrow 0 \quad (j\in\mathcal{H}), \qquad \Lambda_{G,N}\longrightarrow 0, \tag{5.58}$$

and $\Lambda_{j,N}^0\to 0$ for $j\in\mathcal{H}$. Every limit ranges over even N tending to infinity; the thresholds depend on σ.

Proof. Step 1: convergence of the harmonic measures. Let $u\le a<b\le 1$. Mertens' theorem and partial summation give

$$m_N([a,b)) = \log\frac{b}{a}+o(1). \tag{5.59}$$

Replacing $1/(p-1)$ by $1/p$ costs $O(N^{-u})$. There are at most $L/\log 2$ prime divisors of N. Each omitted divisor above N^u has mass $O(N^{-u})$. Their total cost is $O(N^{-u}L)$. The largest endpoint atom is $O(N^{-u})$. The total masses therefore satisfy

$$c_N := m_N(X_0)\longrightarrow c_0 := m(X_0)=\log(1/u)>0. \tag{5.60}$$

Let g be continuous on X_0. Choose a finite interval partition and a step function on it, uniformly within a prescribed error of g. The integrals of the step function converge by (5.59). The integral errors are at most the uniform error times c_N+c_0, which is bounded by (5.60). Hence $m_N\Rightarrow m$ [2, Theorem 2.2 and Example 2.3]. For $k=2,3$, rectangle masses are products of interval masses. The same argument with rectangular step functions gives

$$m_N^k\Rightarrow m^k \qquad (k=1,2,3). \tag{5.61}$$

For a coordinate diagonal $D_{ij}=\{t_i=t_j\}$,

$$m_N^k(D_{ij})\le\left(\max_{p\ge N^u}\frac{1}{p-1}\right)c_N^{k-1}=O(N^{-u}) \qquad (k\ge 2). \tag{5.62}$$

Thus strict prime order costs nothing in the limit, and no orbit factor enters.

Step 2: the source and Euler normalizations. For $p>2$,

$$1-\frac{1}{p-1}=\left(1-\frac{1}{p}\right)\left(1-\frac{1}{(p-1)^2}\right). \tag{5.63}$$

Mertens' product formula and the convergent second product yield

$$V_N(N^t) = \frac{2e^{-\gamma}C(N)}{tL}(1+o(1)), \qquad u/\rho \le t \le 1, \tag{5.64}$$

uniformly in t. The prime 2 is omitted because N is even. The missing divisors $p \mid N$ above N^t contribute $\prod(1+1/(p-2)) = 1+O(N^{-u/\rho}L)$. The ordinary product error is uniform for $N^t \ge N^{u/\rho}$. By Lemma 5.7 and (2.2),

$$r_{D,N} \longrightarrow 1, \qquad \sup_{u\le t\le \mu_*} |r_{G,N}(t)-1| \longrightarrow 0. \tag{5.65}$$

For example, $WK_G = 2e^{-\gamma}(1-\varepsilon)(2I_{11}+I_{12}+I_{13})C(N)N/L^2$. This is exactly the main factor in $tX_GV_N(N^t)$. Thus both the truncation factor and the representation multiplicities cancel once.

Step 3: restriction and bounded kernels. Write $\widehat{m}_N = m_N/c_N$ and $\widehat{m} = m/c_0$. These are probability measures, and their products converge weakly. Let D have $m^k(\partial D) = 0$, and let $g\mathbf{1}_D$ be bounded and m^k-almost everywhere continuous. Apply the mapping form of the portmanteau theorem [2, Theorems 2.1 and 2.7] to $g\mathbf{1}_D$ and the probability measures. Then multiply by $c_N^k \to c_0^k$. This gives

$$\int_D g\, d(m_N^k) \longrightarrow \int_D g\, d(m^k). \tag{5.66}$$

The intervals I_j have finitely many boundary points. The boundary of $\mathcal{D}_{10}$ is piecewise affine, since B_0 is continuous and piecewise affine. Both have zero m^{e_j}-measure. On I_j $(j=4,5,7)$ the argument $s(x)$ lies in a compact subinterval of $(1,\infty)$, where F is continuous. On $\mathcal{D}_{10}$, $y < K - 11\alpha/5 - 5x/2$ gives $q(x,y) > 11/5$, and $q + h/\alpha < 4$; there f is continuous. Hence k_j^0, k_j^h and d_j are bounded and continuous off a null set, so (5.66) applies to them. With (5.65), it gives

$$S_{j,N}^0 \longrightarrow P_j, \qquad B_{j,N} \longrightarrow \Delta_j \qquad (j \in \mathcal{H}). \tag{5.67}$$

Step 4: the ordered three-dimensional kernel. The cone $\mathcal{C}$ has affine boundary faces, so its boundary is m^3-null. The function Θ_* is piecewise affine. Its possible jumps lie on finitely many guard hyperplanes, and its branch switches add finitely many affine faces. On every remaining open piece, a_3 and a_3^s are continuous. The denominator satisfies $t_3 \ge u > 0$, so the arguments of f remain in a fixed compact interval. On it f, extended by zero below 2, is continuous. Since $0 \le f \le 1$, $|d_G| \le K_G/u$. Thus d_G is bounded and m^3-almost everywhere continuous. Apply (5.66) to the zero extension on X_0^3:

$$\iiint d_G\, d(m_N^3) \longrightarrow \iiint d_G\, d(m^3) = \Delta_G. \tag{5.68}$$

The normalization costs at most

$$\left| B_{G,N} - \iiint d_G\, d(m_N^3) \right| \le \sup_{[u,\mu_*]} |r_{G,N}-1|\, \|d_G\|_\infty c_N^3 \longrightarrow 0. \tag{5.69}$$

The cone's coordinate-order inequalities are strict, so $B_{G,N}$ counts strictly ordered triples, as in (5.57). Hence $B_{G,N} \to \Delta_G$, and with Step 3 all limits in (5.58) and $\Lambda_{j,N}^0 \to 0$ follow. □

Whole cells, retained cells and the label cutoff. For the four direct rows, the arithmetic identification is Lemma 5.5. For the joint row it is the proof of (5.52). Both are made at stage eight on Paper II's actual cells, with their inherited signed coefficients. Hence no further whole-cell kernel is needed. For the retained part of S_{10}, the actual whole cells cover $\mathcal{D}_{10}$ up to a strip. Lemma 5.5 absorbs this strip by the retained reserve (5.27). For G_1, the retained cells are those of Paper II, (4.19). By Lemma 5.6, the shifted profile has no positive support outside them. The arrays (5.33) and (5.56) enter only through the nonnegative errors in E', which vanish by Lemma 5.8.

The array (5.57) carries $V_N(p_3)$ at the actual coordinate. The arithmetic lower main term carries $V_N(E)$, $E = N^{t_3}$, at the inherited label. Since $E \le a_3$, $V_N(E) \ge V_N(a_3)$. In the lower part of the joint upper bound this replacement is favourable. We nevertheless bound it absolutely and include it once in Paper II's peel allowance (4.57). No fixed cost is added. By (5.64), uniformly for actual label/coordinate pairs in the same cell with $u/\rho \le t_3 \le x_3 \le 1$ and $x_3 < \rho t_3$,

$$\frac{V_N(N^{t_3})}{V_N(N^{x_3})} = \frac{x_3}{t_3}(1+o(1)) \le \rho(1+o(1)).$$

For a weight $0 \le w \le 1$ on the strictly ordered cone, the replacement therefore costs at most

$$C_{\mathrm{cut},N} \le (\tau + o_\sigma(1))T_N, \qquad T_N = \frac{X_G}{W}\sum_{\mathcal{C}} \frac{V_N(p_3)}{(p_1-1)(p_2-1)(p_3-1)} \longrightarrow K_G I_0,$$
$$I_0 = \iiint_{\mathcal{C}} \frac{d\mathbf{t}}{t_1 t_2 t_3^2} = 500\Big(1 - H_{\log} + \frac{H_{\log}^2}{2}\Big) - \frac{1}{\mu_*}, \tag{5.70}$$

where the sum runs over the primes of (5.57). The limit follows from Steps 2–4 of the proof of Lemma 5.8 applied to the bounded kernel $\mathbf{1}_{\mathcal{C}}K_G/t_3$. We evaluate I_0 by (5.44). Since $1 < H_{\log} < 459/100$ and the quadratic increases for $H_{\log} > 1$,

$$I_0 < \frac{693381}{200}, \qquad K_G I_0 \le K_{\mathrm{hi}} \frac{693381}{200} < 1400.$$

At fixed τ, eventually $o_\sigma(1) \le \tau$ and $T_N \le 1400$, so $C_{\mathrm{cut},N} \le 2800\tau < 4000\tau$. We include 4000τ once. Paper II's peel bound before its final rounding is

$$P_{\mathrm{peel}}\tau = \frac{3 \cdot 502002\,(459/100)^3}{1-\varepsilon}\tau = \frac{8090814677193}{55550}\tau$$

[27, (4.47), (4.56)], and its printed allowance is $P_{\mathrm{cap}}\tau = (485500000000/3333)\tau$ [27, (4.57)]. The per-layer coefficient 502002 in Paper II, (4.53), includes the Selberg remainder. The printed allowance (4.57) uses that coefficient; we therefore use its slack with the Selberg remainder already included. Exact subtraction gives

$$P_{\mathrm{cap}} - P_{\mathrm{peel}} = \frac{2555968421}{166650}, \qquad P_{\mathrm{cap}} - P_{\mathrm{peel}} - 4000 = \frac{1889368421}{166650} > 0. \tag{5.71}$$

Hence the peel and the label replacement together stay below $P_{\mathrm{cap}}\tau < 145664566.4567\tau$, the entry already contained in g_τ. The fixed deduction F_{net} is unchanged.

5.6. **The resulting lower comparison.**

Remark 5.9 (Inputs). The proof of Proposition 5.10 uses the cited statements of Paper I and Paper II. They are applied to Paper II's unchanged sources. It uses Paper I, Theorem 2.8, through Fact 3.22, and the prime Siegel–Walfisz theorem, Paper I, (2.21). It uses Lemma 6.5(ii) at Paper I, (3.48), and the finite enclosures of Propositions B.1 and B.2. The optional endpoint improvement also uses Lemma 5.3. The table of Appendix B.7 is an index, not an input.

Proposition 5.10 (Retained comparison)**.** *Fix $0 < \sigma \le 10^{-10}$ satisfying Paper II, (1.27) and assume Lemma 5.3. Then* (5.2) *holds, and $E'(N;\sigma) \to 0$ as $N \to \infty$. Without that lemma,* (5.2) *holds with g_6 replaced by 0; the same limit for $E'(N;\sigma)$ holds.*

Proof. Write each chain of $\mathcal{I}$, including T5, by (5.4) and each joint group by (5.5). Lemma 5.2 covers the differences at stages one to eight and the remaining Paper II differences at stages nine and ten. Their sum is at least $-(F_{\mathrm{net}} - \delta_{G1}) - E_{\mathrm{van}}$. For $j \in \mathcal{H}$, replace the stage-nine object by $Q^h_{j,9} = P_j + \Delta_j$. Then Lemma 5.5 gives $Q_{j,8} - Q^h_{j,9} \ge 0$. For $j \in \mathcal{J}$, Lemma 5.3 gives the stage-ten difference $P_j - N_j \ge 0$, so the endpoint rises by $N_j - O_j$. Only the endpoint changes, so g_6 enters once. By Proposition B.1, $\sum_{j\in\mathcal{H}} \Delta_j > 572004/10^8 > 5719/10^6$, which is (5.25). For the joint occurrence, (5.52) replaces Paper II's G_1 bound. Its E^D_{G1} is the G_1 part of $F_{\mathrm{net}} + E_{\mathrm{van}}$ and is counted once. These modifications concern disjoint profile terms, so their gains enter once. The endpoint sum (5.6) and the floor (5.13) give $D_{\mathrm{II}} + \overline{J}_6 + 2\overline{H}$. Hence

$$\begin{aligned} \frac{R_{\mathrm{fix}} + S_2}{W} \ge{}& D_{\mathrm{II}} + g_6 + \overline{J}_6 + 2\overline{H} + \frac{5719}{10^6} + M_{5,\mathrm{internal}} \\ &- E_{\mathrm{van}}(N;\sigma) - \sum_{j\in\mathcal{H}} \big(|\Lambda^0_{j,N}| + |\Lambda_{j,N}|\big) - |\Lambda_{G,N}|. \end{aligned} \tag{5.72}$$

The nine conversion errors are nonnegative, so (5.72) is weaker than the same bound without them. This is (5.2). By Lemmas 5.2 and 5.8, $E'(N;\sigma) \to 0$; the threshold depends on the fixed σ. No fixed increment is added. The only new fixed cost is the label replacement, and by (5.71) it lies inside Paper II, (4.57). Without Lemma 5.3, take $N_j = O_j$; then the same argument gives (5.72) with g_6 replaced by 0. □

6. The sixth sum at $N/4$

Put $L = \log N$ and fix $0 < \sigma \le 10^{-10}$ for the common error allocation. No estimate in the sixth-sum construction depends on σ. Fix

$$\begin{gathered} \alpha = \frac{100}{1149}, \qquad t_0 = \frac{57}{224}, \qquad v_{\mathrm{RA}} = \frac{5227}{10000}, \qquad \eta_{\mathrm{m}} = \varepsilon = \frac{1}{10000}, \\ J = [t_0, 1/3), \quad J_t = [t_0, 1/3 - \eta_{\mathrm{m}}), \quad J_{\mathrm{tail}} = [1/3 - \eta_{\mathrm{m}}, 1/3), \\ I_L = [2, N/4], \qquad I_H = (N/4, (1-\varepsilon)N), \qquad z_s = N^\alpha. \end{gathered} \tag{6.1}$$

Write $p \in N^K$ for $p \in \mathbb{P}$ with $\log p/L \in K$. Paper II, (1.7), gives the original sum [27, (1.7)]:

$$S_6 = \sum_{\substack{p\in N^J \\ (p,N)=1}} \sum_{\substack{m\ge 1,\ e=N-pm\in\mathbb{P} \\ 2\le e<(1-\varepsilon)N}} \mathbf{1}_{P^-(m)\ge z_s} \mathbf{1}_{(m,Np)=1}. \tag{6.2}$$

Each prime mark remains a separate occurrence.

6.1. **Three upper weights.** For squarefree $d = \ell_1 \cdots \ell_k$, order its primes decreasingly. The ordinary upper Rosser weight is

$$\mu_d^+(D) = \mu(d) \prod_{\substack{1 \le j \le k \\ j \text{ odd}}} \mathbf{1}_{\ell_1 \cdots \ell_{j-1} \ell_j^3 < D}, \qquad \mu_1^+(D) = 1. \tag{6.3}$$

The weight is zero outside its stated squarefree support. Its divisor sum is a nonnegative upper sieve weight [14, Lemma 1, (3.2), (3.6)] [13, Lemmas 2–3, p. 313].

For $K \subseteq J$ and $p \in N^K$, define

$$c_{R,K}(pd) = \mu(d) \mathbf{1}_{pd<R} \prod_{\substack{1 \le j \le k \\ j \text{ odd}}} \mathbf{1}_{p\ell_1 \cdots \ell_{j-1} \ell_j^3 < R}. \tag{6.4}$$

Require $\ell_j < z_s$ and $(pd, 2N) = 1$. Set the coefficient to zero otherwise. Since $t_0 > \alpha$, the mark is unique. Thus $|c_{R,K}(q)| \le 1$. On supported moduli with $p < R$, the gates give $d < R/p$. For $d = 1$ the gate product is empty, and $p < R$ suffices. Every marked prime has $p < N^{1/3}$. Each sieve level below is $> N^{1/3}$ for large N. Hence

$$c_{R,K}(pd) = \mu_d^+(R/p), \qquad \varphi(pd) = (p-1)\varphi(d). \tag{6.5}$$

The equality is claimed on supported moduli only. Off that support the coefficient is zero.

Choose fixed logarithmic exponents. Put

$$\begin{aligned} R_L &= N^{1/2} L^{-B_L}, & R_{\text{tail}} &= N^{1/2} L^{-B_{\text{tail}}}, \\ Q &= N^{v_{\text{RA}}} L^{-B}, & Q_0 &= N^{1/2} L^{-B_0}, \\ b_{\text{all}} &= c_{R_L,J}, & b_{\text{tail}} &= c_{R_{\text{tail}},J_{\text{tail}}}, \\ c_{\text{trim}} &= c_{Q,J_t}, & c_{\text{lo}}(q) &= c_{\text{trim}}(q) \mathbf{1}_{q \le Q_0}, \\ \Gamma(q) &= c_{\text{trim}}(q) \mathbf{1}_{q > Q_0}. \end{aligned} \tag{6.6}$$

The last two arrays keep the gates at Q.

First omit $(m, Np) = 1$ in (6.2). Sieve the larger sequence by primes $\ell \nmid 2N$, $\ell < z_s$. Use the three weights on

$$I_L \times J, \qquad I_H \times J_{\text{tail}}, \qquad I_H \times J_t. \tag{6.7}$$

Their divisor sum is the finite weighted count

$$\begin{aligned} A_{\text{mix}} = &\sum_q b_{\text{all}}(q) \sum_{\substack{e \in I_L \cap \mathbb{P} \\ e \equiv N \ (q)}} 1 \\ &+ \sum_q \{b_{\text{tail}}(q) + c_{\text{trim}}(q)\} \sum_{\substack{e \in I_H \cap \mathbb{P} \\ e \equiv N \ (q)}} 1. \end{aligned} \tag{6.8}$$

The upper-weight comparison

$$S_6 \le A_{\text{mix}}, \qquad Z_6 := A_{\text{mix}} - S_6 \ge 0, \tag{6.9}$$

is proved in Lemma 6.11(a) below. Positivity is used before expanding the divisor sums.

6.2. **The exact discrepancy identity.** For a bounded interval $I \subset [2, \infty)$ and a finitely supported modulus array $a(q)$, put $\text{Li}(I) = \int_I dt/\log t$ and define

$$\begin{aligned} \Pi_I(q) &= \sum_{e \in I \cap \mathbb{P}} \mathbf{1}_{e \equiv N \pmod q}, \\ B_I(q) &= \frac{1}{\varphi(q)} \sum_{e \in I \cap \mathbb{P}} \mathbf{1}_{(e,q)=1}, \\ D_I(a) &= \sum_q a(q) \left\{ \Pi_I(q) - \frac{\text{Li}(I)}{\varphi(q)} \right\}, \\ P_I(a) &= \sum_q a(q) \{\Pi_I(q) - B_I(q)\}, \qquad C_I(a) = D_I(a) - P_I(a). \end{aligned} \tag{6.10}$$

Also put

$$\begin{aligned} M_{\text{mix}} = &\ \text{Li}(I_L) \sum_q \frac{b_{\text{all}}(q)}{\varphi(q)} \\ &+ \text{Li}(I_H) \sum_q \frac{b_{\text{tail}}(q) + c_{\text{trim}}(q)}{\varphi(q)}. \end{aligned} \tag{6.11}$$

Finite expansion gives

$$A_{\mathrm{mix}} = M_{\mathrm{mix}} + D_{I_L}(b_{\mathrm{all}}) + D_{I_H}(b_{\mathrm{tail}}) + C_{I_H}(c_{\mathrm{trim}}) + P_{I_H}(c_{\mathrm{lo}}) + P_{I_H}(\Gamma). \tag{6.12}$$

Since $c_{\mathrm{trim}} = c_{\mathrm{lo}} + \Gamma$,

$$C_{I_H}(c_{\mathrm{trim}}) + P_{I_H}(c_{\mathrm{lo}}) = D_{I_H}(c_{\mathrm{lo}}) + C_{I_H}(\Gamma). \tag{6.13}$$

Therefore

$$A_{\mathrm{mix}} = M_{\mathrm{mix}} + D_{I_L}(b_{\mathrm{all}}) + D_{I_H}(b_{\mathrm{tail}}) + D_{I_H}(c_{\mathrm{lo}}) + C_{I_H}(\Gamma) + P_{I_H}(\Gamma). \tag{6.14}$$

No low natural-coprime term is estimated separately.

For large N, each prime in I_H exceeds Q. Hence

$$C_{I_H}(\Gamma) = \{\pi(I_H) - \mathrm{Li}(I_H)\} \sum_q \frac{\Gamma(q)}{\varphi(q)}. \tag{6.15}$$

The elementary identity

$$\frac{q}{\varphi(q)} = \sum_{d|q} \frac{\mu(d)^2}{\varphi(d)}, \qquad \sum_{q \le Q} \frac{1}{\varphi(q)} \ll \log(2Q) \tag{6.16}$$

follows by multiplicativity and a convergent Euler product. Thus Paper I, (2.21), bounds (6.15) by $O_A(NL^{-A})$ for each fixed A [26, (2.21)].

For the three D-terms we use the maximal-endpoint prime form of Bombieri–Vinogradov. Here it is conditional on the prime Siegel–Walfisz input [26, (2.21)]. Define the prime progression error by

$$\Delta_\pi(y; q, a) = \pi(y; q, a) - \frac{\mathrm{Li}(y)}{\varphi(q)}. \tag{6.17}$$

Fact 3.2 gives, for each fixed $S > 0$,

$$\sum_{q \le X^{1/2}(\log X)^{-(S+60)}} \max_{(a,q)=1} \sup_{0 \le y \le X} |\Delta_\pi(y; q, a)| \ll_S X(\log X)^{-S}. \tag{6.18}$$

Here $\mathrm{Li}(y) = 0$ for $0 \le y < 2$. The level sits at X, not at the variable endpoint y, so intervals with endpoints chosen separately for each modulus cost at most twice this bound. This is the maximal prime statement. We do not substitute the fixed-endpoint von Mangoldt statement of Paper I, (2.20). The source and the fixed 60 logarithmic buffer are those of Fact 3.2.

Set $X = N$, $S = A + 10$, and choose the half-level buffers so that

$$B_L,\ B_{\mathrm{tail}},\ B_0 \ge A + 72. \tag{6.19}$$

Lemma 6.1 (The three low discrepancies)**.** *Under* (6.19)*,*

$$|D_{I_L}(b_{\mathrm{all}})| + |D_{I_H}(b_{\mathrm{tail}})| + |D_{I_H}(c_{\mathrm{lo}})| \ll_A NL^{-A-10}. \tag{6.20}$$

Proof. The arrays b_{all}, b_{tail} and c_{lo} are supported on $q < R_L$, $q < R_{\mathrm{tail}}$ and $q \le Q_0$, respectively. Each is one-bounded. Since $B_L, B_{\mathrm{tail}}, B_0 \ge A + 72 > S + 60$, every such q has $q \le N^{1/2}L^{-(S+60)}$. This is the level of (6.18) at $X = N$. Every supported q is coprime to N, so N mod q is a reduced class. Since $\pi(y; q, N) = \mathrm{Li}(y) = 0$ for $y < 2$,

$$\Pi_{I_L}(q) - \frac{\mathrm{Li}(I_L)}{\varphi(q)} = \Delta_\pi(N/4; q, N).$$

Write y^- for the left limit at y. Then

$$\Pi_{I_H}(q) - \frac{\mathrm{Li}(I_H)}{\varphi(q)} = \Delta_\pi\big(((1-\varepsilon)N)^-; q, N\big) - \Delta_\pi(N/4; q, N).$$

A left limit is a limit of values at points of $[0, N]$. Hence each value on the right is at most $\sup_{0 \le y \le N} |\Delta_\pi(y; q, N)|$. Therefore the left side of (6.20) is at most five times the left side of (6.18), with $X = N$ and $S = A + 10$. This is the maximal prime statement of Fact 3.2. □

6.3. **The high coefficient.** Put

$$\begin{aligned} U = V = N^{1/3}, \qquad z_0 = N^{1/(\log\log N)^3}, \\ \mathcal{R}(n) = \mathbf{1}_{P^-(n)>z_0}, \qquad \mathcal{R}(1) = 1, \qquad \mathcal{A}(a) = \mu(a)\mathcal{R}(a), \\ \mathcal{C}(t) = \mathcal{R}(t) \sum_{\substack{ab=t \\ a\le U,\ b\le V}} \mu(a)\Lambda(b), \qquad \mathcal{B}(s) = \mathcal{R}(s) \sum_{\substack{b\mid s \\ b>V}} \Lambda(b), \\ \mathcal{K}_q(e) = \mathbf{1}_{e\equiv N \pmod q} - \frac{\mathbf{1}_{(e,q)=1}}{\varphi(q)}. \end{aligned} \tag{6.21}$$

The von Mangoldt identity gives three sums:

$$\begin{aligned} T_1 &= \sum_q \Gamma(q) \sum_{\substack{a\le U,\ c\ge 1 \\ ac\in I_H}} \mathcal{A}(a)\mathcal{R}(c)\frac{\log c}{\log(ac)}\mathcal{K}_q(ac), \\ T_2 &= -\sum_q \Gamma(q) \sum_{\substack{t\le UV,\ c\ge 1 \\ tc\in I_H}} \frac{\mathcal{C}(t)\mathcal{R}(c)}{\log(tc)}\mathcal{K}_q(tc), \\ T_{\mathrm{II}} &= \sum_q \Gamma(q) \sum_{\substack{a>U,\ s>V \\ as\in I_H}} \frac{\mathcal{A}(a)\mathcal{B}(s)}{\log(as)}\mathcal{K}_q(as). \end{aligned} \tag{6.22}$$

These come from

$$\Lambda = \mu_{\le U} * \log - \mu_{\le U} * \Lambda_{\le V} * 1 + \Lambda_{\le V} + \mu_{>U} * \Lambda_{>V} * 1. \tag{6.23}$$

Roughness is multiplicative, including shared prime factors. The short Λ-term vanishes on I_H.

The detector retains proper prime powers whose bases exceed z_0. For $e \in I_H$, hence $e > z_0$,

$$\frac{\mathcal{R}(e)\Lambda(e)}{\log e} = \mathbf{1}_{e\in\mathbb{P}} + \sum_{\substack{P\in\mathbb{P},\ j\ge 2 \\ e=P^j}} \frac{\mathbf{1}_{P>z_0}}{j}. \tag{6.24}$$

Consequently

$$\begin{aligned} P_{I_H}(\Gamma) &= T_1 + T_2 + T_{\mathrm{II}} - E_{\mathrm{pp}}, \\ E_{\mathrm{pp}} &= \sum_q \Gamma(q) \sum_{\substack{P\in\mathbb{P},\ j\ge 2 \\ P^j\in I_H}} \frac{\mathbf{1}_{P>z_0}}{j}\mathcal{K}_q(P^j). \end{aligned} \tag{6.25}$$

Let $e \in I_H$. A supported q with $e \equiv N \pmod q$ divides $N - e$, and $|\Gamma(q)| \le 1$ by unique marking. The comparator part is at most $\sum_{q\le Q} 1/\varphi(q)$. Hence (6.16) gives

$$\sum_q |\Gamma(q)||\mathcal{K}_q(e)| \le d(N-e) + O(L), \qquad E_{\mathrm{pp}} \ll_\nu N^{1/2+\nu}L^2. \tag{6.26}$$

Here $N - e \ge \varepsilon N$, and $0 < \nu < 1/2$ is fixed. The second bound follows from the first, $d(N-e) \ll_\nu N^\nu$ and $\sum_{j\ge 2} \#\{P \in \mathbb{P} : P^j \le N\} \ll N^{1/2}$. The high-coefficient estimate uses the $P > z_0$ mask of E_{pp} in (6.25). That mask excludes bases $P \le z_0$, so the displayed decomposition needs no separate add-back.

Proposition 6.12(3) proves the high-coefficient estimate

$$P_{I_H}(\Gamma) \ll_{A,B,B_0} NL^{-A} \qquad (A > 0 \text{ fixed}). \tag{6.27}$$

Its modulus coefficient is exactly (6.6). The estimate permits no joint arithmetic mask. Table 6.1 gives its decomposition. Each power boundary belongs to the interval on its right.

6.4. **The three distribution estimates.**

Lemma 6.2 (Rough-factor distribution under parent-scale and strict-roughness conditions)**.** *Let $L = \log N$, $z_0 = N^{1/(\log\log N)^3}$, and $Q = N^{5227/10000}L^{-B}$, with fixed B. Assume the two explicit conditions used by the direct rows:*

- (W1) *The original parent satisfies $cN^{.54} \le H \le N$, for a fixed $c > 0$. In the T_1 row $H = N/a \le N$; in the right T_2 row $H = N/t \le N$.*
- (W2) *Use the strict roughness convention $\mathcal{R}(n) = \mathbf{1}_{P^-(n)>z_0}$, as in (6.21). It is not replaced by $P^-(n) \ge z_0$.*

TABLE 6.1. The high-coefficient decomposition

Sum and range	Tested array	Required estimate
T_1, all $a \le U$	$\mathcal{R}(c)$	Rough-factor estimate
T_2, $c < N^{.46}$	$\mathcal{R}(c)$	Two-factor convolution
T_2, $N^{.46} \le c < N^{.50}$	$\mathcal{C}(t)/(2L)$	Central estimate
T_2, $N^{.50} \le c < N^{.54}$	$\mathcal{R}(c)$	Central estimate
T_2, $c \ge N^{.54}$	$\mathcal{R}(c)$	Rough-factor estimate
T_{II}, $a < N^{.46}$	$\mathcal{A}(a)$	Two-factor convolution
T_{II}, $N^{.46} \le a < N^{.50}$	$\mathcal{B}(s)/(2L)$	Central estimate
T_{II}, $N^{.50} \le a < N^{.54}$	$\mathcal{A}(a)$	Central estimate
T_{II}, $a \ge N^{.54}$	$\mathcal{B}(s)/(2L)$	Two-factor convolution

For every fixed $T_{\mathrm{save}} > 0$, uniformly in real intervals $I \subset (0, 2H]$ (so $|I| \le 2H$; open, closed or one-point), supported $q \le Q$ with $(q, N) = 1$, and reduced residues $(b, q) = 1$,

$$\sum_{n \in I} \mathcal{R}(n) \left\{ \mathbf{1}_{n \equiv b \pmod q} - \frac{\mathbf{1}_{(n,q)=1}}{\varphi(q)} \right\} \ll_{T_{\mathrm{save}}} \frac{H}{qL^{T_{\mathrm{save}}}}. \tag{6.28}$$

The interval may be arbitrarily short. The saving is measured at its original parent. The estimate is an integer-sieve statement and uses no prime distribution input.

Proof. Put $\ell = \log\log N$ and $\mathcal{P}_q = \{p \le z_0 : p \nmid q\}$. The Euler product at $s = 1 + 1/\log z_0$ gives $h_q = \sum_{p \in \mathcal{P}_q} p^{-1} \le 3\ell$ for sufficiently large N. Choose the fixed constant K after T_{save}, with

$$K > 3e, \qquad K \log\left(\frac{K}{3e}\right) > T_{\mathrm{save}} + 4. \tag{6.29}$$

Let $k = 2\lceil K\ell/2 \rceil$. The two adjacent Bonferroni truncations bracket the strict exclusion of all primes in $\mathcal{P}_q$. Their normalized gap satisfies

$$k = 2\lceil K \log\log N/2 \rceil, \qquad E_k \le \frac{h_q^k}{k!} \le \left(\frac{3e}{K}\right)^{K\ell} \le L^{-T_{\mathrm{save}}-4}, \qquad (k+1) z_0^k = N^{O(K/\ell^2)}. \tag{6.30}$$

Every divisor in the truncation is at most z_0^k. By CRT, each divisor contributes its expected interval mass with error $O(1)$. This is uniform for open, closed, or singleton intervals. The same expansion applies to the comparator after using

$$\mathbf{1}_{(n,q)=1} = \sum_{\substack{d \mid \mathrm{rad}(q) \\ d \mid n}} \mu(d).$$

Thus the main terms cancel. The accumulated lattice error is $N^{O(K/\ell^2)} q/\varphi(q)$, and $q/\varphi(q) \le \omega(q) + 1 \ll L$. Under (W1), $H/q \gg N^{.0173} L^B$. The Bonferroni remainder is at most $(|I|/q) E_k L^2 \le 2HL^{-T_{\mathrm{save}}-2}/q$ because $|I| \le 2H$. Hence both are absorbed by $H/(qL^{T_{\mathrm{save}}})$ once the fixed truncation precision is raised. This proves (6.28).

The strict convention in (W2) has no threshold atom. An application might instead use $P^-(n) \ge z_0$. Then the only possible difference is the atom $z_0 = P$ when z_0 is prime. That weak convention requires its separate CRT boundary correction. It is not used in (6.28). □

The positive gap $.54 - v_{\mathrm{RA}} = 173/10000$ absorbs the lattice error in (6.30). For T_1, the free scale is at least $N^{2/3}/4$. For the right T_2-part, it exceeds a fixed multiple of $N^{.54}$. Partial summation then gives

$$\sum_{m \le M_{\mathrm{out}}} \sum_{q \le Q} \frac{N}{mqL^{T_{\mathrm{save}}}} \ll NL^{2-T_{\mathrm{save}}}. \tag{6.31}$$

Both kernel terms remain in this estimate. Here M_{out} is the outer endpoint of the current row. It is U in the T_1 row and the corresponding low-row endpoint in the T_2 decomposition.

The other tested arrays need uniformity under coprimality with an auxiliary integer. Write $\tau(h) = \sum_{d \mid h} 1$ and $\omega(h) = \#\{p \in \mathbb{P} : p \mid h\}$. For any sequence $f(n)$ on its original parent $(H, 2H]$, define

$$\begin{aligned} \Delta_f(I; k, b; h) = &\sum_{\substack{n \in I,\ n \equiv b \pmod k \\ (n,h)=1}} f(n) \\ &- \frac{1}{\varphi(k)} \sum_{\substack{n \in I \\ (n,kh)=1}} f(n). \end{aligned} \tag{6.32}$$

Lemma 6.3 (Prime-variable transfer)**.** *Let $N^{1/3}/8 \le H \le 2N$, $L = \log N$, and $z_0 = N^{1/(\log\log N)^3}$. Let $y_0 \ge z_0$ and*

$$f(n) = \sum_{\substack{mp=n \\ p\in J_m}} \alpha(m) w(p),$$

where p is prime and $|\alpha(m)| \le 1$. For each m the allowed primes lie in a real interval $J_m \subset [y_0, \infty)$. Assume $\|w\|_\infty + \mathrm{Var}(w) \le W$ on $[y_0, 2H]$ and $|f(n)| \le 1$ on $(H, 2H]$. For every fixed $P > 0$, uniformly in intervals $I \subset (H, 2H]$, $k, h \ge 1$, and $(b, k) = 1$, prime Siegel–Walfisz, Paper I (2.21), implies

$$|\Delta_f(I; k, b; h)| \ll_P WH\left(L^{-P} + \frac{\omega(h)L}{y_0}\right) \ll_P WH\tau(h)L^{-P}. \tag{6.33}$$

Proof. Put $K = 2P + 4$. If $k > L^K$, the two terms in the discrepancy are bounded by a constant times $\|f\|_\infty(H/k + 1 + H/\varphi(k))$. For each represented n, a prime divisor $p \ge y_0$ determines at most one pair (m, p), since $m = n/p$. Hence

$$|f(n)| \le W\frac{\log(2H)}{\log y_0} \ll W(\log L)^3,$$

using $y_0 \ge z_0$, $H \le 2N$, and $\log z_0 = L/(\log L)^3$. Since $\varphi(k) \gg \sqrt{k}$ and $H \ge N^{1/3}/8$, the resulting discrepancy is $\ll_P WHL^{-P}$. The two spare powers in $K/2 = P + 2$ absorb $(\log L)^3$. The additive 1 is absorbed because $H \ge N^{1/3}/8 \gg L^P$.

Now let $k \le L^K$. Terms with $(m, kh) > 1$ vanish from both sides. Otherwise the prime progression is $p \equiv bm^{-1} \pmod k$ on $J_m \cap I/m$. Primes dividing h are excluded on the progression side and in the comparator. For each m the two exclusions differ by at most $2W\omega(h)$. There are at most $2H/y_0$ possible values of m, so these terms contribute $\ll WH\omega(h)/y_0$.

For the remaining primes define

$$E(t) = \pi(t; k, c) - \frac{1}{\varphi(k)} \sum_{\substack{a \bmod k \\ (a,k)=1}} \pi(t; k, a).$$

The reduced-class sum equals $\pi(t) - \#\{p \le t : p \mid k\}$. Thus this is the centered prime discrepancy without the primes dividing k. Apply (2.21) to the class c and average its uniform error over all reduced classes. This gives $E(t) \ll_{A'} t(\log t)^{-A'}$, with no factor of k. Since $t \ge y_0 \ge z_0$, $\log t \ge L/(\log L)^3$. Hence, for all sufficiently large N, $k \le (\log t)^{2K}$. Therefore (2.21), with modulus exponent $2K$ and fixed saving A', applies throughout each interval of integration. Stieltjes partial summation bounds each m-contribution by $O(W(H/m)(\log y_0)^{-A'})$. Sum over $m \le 2H/y_0$ and take $A' = 2(P + K + 2)$. This gives $O_P(WHL^{-P})$. The estimate is measured at the original parent H, including for short intervals. Finally, $L/y_0 \ll_P L^{-P}$ because $y_0 \ge z_0$, and $\omega(h) \le \tau(h)$. This proves (6.33). □

Lemma 6.4 (Four-family auxiliary-coprime Siegel–Walfisz)**.** *By prime Siegel–Walfisz, Paper I (2.21), for each of $\mathcal{R}, \mathcal{A}, \mathcal{C}/(2L), \mathcal{B}/(2L)$, the bound*

$$|\Delta_f(I; k, b; h)| \ll_P H\tau(h)L^{-P}, \qquad (b, k) = 1, \tag{6.34}$$

holds uniformly for every $I \subset (H, 2H]$ and $k, h \ge 1$. The parents are the central ranges in Table 6.1 and the convolution parents $N^{1/3}/8 \le H \le 2N^{.46+O(1/L)}$. The proof uses the lower bound only to absorb the large-modulus lattice term and to exclude $n = 1$. It uses $H \le 2N^{.54}$ for the remaining weight bounds. The divisor exponent is the single fixed value $C_{\mathrm{SW}} = 1$, independent of P. The implied constant may depend on P and the fixed family parameters, but not on N, H, I, k, h, b. The error is measured at H, even when I is short.

Proof. For sufficiently large N, every parent has $H \ge N^{1/3}/8 > 1$, so $n = 1$ is excluded. The rough and signed-rough arrays have exact prime-variable decompositions. For $\mathcal{R}(n)$, take $\alpha(m) = \mathcal{R}(m)$, $J_m = [P^+(m), \infty) \cap (z_0, \infty)$, and $w(p) = 1$. Use $P^+(1) = 1$. The largest prime factor gives the unique term. For $\mathcal{A}(n) = \mu(n)\mathcal{R}(n)$, take $\alpha(m) = -\mu(m)\mathcal{R}(m)$, $J_m = (\max(P^+(m), z_0), \infty)$, and $w(p) = 1$. The strict inequality excludes repeated largest primes, where $\mu(n) = 0$.

For $\mathcal{B}(s)/(2L)$, the $j = 1$ terms in $b = p^j \mid s$, $b > V$, use $\alpha(m) = \mathcal{R}(m)$, $J_m = (V, \infty)$, and $w(p) = \log p/(2L)$. For $\mathcal{C}(t)/(2L)$, the $j = 1$ terms use $\alpha(a) = \mu(a)\mathcal{R}(a)\mathbf{1}_{a\le U}$, $J_a = (z_0, V]$, and the same weight. In each case $\|w\|_\infty + \mathrm{Var}(w) \le 1$. The four tested arrays are one-bounded on their parent ranges. For the last two arrays this follows from $\sum_{d|n} \Lambda(d) = \log n$ and $\log(2H) < 2L$. On the central ranges $H \le 2N^{.54}$. There the weight bound follows from $y_0 = V = N^{1/3}$ for $\mathcal{B}$ and $y_0 = z_0$ for $\mathcal{C}$, since $\log(2H) \le .54L + O(1)$.

For $\mathcal{B}$ and $\mathcal{C}$, write the remaining $j \ge 2$ terms as $f_{\ge 2}$. They need not be nonnegative. For instance, the $\mu(a)$ coefficient in $\mathcal{C}$ may change sign. Take absolute values before summing:

$$\sum_{n\le 2H} |f_{\ge 2}(n)| \ll \frac{H}{L} \sum_{p > z_0} \sum_{j\ge 2} \frac{\log p}{p^j} \ll \frac{H \log z_0}{Lz_0} \ll \frac{H}{z_0}.$$

For any sequence g, $|\Delta_g| \le 2\sum_{n\le 2H}|g(n)|$. Hence this remainder is $O_P(HL^{-P})$ for every fixed P. Applying Lemma 6.3 to each prime term proves (6.34) with the same exponent $C_{\mathrm{SW}} = 1$. □

The mean-square estimate below has five uses. Section 7 uses it for the zero mode of the exterior sum. Section 5 uses it for the Class II estimate of Paper II inside the retained side. Section 8 uses it for the row-three dispersion estimate of the switched sum. These three apply Paper I, Theorem 3.2. In the large-norm case of Paper I, (3.48), part (ii) below replaces [3, Theorem 0(a)]. The other two uses are the central rectangular estimate of Lemma 6.9 and the small-modulus norm-relative step of Lemma 6.8. Part (i) derives the conclusion of that published theorem (Paper I, Theorem 2.4). Its inputs are the norm-relative condition (6.35) and an elementary large sieve, so the argument is self-contained at this step. We do not correct [3]. Paper I's use of it is unaffected. The norm-relative premise is not deduced from the absolute parent-scale estimate (6.34) alone. Let ξ be complex and supported in $[1, Z]$, $Z \ge 3$. Fix $B_{\mathrm{div}} \ge 0$. Consider the following condition. For every $P > 0$, uniformly in integers $h, k \ge 1$ and reduced classes $b \bmod k$,

$$\left|\sum_{\substack{n\equiv b\ (k)\\(n,h)=1}}\xi_n - \frac{1}{\varphi(k)}\sum_{(n,kh)=1}\xi_n\right| \ll_P \|\xi\|_2 Z^{1/2}\tau(h)^{B_{\mathrm{div}}}(\log(2Z))^{-P}, \tag{6.35}$$

The constants are common to the family of pairs (Z, ξ) under consideration. Write $\Delta_\xi(k, b)$ for the centered discrepancy at $h = 1$. The norm conclusion is

$$\sum_{k\le Z(\log Z)^{-B_1(A)}}{\sum_{b \bmod k}}^{*}|\Delta_\xi(k,b)|^2 \ll_A \|\xi\|_2^2 Z(\log Z)^{-A}. \tag{6.36}$$

In the selected-factor form of Paper I, (3.48), $\xi_n = \alpha_n 1_{(n,r)=1}$, $Z = \lceil 2T_{\mathrm{int}}\rceil$, and

$$\sum_{q\le 3Q}{\sum_{b \bmod q}}^{*}|\Delta_\xi(q,b)|^2 \ll N_{\mathcal{I}}T_{\mathrm{int}}L^{-A_3}, \qquad 3Q \le Z(\log Z)^{-B_F}, \quad B_F = A_3 + 15, \tag{6.37}$$

where $N_{\mathcal{I}}$ is the interval mass in Paper I, (3.8).

Lemma 6.5 (Transferred mean-square estimate)**.** (i) *If* (6.35) *holds, then* (6.36) *holds for every fixed $A > 0$. Here $B_1(A)$ and the implied constant depend only on A, B_{div} and the common constants.*

(ii) *Assume the hypotheses (3.1), (3.2), (3.6), (3.8) with its length condition, (3.11) and (3.25)–(3.27) of Paper I. Then the estimate* (6.37) *holds uniformly over the parameter boxes and selected factors of Paper I, Theorem 3.2. Hence that theorem holds with this input in place of* [3, Theorem 0(a)] *at (3.48).*

Proof. (i) Expand $\Delta_\xi(k, b)$ in Dirichlet characters modulo k and pass to primitive characters. For conductors up to a fixed power of $\log Z$, apply (6.35). Take the auxiliary divisor h to be the imprimitive part of the modulus. For larger conductors use the multiplicative large sieve inequality. Its elementary proof uses finite Fourier identities and the spacing of reduced fractions. Summing over the moduli gives (i). (ii) If $\|\xi\|_2^2 \le N_{\mathcal{I}}L^{-A_3-1}$, the elementary bound in Paper I, (3.48), applies. Otherwise (3.25), with auxiliary divisor dr, and (3.26) give (6.35) for $\xi_n = \alpha_n 1_{(n,r)=1}$ with divisor exponent one. Also (3.6) gives the modulus range in (6.37). Part (i) then applies. The complete argument is in Appendix A.2, with its row index in Appendix B.8. We assert no implication from (6.34) alone to the norm bound. □

For the three convolution rows, let

$$D_1 = HN^{-\sigma_{\mathrm{p}}}, \qquad D_2 = QN^{w_{\mathrm{RA}}+\sigma_{\mathrm{p}}}/H, \qquad \sigma_{\mathrm{p}} = 10^{-6}, \quad w_{\mathrm{RA}} = 9/1000, \tag{6.38}$$

where $N^{1/3}/8 \le H \le 2N^{.46+O(1/L)}$. A supported modulus has the unique form $q = pd$ of (6.4). Here p is the marked prime and d the squarefree cofactor.

$$r < D_1, \qquad s < D_2, \qquad rs = q < Q. \tag{6.39}$$

Lemma 6.6 (Deterministic allocation)**.** *For each supported $q = pd$, the following rule defines a unique coprime split $q = rs$ satisfying* (6.39)*. Put p in r. Order the cofactor primes exceeding $N^{w_{\mathrm{RA}}}$ as $b_1 > \cdots > b_m$. Here $m \le (v_{\mathrm{RA}} - t_0)/w_{\mathrm{RA}} = 37553/1260 < 30$. Process them in this order. Put b_i in r exactly when the current $rb_i < D_1$, and put it in s otherwise. Let r_{big} and s_{big} be the resulting factors. Order the remaining cofactor primes as $l'_1 > \cdots > l'_k$. Set $r_0 = r_{\mathrm{big}}\prod_{j=1}^{k} l'_j$. If $r_0 < D_1$, take $r = r_0$, $s = s_{\mathrm{big}}$. Otherwise let j_* be the least index for which $r_0/(l'_1\cdots l'_{j_*}) < D_1$, and take*

$$r = r_0/(l'_1\cdots l'_{j_*}), \qquad s = s_{\mathrm{big}}l'_1\cdots l'_{j_*}.$$

Call the resulting pair $\mathrm{Sel}_H(q) = (r, s)$.

Proof. The marked prime fits, since $p < N^{1/3-\eta_{\mathrm{m}}} < D_1$, even at the fixed lower endpoint $H \geq N^{1/3}/8$. The accepted big-prime partial products stay below D_1. Thus $r_{\mathrm{big}} < D_1$, and the least index j_* exists when needed. Write all cofactor primes in decreasing order as $l_1 > l_2 > \cdots$. The odd cube gates in (6.4) imply

$$pl_1 \cdots l_{i-1} l_i^2 < Q \qquad \text{for every } i :$$

For odd i, drop one factor from the cube gate. For even i, use the preceding odd gate and $l_i \leq l_{i-1}$.

Suppose $r_0 < D_1$, and let b_i be the last big prime rejected into s. The rejection test gives $r^{(i)} b_i \geq D_1$. The two-gate inequality then gives

$$s = s_{\mathrm{big}} = \frac{pb_1 \cdots b_i}{r^{(i)}} \leq \frac{pb_1 \cdots b_{i-1} b_i^2}{D_1} < \frac{Q}{D_1} < D_2.$$

If no big prime is rejected, $s = 1$. In the prefix case, minimality gives $rl'_{j_*} \geq D_1$, hence $r \geq D_1 N^{-w_{\mathrm{RA}}}$ and

$$s = q/r < QN^{w_{\mathrm{RA}}}/D_1 = D_2.$$

The factors partition the squarefree prime factors of q. Thus $(r, s) = 1$, $rs = q < Q$, and $r < D_1$. □

Define the exact joint coefficient on each dyadic Cartesian modulus shell by

$$F_H(r,s) = \begin{cases} \Gamma(rs), & rs \text{ is supported and } \mathrm{Sel}_H(rs) = (r,s), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_q \Gamma(q)\Phi(q) = \sum_{r,s} F_H(r,s)\Phi(rs), \tag{6.40}$$

$$F_H(r,s) = \sum_\nu c_\nu f_\nu(r) g_\nu(s), \qquad |f_\nu|, |g_\nu| \leq 1, \qquad \sum_\nu |c_\nu| \ll L^{C_{\mathrm{tensor}}}, \quad C_{\mathrm{tensor}} = 2^{30} + 91.$$

Here Sel_H is the rule in Lemma 6.6. The expansion is exact on the full Cartesian parent, including zero values off the selected pairs.

Lemma 6.7 (Global tensor for the selector)**.** *The coefficient F_H has a rank-one expansion satisfying* (6.40)*, with the sharper bound*

$$\sum_\nu |c_\nu| \leq 4(2^{30} - 1)(3 + L)^{78} \ll L^{79}. \tag{6.41}$$

Proof. Fix a pair of dyadic shells. A branch fixes the merge order $\pi \in \{R, S\}^m$ of the at most 29 big cofactor primes. It also fixes the final large-prime/prefix case. In the prefix case it fixes the parity of the number of factors assigned to s. There are at most $4 \sum_{m=0}^{29} 2^m = 4(2^{30} - 1)$ branches. Within a branch, unary conditions impose squarefreeness, coprimality with $2N$ and the unique marked prime p in r. They also impose the bounds $P^+(r/p) < z_s$ and $P^+(s) < z_s$, assigned big-prime counts, $r < D_1$, $s < D_2$, and $r_{\mathrm{big}} < D_1$. The remaining unary data are the case indicator, the parity of the small-prime count, and the sign $\Gamma(rs) = \mu(rs/p) = -\mu(r)\mu(s)$.

Every cross-condition is a comparison $\phi(r) < \psi(s)$ or its weak counterpart. Let K_0 be the number of distinct values of ϕ on the r-shell. For ranks $1 \leq a \leq K_0$, $0 \leq b \leq K_0$, put $M = 2K_0 + 1$ and

$$c_t = M^{-1} \sum_{u=0}^{K_0} e(-tu/M).$$

Finite Fourier orthogonality gives the exact separator

$$\mathbf{1}_{a \leq b} = \sum_{t \bmod M} c_t e(tb/M) e(-ta/M), \qquad \sum_{t \bmod M} |c_t| \leq 3 + \log M. \tag{6.42}$$

Rank $\phi(r)$ among its shell values, and count the values below $\psi(s)$. Each shell has at most Q values, so $\log M \ll L$. The cross-comparisons are exactly:

(1) the adjacent strict comparisons in the fixed merge order of the big primes (at most 28);
(2) for each S-letter b_i, the greedy rejection test $p(\prod_{b_j > b_i,\ b_j \in r} b_j) b_i \geq D_1$ (at most 29);
(3) the odd cube gates $pl_1 \cdots l_{i-1} l_i^3 < Q$ at odd indices among the big primes (at most 15);
(4) the remaining odd cube gates on the small-prime part, grouped by the fixed parity into the two maxima of the r- and s-side partial products (two comparisons);
(5) in the prefix case, the prefix-order test $\min P > \max R_{\mathrm{small}}$ and the least-prefix test $r \min P \geq D_1$ (two comparisons);
(6) the support range $Q_0 < rs < Q$ (two comparisons).

There are at most 78 comparisons in total. Once a merge branch is fixed, each displayed test compares an r-only quantity with an s-only quantity. The small-prime maxima combine all gates of the same parity and side. The branch conditions are exactly the greedy tests, the cube gates and the least-prefix condition. Hence their union is precisely F_H. Multiplying the separators in (6.42) gives (6.41). The unary factors remain one-bounded. The bound in (6.41) is per dyadic shell pair. The resulting $O(L^2)$ shell pairs are absorbed by the chosen logarithmic precision. □

For the convolution rows, split $s = dt$ with $P^+(d) \le z < P^-(t)$, where $z = N^{1/(\log\log N)^3}$ and $y = N^{1/\log\log N}$. The aggregate with $d > y$ is bounded by Rankin's method. The mean-square modulus in the proof of Lemma 6.8 is $k = rd$, not the full q. Its scale is

$$Q' = D_1 y = HN^{-\sigma_{\mathrm{p}}+1/\log\log N}, \qquad k \le 5Q', \qquad Z = \lceil 2H \rceil. \tag{6.43}$$

The inner factor t is z-rough and is removed in L^2, with cost H^2L^C/z. No bound for $\tau(r)$ and no roughness condition on r is used. The required range is

$$5Q' \le Z(\log Z)^{-B_F}, \qquad B_F = P_{\mathrm{MS}} + 15, \tag{6.44}$$

where P_{MS} is the fixed source-level precision in the supplementary Lemma 6.5(i). This holds since $Q'/H = N^{-\sigma_{\mathrm{p}}+o(1)}$. The full-modulus range $3Q \le Z(\log Z)^{-B_F}$ is not asserted for these parents.

Lemma 6.8 (Two-factor convolution estimate)**.** *Let $N^{1/3}/8 \le H \le 2N^{.46+O(1/L)}$ and $HM \asymp N$. Let a_n, $n \sim H$, be one-bounded, supported on $P^-(n) > z$, and satisfy* (6.34) *at the original parent H, where $z = N^{1/(\log\log N)^3}$. Let b_m, $m \sim M$, be one-bounded. Let U_1, V_1 be dyadic modulus scales, allowing the singleton scale* $1/2$ *for the integer* 1*, such that*

$$U_1 \le D_1, \qquad V_1 \le D_2, \qquad U_1V_1 \le N^{v_{\mathrm{RA}}}.$$

Let $f(r)$, $r \sim U_1$, and $g(s)$, $s \sim V_1$, be arbitrary one-bounded arrays. Keep the unary restrictions $r < D_1$, $s < D_2$, $(r, N) = (s, N) = 1$. Neither a pointwise condition $rs < Q$ nor a joint condition $(r, s) = 1$ is imposed on the pair. Then, for every fixed $A > 0$, uniformly in these occupied-parent scales, f, g, unary interval cuts of a_n and b_m, and intervals $I \subset [N/8, 2N]$,

$$\sum_{r,s} f(r)g(s) \sum_{\substack{n,m \\ nm\in I}} a_n b_m \mathcal{K}_{rs}(nm) \ll_A NL^{-A}. \tag{6.45}$$

Proof. We use the dispersion method of Paper I, Theorem 3.2. It runs on the whole Cartesian shell $r \sim U_1$, $s \sim V_1$, with residue class $N \bmod rs$. Its occupied-parent scale conditions are $U_1 \le HN^{-\sigma_{\mathrm{p}}}$, $V_1 \le N^{v_{\mathrm{RA}}+w_{\mathrm{RA}}}/(HN^{-\sigma_{\mathrm{p}}})$, and $U_1V_1 \le N^{v_{\mathrm{RA}}}$. The second follows from the displayed bound by D_2 after dropping L^{-B}. The tested sequence is strictly z-rough. The unary supports carry $(rs, N) = 1$ and the individual caps. No pointwise product cap is needed for the rank-one arrays. After Cauchy's inequality in m, the split $s = dt$ above reduces the modulus of the mean square to $k = rd$ in the range (6.44). The mean square is split at a fixed power of L below its target. Below it the trivial bound suffices. Above it (6.34) gives (6.35) with a loss of that power, so Lemma 6.5(i) applies. The prime Siegel–Walfisz input is Paper I, (2.21). No joint condition $(r, s) = 1$ is used, because r, s are not the moduli of Paper I, Theorem 2.8. Write $s = dt$ as above and $k = rd$. A prime $p \mid (r, s)$ with $p \le z$ divides d and only enlarges k. The Chinese remainder step needs only that k, t_1, t_2 are pairwise coprime. The smooth–rough split gives $(d, t_1t_2) = 1$. A prime $p \mid (r, s)$ with $p > z$ divides k and a rough t_i. Such tuples are the bad type $(k, t_1t_2) > 1$. If $p \mid t_2$, the second progression $kt_2 \mid mn_2 - N$ places n_2 in one class modulo kp. If $p \mid t_1$ only, exchange the two copies first; the absolute square majorant is symmetric, and the same statement then applies. We therefore remove these tuples before the Chinese remainder step. The cost in the square is $\ll MHL^C[H/(zK) + 1]$. After the factor KM of the first Cauchy step this is $\ll NL^C(z^{-1/2} + (K/H)^{1/2})$. That is $O_A(NL^{-A})$, since $z = N^{1/(\log\log N)^3}$ and $K/H \le N^{-\sigma_{\mathrm{p}}/2}$. The zero-mode identity holds for all tuples, including those with gcd failures. After the second Cauchy step, Paper I, Theorem 2.8 bounds the nonzero frequencies. It is applied with $r_{\mathrm{int}} = tt'$, the product of two z-rough inner variables, and $s_{\mathrm{int}} = p_0$, the least prime not dividing N. Since $p_0 \ll \log N < z$, $(r_{\mathrm{int}}, s_{\mathrm{int}}) = 1$, exactly as in Paper I, (3.82)–(3.83). This is the form of Paper I, Theorem 2.8 (Sections 5.3 and 7.4) used here. Its statement allows arbitrary one-bounded f, g with only $(rs, N) = 1$. The product restriction $nm \in I$ is kept as in Paper I, Theorem 3.2. The finite product partition used there also handles the endpoints. Neither (6.37) at the full modulus $3Q$ nor a norm-relative consequence of (6.34) for small norms is used. The complete argument is in Appendix A.3, with its row index in Appendix B.8. □

The central rows use $Y = N^y$, $y = 341/5000$. Choose the largest divisor $s \mid d$ with $s \le Y$. Since $p < N^{1/3-\eta_{\mathrm{m}}}$ and $q = pd > Q_0$, $d > Q_0/p \gg N^{1/6+\eta_{\mathrm{m}}}L^{-B_0} > Y$. Hence a next divisor exists. The cube gates control adjacent divisors and give

$$\frac{Yq}{Q} < s \le Y, \qquad r = q/s < Q/Y. \tag{6.46}$$

This allocation keeps a one-bounded joint modulus coefficient. The required central estimate has

$$\begin{aligned} H_{\mathrm{c}}M_{\mathrm{c}} &\asymp N, \qquad N^{.50} \ll H_{\mathrm{c}} \ll N^{.54}, \\ Q_1 &\ll N^{.4545}, \qquad Q_2 \ll N^{.0682}. \end{aligned} \tag{6.47}$$

Lemma 6.9 (Central rectangular distribution)**.** *Let x be even, $H_{\mathrm{c}}M_{\mathrm{c}} \asymp x$, and let the bounds in* (6.47) *hold with N replaced by x, with fixed comparison constants. Let a_n, b_m be one-bounded on the complete dyadic rectangle $n \sim H_{\mathrm{c}}, m \sim M_{\mathrm{c}}$. Let the tested a_n satisfy* (6.34) *at its original parent. The divisor exponent is fixed, and the constants are uniform at every requested precision. Write*

$$\mathcal{D}(q,a) = \sum_{n\sim H_{\mathrm{c}}, m\sim M_{\mathrm{c}}} a_n b_m \left(1_{nm\equiv a\ (q)} - \frac{1_{(nm,q)=1}}{\varphi(q)} \right).$$

Then, for every fixed $A > 0$,

$$\sum_{r\sim Q_1} \max_{(b,r)=1} \sum_{\substack{s\sim Q_2 \\ \mu(rs)^2=1,\ (rs,x)=1}} \max_{\substack{a \bmod rs,\ (a,rs)=1 \\ a\equiv b\ (r)}} |\mathcal{D}(rs,a)| \ll_A x(\log x)^{-A}. \tag{6.48}$$

The constant is uniform over these families, rectangles, squarefree modulus supports and unary interval restrictions of a_n and b_m. All errors are measured at the original parents. One outer residue b is chosen before s varies. In the application it is $N \bmod r$.

Proof. This is the rectangular dispersion theorem proved in Appendix A.4 and indexed in Appendix B.8. The restriction $(rs,x) = 1$ only removes nonnegative terms. Its proof opens the inner modulus s by dispersion with the outer residue fixed. It uses the mean square over all moduli of Lemma 6.5(i), which gives Paper I, (2.17), under its hypothesis (2.16). The supplementary proof applies the mean square twice. The second application is repeated over two correlations. At each application a constant sequence on the same parent is added. Its sign is chosen by the parallelogram identity to make the tested sequence norm-large. The resulting sequence and the constant sequence satisfy (6.35) uniformly. Their squared norms are at most the original parent times a fixed power of L. Fix the requested logarithmic modulus buffer B_1 first. Then choose the source-level parameter so that the moduli lie in the range of (6.36). Apply part (i) to both sequences, subtract by linearity, and absorb that fixed power by increasing the requested precision. This uses neither part (ii) nor a mean-square conclusion for an arbitrary small-norm sequence from absolute Siegel–Walfisz alone. The other inputs are Paper I, (2.21) (prime Siegel–Walfisz), (2.35) (Weil–Estermann) and Lemma 3.4 (completed hyperbola count). It does not use Paper I, Theorem 2.8. For the four tested arrays of Table 6.1, Lemma 6.4 supplies (6.34) with divisor exponent one. □

Lemma 6.9 concerns complete dyadic rectangles and one-bounded sequences. Lemma 6.10 below reduces the four central rows, with their product cuts, to it. The raw exponent checks are

$$\begin{aligned} 7v_{\mathrm{RA}} + 5y &= 3.9999 < 4, \\ 3v_{\mathrm{RA}} - y - 1 &= .4999 < .50, \\ 1 - v_{\mathrm{RA}} + y &= .5455 > .54. \end{aligned} \tag{6.49}$$

The three margins verify the numerical range only. They do not establish the rectangular-support or family hypotheses for a selected product region.

Product cuts require an additional finite reduction. For $e \in I_H$, put

$$\begin{aligned} w(e) &= \frac{2L}{\log e}, \\ \mathcal{A}(a)\mathcal{R}(c)\frac{\log c}{\log(ac)} &= \left(\mathcal{A}(a)\mathcal{R}(c)\frac{\log c}{2L} \right) w(ac), \\ -\frac{\mathcal{C}(t)\mathcal{R}(c)}{\log(tc)} &= \left(-\frac{\mathcal{C}(t)}{2L}\mathcal{R}(c) \right) w(tc), \\ \frac{\mathcal{A}(a)\mathcal{B}(s)}{\log(as)} &= \left(\mathcal{A}(a)\frac{\mathcal{B}(s)}{2L} \right) w(as). \end{aligned} \tag{6.50}$$

Here $\log c/(2L)$ is unary, and $\|w\|_\infty \le 2L/(L - \log 4) = O(1)$, and

$$\mathrm{Var}_{I_H}(w) = \frac{2L}{L - \log 4} - \frac{2L}{L + \log(1-\varepsilon)} = O(1).$$

Thus uniform product-prefix estimates restore the exact weights by Stieltjes summation, including the minus sign of T_2. Use multiplicative cells of relative width $L^{-D_{\mathrm{m}}}$. The original-source boundary strips satisfy

$$E_{\mathrm{strip}} \ll \big(NL^{-D_{\mathrm{m}}/2} + N^{1/2} \big) L^{15/2}. \tag{6.51}$$

Each unnormalized source multiplicity is at most $3d(e)$. Cauchy and $d(n)^4 \le d_{16}(n)$ give (6.51). Sum the boundary strips before the modulus expansion. For a final saving A, fix

$$D_{\mathrm{m}} = 2A + 40, \qquad T_{\mathrm{save}} > A + C_{\mathrm{tensor}} + 2D_{\mathrm{m}} + 40. \tag{6.52}$$

Then fix every Siegel–Walfisz and half-level precision. Choose the common threshold last.

Lemma 6.10 (Product-region reduction for the central rows)**.** *Fix $A > 0$ and choose D_{m} and T_{save} as in* (6.52)*. Take one of the four central rows of Table 6.1. Keep its unary cuts, its normalized arrays from* (6.50) *and the coefficient Γ. For $N/4 < u \le (1-\varepsilon)N$ let $\mathcal{S}(u)$ be its normalized discrepancy sum over $N/4 < nm \le u$, without the factor w. Then*

$$\sup_{N/4<u\le(1-\varepsilon)N} |\mathcal{S}(u)| \ll_A NL^{-A}, \tag{6.53}$$

and the same bound holds for the sum over $nm \in I_H$ with the factor $w(nm)$.

Proof. Selector. Let $q = pd$ be supported, with the primes of d in decreasing order $\ell_1 > \cdots > \ell_k$. As in the proof of Lemma 6.6, the cube gates give $p\ell_1 \cdots \ell_{i-1}\ell_i^2 < Q$ for every i, hence

$$\frac{\ell_i}{\ell_{i+1}\cdots\ell_k} < \frac{Q}{q} \qquad (1 \le i \le k).$$

Consecutive divisors of a squarefree d have ratio at most $\max_i \ell_i/(\ell_{i+1}\cdots\ell_k)$. Write $d = \ell_1 M$. The divisors of d form the two chains $\mathcal{D}(M)$ and $\ell_1\mathcal{D}(M)$. If $\ell_1 > M$, the only new gap is ℓ_1/M. If $\ell_1 \le M$, the ranges $[1, M]$ and $[\ell_1, \ell_1 M]$ overlap. A gap of the union below M lies inside a gap of $\mathcal{D}(M)$. One above M lies inside a gap of $\ell_1\mathcal{D}(M)$. Induction on k gives the claim. Since $d > Y$, the largest divisor $s \le Y$ of d has a successor $s' > Y$ with $s'/s < Q/q$. This proves (6.46). The pair (r, s) is determined by q, and $q = rs$. Put $F(r, s) = \Gamma(rs)$ on selected pairs and $F(r, s) = 0$ otherwise. This coefficient is one-bounded and supported on squarefree rs with $(rs, 2N) = 1$. Its dyadic shells satisfy $Q_1 \ll N^{v_{\mathrm{RA}}-y}$ and $Q_2 \ll N^y$, where $v_{\mathrm{RA}} - y = 909/2000$ and $y = 341/5000$. There are $O(L^2)$ shell pairs.

Parents. In the first central T_2 row the tested variable and array are $n = t$ and $\mathcal{C}(t)/(2L)$. In the second they are $n = c$ and $\mathcal{R}(c)$. In the central T_{II} rows they are $n = s$ and $\mathcal{B}(s)/(2L)$ in the first. In the second they are $n = a$ and $\mathcal{A}(a)$. The other variable carries the remaining one-bounded array, with the sign of T_2. On I_H, every relevant pair of dyadic parents has $H_{\mathrm{c}}M_{\mathrm{c}} \asymp N$ and $N^{.50} \ll H_{\mathrm{c}} \ll N^{.54}$ with fixed constants. There are $O(L^2)$ such pairs. Lemma 6.4 gives (6.34) for the tested arrays at these parents.

Restriction. Let a_n be a tested array on its parent and J an interval. On the same parent, $a_n\mathbf{1}_J(n)$ satisfies $\Delta_{a\mathbf{1}_J}(I; k, b; h) = \Delta_a(I \cap J; k, b; h)$ for every interval I. It therefore satisfies (6.34) with the same constants, measured at the original parent. The other array may be restricted to any interval.

Mesh. Put $\delta = L^{-D_{\mathrm{m}}}$. Divide each parent into $O(\delta^{-1})$ intervals with endpoint ratio at most $1 + \delta$. All unary cut points of the row are endpoints. For fixed u, call a cell complete if it lies inside $\{N/4 < nm \le u\}$. Any other cell meeting this region has all its products within a factor $(1 + \delta)^2$ of $N/4$ or of u. The part of the region it covers lies in the set $\mathcal{E}_u$ of integers $e \in (N/4, (1-\varepsilon)N]$ with $|e - N/4| \le 3\delta N$ or $|e - u| \le 3\delta N$. Here $\#\mathcal{E}_u \ll \delta N + 1$. For a complete cell apply Lemma 6.9 with $x = N$, saving T_{save}, the restricted arrays, $b = N \bmod r$ and $a = N \bmod rs$. Both classes are reduced since $(rs, N) = 1$, and $a \equiv b \pmod r$. Thus

$$\Big|\sum_{r,s} F(r,s)\mathcal{D}_{\mathrm{cell}}(rs, N)\Big| \le \sum_{r\sim Q_1} \max_{(b,r)=1} \sum_{\substack{s\sim Q_2 \\ \mu(rs)^2=1,\ (rs,N)=1}} \max_{\substack{a \bmod rs \\ a\equiv b\ (r) \\ (a,rs)=1}} |\mathcal{D}_{\mathrm{cell}}(rs, a)| \ll_{T_{\mathrm{save}}} NL^{-T_{\mathrm{save}}}.$$

There are $O(L^{2D_{\mathrm{m}}+4})$ choices of cell and shell pair, so the complete cells contribute $O(NL^{-T_{\mathrm{save}}+2D_{\mathrm{m}}+4})$.

Strips. Sum the incomplete cells together before expanding the modulus coefficient. For fixed e, the factorizations $e = nm$ in the row carry total normalized weight at most $d(e)$. Also (6.26) bounds the modulus sum by $d(N-e)+O(L)$. Their contribution is therefore

$$\ll \sum_{e\in\mathcal{E}_u} d(e)\{d(N-e) + L\} \ll (\#\mathcal{E}_u)^{1/2}N^{1/2}L^{15/2} \ll \big(N\delta^{1/2} + N^{1/2}\big)L^{15/2},$$

by Cauchy's inequality twice, $d(n)^4 \le d_{16}(n)$ and $\sum_{n\le N} d_{16}(n) \ll NL^{15}$. This is (6.51).

Conclusion. With $D_{\mathrm{m}} = 2A+40$ and $T_{\mathrm{save}} > A+C_{\mathrm{tensor}}+2D_{\mathrm{m}}+40$, the two contributions are $O(NL^{-A-C_{\mathrm{tensor}}-36})$ and $O(NL^{-A-25/2} + N^{1/2}L^{15/2})$, uniformly in u. This proves (6.53). Since $\|w\|_\infty + \mathrm{Var}_{I_H}(w) = O(1)$, Stieltjes summation against $\mathcal{S}$ gives the weighted bound, with the sign of T_2 kept. □

6.5. **The main coefficient.** For primes $\ell \nmid 2N$, set $g_N(\ell) = 1/(\ell - 1)$, and put $g_N(\ell) = 0$ for $\ell \mid 2N$. Extend g_N multiplicatively to squarefree integers, with $g_N(\ell^j) = 0$ for $j \ge 2$. For primes $e \equiv N \pmod p$ in the fixed reduced class modulo p, and primes $\ell \nmid 2Np$, the condition $\ell \mid (N - e)/p$ has density $1/\varphi(\ell) = g_N(\ell)$. Put

$$V_N(z_s) = \prod_{\substack{2<\ell<z_s \\ \ell\nmid N}} \left(1 - \frac{1}{\ell - 1}\right) = \frac{2e^{-\gamma}C(N)}{\alpha L}(1 + o(1)), \tag{6.54}$$

with $C(N)$ as in Paper II, (1.2). For a marked prime $p \in N^K$ and a level R, put $D = R/p$ and $s_p = \log D/(\alpha L)$. Let $P = P_N(z_s)$ be the product of the primes $\ell < z_s$ with $\ell \nmid 2N$.

Lemma 6.11 (Upper-sieve application)**.** (a) *The comparison* (6.9) *holds.*

(b) *There is an absolute constant* $K_{\dim}$ *such that, for every even* N*,*

$$\prod_{w \le \ell < z} (1 - g_N(\ell))^{-1} \le \frac{\log z}{\log w}\left(1 + \frac{K_{\dim}}{\log w}\right) \quad (z > w \ge 2). \tag{6.55}$$

(c) *For the three pairs* $(R, K) \in \{(R_L, J), (R_{\mathrm{tail}}, J_{\mathrm{tail}}), (Q, J_t)\}$*, uniformly in the mark and in even* N*,*

$$\sum_{d \mid P} \frac{\mu_d^+(R/p)}{\varphi(d)} \le V_N(z_s)\{F(s_p) + O(L^{-1/3})\}, \tag{6.56}$$

and, with $R = N^r L^{-B_r}$*,*

$$\sum_{q} \frac{c_{R,K}(q)}{\varphi(q)} \le \frac{2e^{-\gamma} C(N)}{\alpha L}\left\{\int_K F\left(\frac{r-t}{\alpha}\right)\frac{dt}{t} + o(1)\right\}. \tag{6.57}$$

Proof. We use Iwaniec's upper bound for the linear sieve [13]. Let $\mathcal{A}$ be a finite sequence and $D > 1$ a level. Let $P(z)$ be the product of the sifting primes below z. The upper weight μ_d^+ of [13, p. 313] is $(-1)^k$ when $d = \ell_1 \cdots \ell_k \mid P(z)$, $\ell_1 > \cdots > \ell_k$, and $\ell_1 \cdots \ell_{2l} \ell_{2l+1}^3 < D$ for every $0 \le l \le (k-1)/2$. Otherwise it is 0. With $j = 2l+1$ these are the gates of (6.3), so (6.3) is this weight on $d \mid P$. By the definition of W^+ [13, p. 310] and Lemma 2 there [13, p. 313],

$$\sum_{d \mid P(z)} \mu_d^+ |\mathcal{A}_d| = W^+ \ge S(\mathcal{A}, P, z)$$

for every $D > 1$, since W^+ is $S(\mathcal{A}, P, z)$ plus nonnegative sifted counts. If the density satisfies (6.55) with a constant K and $z \le D$, Lemma 3, (12), of [13, p. 313] gives, with $s = \log D/\log z$,

$$\sum_{d \mid P(z)} \mu_d^+ g(d) \le V(z)\{F(s) + O(e^{\sqrt{K}-s}(\log D)^{-1/3})\}.$$

(a) Let e be a prime in one of the pieces (6.7) and p a mark with $pm = N - e$. If $P^-(m) \ge z_s$, then $(m, P) = 1$. The inequality $W^+ \ge S$ for the one-term sequence $\{m\}$ at level $D = R/p$ gives

$$\sum_{\substack{d \mid P \\ d \mid m}} \mu_d^+(D) \ge \mathbf{1}_{(m,P)=1} \ge \mathbf{1}_{P^-(m) \ge z_s}. \tag{6.58}$$

This holds for every $D > 1$, in particular when $s_p = 383/200 < 2$. Omitting $(m, Np) = 1$ in (6.2) only enlarges the count. The condition $d \mid m$ is $e \equiv N \pmod{pd}$, and $c_{R,K}(pd) = \mu_d^+(R/p)$ by (6.5). The pieces (6.7) partition $[2, (1-\varepsilon)N) \times J$. The point $e = N/4$ lies in I_L and $t = 1/3 - \eta_{\mathrm{m}}$ lies in J_{tail}. Summing (6.58) over e and p, each mark with its multiplicity, gives $S_6 \le A_{\mathrm{mix}}$ by (6.8).

(b) For odd ℓ,

$$\Big(1 - \frac{1}{\ell - 1}\Big)^{-1} = \Big(1 - \frac{1}{\ell}\Big)^{-1}\Big(1 - \frac{1}{(\ell-1)^2}\Big)^{-1}.$$

The factors with $\ell \mid 2N$ equal one. Omitting them only lowers the product. Mertens' theorem, with its uniform error term, bounds the product of the first factors over $w \le \ell < z$ by $(\log z/\log w)(1 + K_1/\log w)$. The product of the second factors over $\ell \ge \max(w, 3)$ is at most $1 + K_2/\log w$. The constants K_1, K_2 are absolute. This gives (6.55) with one $K_{\dim}$, independent of N and of $\omega(N)$.

(c) The limiting ranges of s_p for the three pairs are

$$\Big(\frac{383}{200}, \frac{12639}{4480}\Big], \qquad \Big(\frac{383}{200}, \frac{1916149}{1000000}\Big], \qquad \Big(\frac{544243}{250000}, \frac{43148397}{14000000}\Big],$$

all inside $[383/200, 3.083) \subset (1, \infty)$. The factors L^{-B_r} move s_p by $O(\log L/L)$. Hence $z_s \le D$ and $\log D \asymp L$ for $N \ge N_0(B_r)$. Then (12) gives (6.56), with an error $O(L^{-1/3})$ uniform in the mark, the pair and N. For (6.54), use the factorization in (b) for odd $\ell < z_s$ and restore the factors with $\ell \mid N$. The primes $\ell \mid N$ with $\ell \ge z_s$ number at most $1/\alpha$ and change the product by a factor $1 + O(1/z_s)$. By (6.5),

$$\sum_{q} \frac{c_{R,K}(q)}{\varphi(q)} = \sum_{\substack{p \in N^K \\ p \nmid N}} \frac{1}{p-1} \sum_{d \mid P} \frac{\mu_d^+(R/p)}{\varphi(d)}.$$

The outer weights are positive, so (6.56) may be summed termwise. Since F is Lipschitz on the compact range above, the prime number theorem and partial summation give (6.57). Marks dividing N number at most $1/t_0$, and endpoint atoms and the logarithmic buffers contribute $o(1)$. □

The delayed branch $s_p > 3$ of F occurs only for c_{trim}, when $t < t_{\mathrm{RA}}$, where $t_{\mathrm{RA}} = v_{\mathrm{RA}} - 3\alpha$.

Define

$$J_K(r) = \frac{1}{2e^{\gamma}\alpha} \int_K F\left(\frac{r-t}{\alpha}\right) \frac{dt}{t}. \tag{6.59}$$

The rational endpoint $\overline{J}_6$ is as in Paper II [27, (A.13)]:

$$\begin{aligned} \overline{J}_6 &= \frac{1314858195915732153351324727678228531075877046888623761958 5}{10^{58}}, \\ J_J(1/2) &= 2\log(110/57) \le \overline{J}_6. \end{aligned} \tag{6.60}$$

The inequality is certified by outward interval evaluation. No central approximation is rounded. Since $W = 4(1-\varepsilon)C(N)N/L^2$, the interval masses give

$$\begin{aligned} \frac{M_{\mathrm{mix}}}{W} &\le w_L J_J(1/2) + w_H\{J_{J_{\mathrm{tail}}}(1/2) + J_{J_t}(v_{\mathrm{RA}})\} + o(1), \\ w_L &= \frac{2500}{9999}, \qquad w_H = \frac{7499}{9999}. \end{aligned} \tag{6.61}$$

Here $w_L + w_H = 1$. No mark multiplicity or singular-series factor is suppressed.

On J_{tail}, the explicit upper-sieve formula gives

$$\begin{aligned} 0 &\le w_H\{J_{J_{\mathrm{tail}}}(1/2) - J_{J_{\mathrm{tail}}}(v_{\mathrm{RA}})\} \\ &= w_H \int_{1/3-\eta_{\mathrm{m}}}^{1/3} \frac{v_{\mathrm{RA}} - 1/2}{t(1/2-t)(v_{\mathrm{RA}} - t)}\, dt \\ &\le \frac{w_H \eta_{\mathrm{m}}(v_{\mathrm{RA}} - 1/2)}{(1/3)(1/2-1/3)(v_{\mathrm{RA}} - 1/3)} \\ &=: d_{\mathrm{m}} = \frac{5106819}{31557955000}. \end{aligned} \tag{6.62}$$

The last integrand is increasing on this interval. Put

$$\mathfrak{g}_{\mathrm{RA}} = w_H\{J_J(1/2) - J_J(v_{\mathrm{RA}})\} - d_{\mathrm{m}}. \tag{6.63}$$

Combining (6.61) with (6.60) gives

$$\frac{M_{\mathrm{mix}}}{W} \le \overline{J}_6 - \mathfrak{g}_{\mathrm{RA}} + o(1). \tag{6.64}$$

The remaining integral is elementary. The linear-sieve functions F and f are as in [14, (1.8)–(1.9), pp. 172–173]. Set $t_{\mathrm{RA}} = v_{\mathrm{RA}} - 3\alpha$. On the ranges here,

$$\frac{F(s)}{2e^{\gamma}} = \begin{cases} 1/s, & 1 \le s \le 3, \\ \dfrac{1}{s}\left\{1 + \displaystyle\int_2^{s-1} \frac{\log(\xi - 1)}{\xi}\, d\xi\right\}, & 3 < s < 5. \end{cases} \tag{6.65}$$

This is the linear-sieve normalization [14, p. 202][26, (2.23)]. Write the delayed contribution as

$$\mathcal{I}_{\mathrm{RA}} = \int_{t_0}^{t_{\mathrm{RA}}} \frac{dt}{t(v_{\mathrm{RA}} - t)} \int_2^{(v_{\mathrm{RA}}-t)/\alpha - 1} \frac{\log(\xi - 1)}{\xi}\, d\xi. \tag{6.66}$$

Substitution gives

$$\begin{aligned} \mathfrak{g}_{\mathrm{RA}} + d_{\mathrm{m}} = w_H \Bigg\{ &\int_{t_0}^{1/3} \left(\frac{1}{t(1/2-t)} - \frac{1}{t(v_{\mathrm{RA}} - t)}\right) dt \\ &- \mathcal{I}_{\mathrm{RA}} \Bigg\}. \end{aligned} \tag{6.67}$$

Swap the order in (6.66) to obtain

$$\begin{aligned} \mathcal{I}_{\mathrm{RA}} &= \frac{1}{v_{\mathrm{RA}}} \int_3^{s_{\max}} h(\zeta)\, d\zeta, \qquad s_{\max} = \frac{v_{\mathrm{RA}} - t_0}{\alpha}, \\ h(\zeta) &= \frac{\log(\zeta - 2)}{\zeta - 1} \log \frac{(v_{\mathrm{RA}} - \alpha\zeta)(v_{\mathrm{RA}} - t_0)}{\alpha\zeta t_0}. \end{aligned} \tag{6.68}$$

For $n = 20000$, set $d = (s_{\max} - 3)/n$. The composite midpoint sum has error at most

$$\left| \int_3^{s_{\max}} h(\zeta)\, d\zeta - d \sum_{j=0}^{n-1} h(3 + (j + 1/2)d) \right| \le \frac{(s_{\max} - 3)d^2}{24} \sup |h''|.$$

Directed interval differentiation on 400 covering boxes gives $\sup |h''| \le 0.720711111$. An independent directed-interval replay encloses $\mathfrak{g}_{\mathrm{RA}}$ in $[0.09895983533531812, 0.09895983533543704]$. Outward rounding therefore gives

$$0.0989598353353 \le \mathfrak{g}_{\mathrm{RA}} \le 0.0989598353355. \tag{6.69}$$

Thus we may use the certified rational floor

$$g_{\mathrm{RA,lo}} = \frac{989598353}{10^{10}} = 0.0989598353, \qquad \mathfrak{g}_{\mathrm{RA}} \ge g_{\mathrm{RA,lo}}. \tag{6.70}$$

The exact integral $\mathfrak{g}_{\mathrm{RA}}$ and its certified floor $g_{\mathrm{RA,lo}}$ are distinct.

Proposition 6.12 (Sixth-sum source applications)**.** *On the exact arrays above, the following hold.*

1. *The comparison* (6.9)*, and the density bound* (6.57) *for the three pairs* (R, K)*.*
2. *The three low discrepancy bounds* (6.20)*.*
3. *For every fixed* $A > 0$*, each of the nine rows of Table 6.1, with its exact weight, is* $O_A(NL^{-A})$*. Consequently the high-coefficient estimate* (6.27) *holds.*

Proof. Items 1 and 2 are Lemmas 6.11 and 6.1. For item 3 fix A. Choose in this order: $D_{\mathrm{m}} = 2A + 40$; $T_{\mathrm{save}} > A + C_{\mathrm{tensor}} + 2D_{\mathrm{m}} + 40$; the saving $A + C_{\mathrm{tensor}}$ in Lemma 6.8 and the saving T_{save} in Lemma 6.9. Then fix the Siegel–Walfisz precisions that these require, the buffers (6.19) and B, and the constant $K_{\dim}$. Choose the common threshold last. The rows partition $T_1 + T_2 + T_{\mathrm{II}}$: T_1 is all of $a \le U$. The sums T_2 in c and T_{II} in $a > U$ are split at $N^{.46}$, $N^{.50}$ and $N^{.54}$. Each boundary goes to the interval on its right.

Direct rows. These are T_1 and the part of T_2 with $c \ge N^{.54}$. Fix the outer variable a, respectively t. Terms with $(a, q) > 1$, respectively $(t, q) > 1$, vanish in $\mathcal{K}_q$. Otherwise $\mathcal{K}_q(ac)$ is the kernel of (6.28) for c in the reduced class $N\bar{a} \bmod q$, and similarly for t. The parent is $H = N/a$, respectively $H = N/t$. The free variable runs over $H/4 < c < (1 - \varepsilon)H$, intersected with $c \ge N^{.54}$ in the T_2-row and with any prefix cut. This interval lies in $(0, 2H]$. Condition (W1) holds. For $a \le U$ we have $N^{2/3} \le H \le N$. If the right T_2-row is nonempty, then $N^{.54} < N/t = H \le N$, since $N^{.54} \le c < (1-\varepsilon)N/t$. Thus Lemma 6.2 applies with this H. (Equivalently, split at $H/2$ into $(H/4, H/2]$ and $(H/2, (1 - \varepsilon)H)$. Each lies inside a dyadic parent $(H', 2H']$ with $H' \in \{H/4, H/2\} \ge N^{.54}/4$. This only doubles the constant.) The weights $\log c/\log(ac)$ and $\mathcal{C}(t)/\log(tc)$ are bounded and monotone in c. Partial summation and (6.31) with $T_{\mathrm{save}} > A + 2$ bound both rows by $O(NL^{-A})$.

Convolution rows. These are T_2 with $c < N^{.46}$ (tested $\mathcal{R}(c)$, $c > N^{1/3}/4$), T_{II} with $a < N^{.46}$ (tested $\mathcal{A}(a)$, $a > N^{1/3}$), and T_{II} with $a \ge N^{.54}$ (tested $\mathcal{B}(s)/(2L)$, $N^{1/3} < s < 2N^{.46+O(1/L)}$). Each tested array contains the strict roughness factor $\mathcal{R}(n) = 1_{P^-(n) > z}$. This is explicit for $\mathcal{R}(c)$. It follows from $\mathcal{A}(a) = \mu(a)\mathcal{R}(a)$ and from $\mathcal{B}(s) = \mathcal{R}(s) \sum_{b|s,\ b>V} \Lambda(b)$. Lemma 6.4 gives (6.34) on these original parents. Write $\sum_q \Gamma(q)\Phi(q) = \sum_{r,s} F_H(r, s)\Phi(rs)$ as in (6.40). Lemma 6.6 places the joint support in (6.39). Discard a dyadic shell pair $r \sim U_1$, $s \sim V_1$ if F_H vanishes identically there. Otherwise a supported pair in it satisfies $U_1V_1 < rs < Q$, $U_1 < D_1$, and $V_1 < D_2$. Hence this occupied shell satisfies all scale conditions of Lemma 6.8. Lemma 6.7 expands F_H on the whole Cartesian shell into rank-one terms with $\sum_\nu |c_\nu| \ll L^{79}$. The unary caps and coprimality are kept. Each rank-one term is estimated on that whole shell. Its factors need not preserve the joint condition $rs < Q$ or the condition $(r, s) = 1$, since Lemma 6.8 imposes neither. Apply Lemma 6.8 with saving $A + C_{\mathrm{tensor}}$ to each term on each occupied shell pair and each product prefix $I = (N/4, u]$. Since $C_{\mathrm{tensor}} \ge 83$, the total is $O(NL^{-A})$, uniformly in u. The factor w is then restored by Stieltjes summation, as in (6.50).

Central rows. The four central rows are bounded by Lemma 6.10.

Proper powers. By (6.25) and (6.26), $P_{I_H}(\Gamma) = T_1 + T_2 + T_{\mathrm{II}} + O(N^{1/2+\nu}L^2)$. This proves (6.27). □

Proposition 6.13 (Sixth-sum estimate)**.** *Fix* σ *in the range stated above before taking* $N \to \infty$*. Then, along even integers,*

$$\frac{S_6}{W} \le \overline{J}_6 - \mathfrak{g}_{\mathrm{RA}} + o(1). \tag{6.71}$$

In particular, the certified floor in (6.70) *gives*

$$\frac{S_6}{W} \le \overline{J}_6 - g_{\mathrm{RA,lo}} + E_{\mathrm{RA}}(N; \sigma), \qquad E_{\mathrm{RA}}(N; \sigma) \ge 0, \qquad E_{\mathrm{RA}}(N; \sigma) \longrightarrow 0. \tag{6.72}$$

Proof. By Proposition 6.12(1), $S_6 \le A_{\mathrm{mix}}$. Use (6.14), obtained from (6.12) and (6.13). Fix $A > 2$, so that $NL^{-A} = o(W)$. The three low discrepancies are $O(NL^{-A-10})$ by Proposition 6.12(2). The term (6.15) is $O_A(NL^{-A})$. Also $P_{I_H}(\Gamma) = O_A(NL^{-A})$ by Proposition 6.12(3). Hence $S_6/W \le M_{\mathrm{mix}}/W + o(1)$, and (6.64) gives (6.71). Enlarge the vanishing error to make it nonnegative. The certified floor (6.70) gives (6.72). □

The precisions are fixed in the order (6.52). All constants precede the common threshold. Their dependence through prime Siegel–Walfisz may be ineffective. The final section uses only (6.72). It uses the certified rational floor in (9.2). Its gain is independent of the common coefficient used in the balanced-semiprime and switched comparisons.

7. The balanced semiprime term

We bound the complete occurrence sum $B_b(N)$. The two prime orientations keep their multiplicities. In this section, fix

$$\begin{gathered}\frac{1}{3}<b<\frac{1}{2},\qquad \varepsilon=\eta_L=10^{-4},\qquad u=\frac{1}{500},\qquad \mu_*=\frac{25}{128},\qquad L=\log N,\\ 0<\sigma\le 10^{-10},\qquad \tau=\sigma^9,\qquad \rho=1+\tau,\\ N'=N(1-L^{-333}),\\ \mathcal{N}_K=(1-\varepsilon)U_C=\frac{W(N)}{4},\\ I(b)=\int_b^{1/2}\frac{dt}{t(1-t)}\\ =\log\frac{1-b}{b}.\end{gathered}\tag{7.1}$$

The final argument uses (1.2). Every limit below fixes b and σ first. For row three, write $\ell_0=53/412-\eta_L/2=264897/2060000>0$. This is the fixed floor of Paper I, §3.1.

7.1. The convolution source.

Define

$$\begin{gathered}A_b(e)=\#\{(m,p')\in\mathbb{P}^2: N^b\le m\le\sqrt{N},\quad \varepsilon N<mp'=e\le N'\},\\ X_b=\sum_e A_b(e),\qquad B_b'=\sum_e A_b(e)1_{N-e\in\mathbb{P}},\\ E_{\mathrm{top}}=B_b-B_b',\qquad 0\le A_b(e)\le 2,\qquad 0\le E_{\mathrm{top}}\le 2(NL^{-333}+1).\end{gathered}\tag{7.2}$$

The last bound counts integers in the removed interval. A prime square contributes one occurrence. No new ordering or coprimality restriction is imposed.

For a finite sequence F, put

$$\Delta_F(q,a)=\sum_e F(e)\left(1_{e\equiv a\ (q)}-\frac{1_{(e,q)=1}}{\varphi(q)}\right),\qquad (a,q)=1.\tag{7.3}$$

Both terms use the full support. For this source, PNT and partial summation give

$$X_b=\frac{(1-\varepsilon)N}{L}I(b)+o_b(N/L).\tag{7.4}$$

This is the calculation of Paper II, [27, (5.44)–(5.51)]. The moving top removes at most $2(NL^{-333}+1)$ mass.

The required convolution estimate is

$$\sum_{q\le Q}\max_{(a,q)=1}|\Delta_{A_b}(q,a)|\ll_{A,b} NL^{-A},\qquad Q\le N^{1/2}L^{-B_{\mathrm{BV}}(A,b)},\tag{7.5}$$

for every fixed $A>0$. Its proof keeps the complete occurrence source and the moving top. Fix a top $T\in\{N,N'\}$. Before the discrepancy estimate, compare the exact cell sources (7.10) with their upper and lower convolutions. Each signed child keeps the natural coprimality comparator $1_{(e,q)=1}/\varphi(q)$. It is not replaced after summing the blocks. The step from $T=N$ to $T=N'$ costs the endpoint bands (7.11). It also costs the removed top mass in (7.2). The actual moving-top application is proved below.

We use the following convolution input. Let a_m,b_n have supports $m\sim M,n\sim Y$. Put $x=MY\ge 3$, and fix $\eta,B_{\mathrm{div}}>0$. Assume

$$\begin{gathered}x^\eta\le M,Y\le x^{1-\eta},\\ \left|\sum_{\substack{n\equiv l\ (k)\\(n,h)=1}}b_n-\frac{1}{\varphi(k)}\sum_{(n,hk)=1}b_n\right|\ll_{A_2}\|b\|_2Y^{1/2}d(h)^{B_{\mathrm{div}}}(\log(2Y))^{-A_2}\end{gathered}\tag{7.6}$$

uniformly for every fixed $A_2>0$, $h,k\ge 1$, and $(l,k)=1$. Here $d(n)$ is the divisor function. Then Fact 3.3 gives

$$\sum_{q\le Q}\max_{(a,q)=1}|\Delta_{a*b}(q,a)|\ll_A\|a\|_2\|b\|_2x^{1/2}(\log x)^{-A},\qquad Q\le x^{1/2}(\log x)^{-B_{\mathrm{BV}}}.\tag{7.7}$$

The constants depend on all family constants in (7.6). This specializes Fact 3.3. It is Paper II, Lemma 5.1 [27, (5.23)–(5.28)]. It applies to complete rectangular supports.

We make the interval reduction explicit. Fix a target $A > 2$, and put $H_g = A + 4$. Choose

$$R_g = \left\lceil \frac{(1/2-b)L}{\log(1+L^{-H_g})} \right\rceil, \qquad \xi = N^{(1/2-b)/R_g}, \qquad y_j = N^b \xi^j, \tag{7.8}$$
$$J_j = [y_{j-1}, y_j), \qquad Y_j = y_j.$$

Close the final upper endpoint. For the fixed top $T \in \{N, N'\}$, put

$$\begin{aligned} b_j(m) &= 1_{m\in\mathbb{P}\cap J_j}, \\ a^+_{j,T}(p') &= 1_{p'\in\mathbb{P}} 1_{\varepsilon N/Y_j \le p' \le T\xi/Y_j}, \\ a^-_{j,T}(p') &= 1_{p'\in\mathbb{P}} 1_{\varepsilon N\xi/Y_j < p' < T/Y_j}. \end{aligned} \tag{7.9}$$

For F, G on the positive integers write $(F*G)(e) = \sum_{mn=e} F(m)G(n)$, and put $A^\pm_{j,T} = a^\pm_{j,T} * b_j$. The exact cell source is

$$A_{j,T}(e) = 1_{\varepsilon N < e \le T} \sum_{\substack{m\in\mathbb{P}\cap J_j,\ p'\in\mathbb{P} \\ mp'=e}} 1. \tag{7.10}$$

The definitions give the pointwise envelopes $A^-_{j,T} \le A_{j,T} \le A^+_{j,T}$. Their difference is supported in multiplicative boundary bands at εN and T. Put $\mathcal{E}_Q(F) = \sum_{q\le Q} \max_{(a,q)=1} |\Delta_F(q,a)|$. Let $B_T(e) = \sum_j (A^+_{j,T}(e) - A^-_{j,T}(e))$. Then $B_T \ge 0$, with support in $[\varepsilon N/\xi, \varepsilon N\xi] \cup [T/\xi, T\xi]$. Each integer has at most two ordered factorizations into primes. Since $1 < \xi \le 1 + L^{-H_g}$ and $T \in \{N, N'\}$ satisfies $T \le N$,

$$\sum_e B_T(e) \le 2\{\varepsilon N(\xi - \xi^{-1}) + T(\xi - \xi^{-1}) + 4\} \ll NL^{-H_g} + 1.$$

Since $\xi - \xi^{-1} \le 2(\xi - 1) \le 2L^{-H_g}$, counting integers in the two complete boundary bands gives the displayed mass bound. Paper II, (5.47), gives the aggregate boundary discrepancy

$$\mathcal{E}_Q\left(\sum_j (A^+_{j,T} - A^-_{j,T})\right) \ll NL^{-H_g+1} + Q. \tag{7.11}$$

The Q term includes the bounded-multiplicity changes at exact endpoints. Split the second factor into dyadic blocks $(M, 2M]$:

$$a^\pm_{j,T,M}(p') = a^\pm_{j,T}(p') 1_{M < p' \le 2M}.$$

For each block meeting the support, $\varepsilon N/(2Y_j) < M \le T\xi/Y_j$, so $M \asymp_\varepsilon N/Y_j$, with only $O_\varepsilon(1)$ blocks. Here $n \sim Z$ means $Z < n \le 2Z$, as in Section 3. For the short factor, the theorem-scale parent is $Y = Y_j/2$; since $\xi < 2$ eventually, $J_j \subset (Y_j/2, Y_j]$ lies in that parent. The norms are

$$\|a^+_{j,T,M}\|_2 \ll \sqrt{M}, \qquad \|a^-_{j,T,M}\|_2 \le \|a^+_{j,T,M}\|_2, \qquad \|b_j\|_2 \ll \sqrt{Y_j} L^{-(H_g+1)/2}. \tag{7.12}$$

The prime number theorem on these intervals also gives a matching lower bound. It is $\|b_j\|_2 \gg \sqrt{Y_j} L^{-(H_g+1)/2}$ for all sufficiently large N. This lower bound normalizes the conversion of the absolute prime Siegel–Walfisz error to (7.6). The prime-interval form of (7.6) holds uniformly in j, T, M. For a fixed target $A > 2$, write $K_{\mathrm{SW}}(A)$ for the chosen logarithmic level. For auxiliary moduli $k \le L^{K_{\mathrm{SW}}(A)}$, this is prime Siegel–Walfisz at arbitrarily high fixed logarithmic precision. The divisor loss from excluding primes dividing the auxiliary integers is absorbed into $d(h)^{B_{\mathrm{div}}}$. For $k > L^{K_{\mathrm{SW}}(A)}$, use

$$|\Delta| \le |J|/k + 1 + \pi(J)/\varphi(k)$$

and $\sup_{k > L^{K_{\mathrm{SW}}(A)}} \varphi(k)^{-1} \ll L^{-K_{\mathrm{SW}}(A)} \log\log(3L^{K_{\mathrm{SW}}(A)})$, taking $K_{\mathrm{SW}}(A)$ larger than the requested saving. For every fixed target $A > 2$, choose $H_g = A + 4$. The grid has $O_b(L^{H_g+1})$ intervals. The prime number theorem gives the norm bounds (7.12). With $A_{\mathrm{block}} = A + H_g/2 + 2$, Fact 3.3 and Paper II, (5.23)–(5.28), bound the rectangular blocks by $O_{A,b}(NL^{-A-3/2})$. The boundary term in (7.11) is $O(NL^{-A-3} + Q)$. Increasing the single fixed logarithmic level buffer absorbs the $x \asymp_b N$ constants. Thus the complete moving-top source, including the actual top $T = N'$, satisfies (7.5). No fixed-width truncation estimate is substituted.

7.2. **Four directed sieve rows.** Let $\mathcal{P}_N = \{p \in \mathbb{P} : p \nmid 2N\}$. Put

$$S_b(d;z) = \sum_{d \mid N-e} A_b(e)\, 1_{((N-e)/d,\, P_N(z))=1}, \qquad P_N(z) = \prod_{p<z,\ p\in\mathcal{P}_N} p. \tag{7.13}$$

There is no prefix–cofactor coprimality condition. Both source primes exceed N^{μ_*} eventually. Thus, for every supported modulus d,

$$\sum_{d \mid N-e} A_b(e) = \frac{X_b}{\varphi(d)} + \Delta_{A_b}(d,N), \qquad (d,2N)=1, \quad P^+(d) < N^{\mu_*}. \tag{7.14}$$

Here $P^+(d)$ is the largest prime factor, with $P^+(1)=1$. For the squarefree sieve divisors, use the multiplicative density

$$g(p) = \frac{1}{p-1} \quad (p \in \mathcal{P}_N), \qquad g(p^j) = 0 \quad (j \ge 2), \qquad g(p) = 0 \quad (p \notin \mathcal{P}_N). \tag{7.15}$$

This is the squarefree-divisor density of the linear-sieve application. That input requires its zero prime-power tail. The full arithmetic density is $g(p^j) = 1/\varphi(p^j)$ for $p \in \mathcal{P}_N$ and $j \ge 1$. No estimate uses its prime-power tail. The two densities agree at active primes. These are the only local factors used by the main-mass calculation. In Paper I, Theorem 2.6 notation, this density is

$$\omega(p) = \frac{p}{p-1} \quad (p \in \mathcal{P}_N), \qquad \omega(p) = 0 \quad (p \notin \mathcal{P}_N), \qquad \omega(p^a) = 0 \quad (a \ge 2).$$

Thus $\omega(p)/p = g(p)$, and condition (2) has zero left side. For $p \ge 3$,

$$\left(1 - \frac{1}{p-1}\right)^{-1} = \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{1}{p(p-2)}\right).$$

The product of the second factors over $p \ge w$ is $1 + O(1/w) = 1 + O(1/\log w)$. Hence Mertens' product estimate gives condition (1) with exponent one and one fixed $K_{\dim} \ge 2$. We may take $L_\omega = 2$. These are the dimension and prime-power inputs of Iwaniec's linear sieve [13].

If $N - e$ is prime, it is at least NL^{-333}. It therefore exceeds N^{μ_*} eventually. Hence

$$B_b' \le S_b(1; N^{\mu_*}). \tag{7.16}$$

Use the ordered primes $N^u \le a_j < \cdots < a_1 < N^{\mu_*}$ in $\mathcal{P}_N$. The exact decomposition of (4.26)–(4.30) gives

$$B_b \le E_{\mathrm{top}} + T_0 - T_1 + T_2 - R_3 + P_3. \tag{7.17}$$

Here

$$\begin{aligned} T_0 &= S_b(1; N^u), \\ T_j &= \sum_{N^u \le a_j < \cdots < a_1 < N^{\mu_*}} S_b(a_1 \cdots a_j; N^u) \quad (j = 1, 2), \\ R_3 &= \sum_{\mathrm{ret}} S_b(a_1 a_2 a_3; N^{t_3}), \\ P_3 &= \sum_{\mathrm{ret}} \sum_{\substack{N^{t_3} \le r < a_3 \\ r \in \mathcal{P}_N}} S_b(a_1 a_2 a_3 r; N^{t_3}). \end{aligned} \tag{7.18}$$

Every prime in these sums belongs to $\mathcal{P}_N$. The last sum keeps the half-open cells it inherits. The symbol $\sum_{\mathrm{ret}}$ restricts by

$$\Theta_{\mathfrak{c}} = \inf_{\bar{\mathfrak{c}}} H, \qquad \ell_{\mathfrak{c}} = \Theta_{\mathfrak{c}} - \eta_L - t_1 - t_2 - t_3, \qquad \ell_{\mathfrak{c}} \ge 2\rho t_3. \tag{7.19}$$

Here H is the full eight-branch packing function, (3.29). The infimum uses the complete closed ordered cell. It does not use a quadrature node.

The four directed applications are

term	row sign	sieve direction	level exponent
T_0	$+$	upper F	$1/2 - \eta_L$
T_1	$-$	lower f	$1/2 - \eta_L - \rho t_1$
T_2	$+$	upper F	$1/2 - \eta_L - \rho(t_1 + t_2)$
R_3	$-$	lower f	$\ell_{\mathfrak{c}}$

(7.20)

The first three cutoffs are N^u. The fourth cutoff is N^{t_3}. The canonical functions are Iwaniec's linear-sieve functions [13]. Extend f by zero for $s \le 2$. No shifted beta-sieve profile is substituted.

For each prefix product h, let $\mathcal{A}_h$ be the cofactor sequence indexed by $n = (N-e)/h$, with multiplicity $A_b(e)$ when $h \mid N - e$. Put $X_h = X_b/\varphi(h)$. Write z_h for the row cutoff, D_h for its level, and $s_h = \log D_h / \log z_h$. Every

prefix prime is at least z_h, and each squarefree sieve modulus $t \mid P_N(z_h)$ has prime factors below z_h. Thus $(h,t)=1$, and the local identity (7.14) gives the exact remainder identity

$$r(\mathcal{A}_h,t)=\Delta_{A_b}(ht,N). \tag{7.21}$$

Since $\omega(t)/t=1/\varphi(t)$ and $(h,t)=1$, the local main term is $X_h\omega(t)/t=X_b/\varphi(ht)$, as required by Theorem 2.6. For rows $j=0,1,2$, put $S=t_1+\cdots+t_j$, with $S=0$ for row zero, and $\theta=1/2-\eta_L$. The clipped exponent cells give

$$h\le N^{\rho S},\qquad D_h=N^{\theta-\rho S},\qquad S\le 2\mu_*.$$

Hence $\log D_h/\log N\ge\theta-2\rho\mu_*>0.1092>2u$. The upper rows satisfy $N^u\le D_h$. The lower row satisfies $N^u\le D_h^{1/2}$. For sufficiently large N, with σ fixed, $D_h\ge 2$. Apply Paper I, Theorem 2.6, (2.30), with density $g(p)=1/(p-1)$ on $\mathcal{P}_N$, $g(p^a)=0$ for $a\ge 2$, and sieve parameter $\epsilon_s=\sigma$. With $\overline{E}=2C_{\mathrm{AA}37}\sigma$, the resulting directed inequalities are

$$\begin{aligned}
S(\mathcal{A}_h,z_h)&\le X_hV_N(z_h)\{F(s_h)+E_{s_h}^{+}\}+R_1^{+}(h)+R^{+}(h),\\
S(\mathcal{A}_h,z_h)&\ge X_hV_N(z_h)\{f(s_h)-E_{s_h}^{-}\}-R_1^{-}(h)-R^{-}(h),\\
|E_{s_h}^{\pm}|&\le\overline{E}.
\end{aligned} \tag{7.22}$$

Only the upper bound $|E_{s_h}^{\pm}|\le\overline{E}$ is used. No sign of $E_{s_h}^{\pm}$ is assumed. Since $X_hV_N(z_h)\ge 0$, replacing $E_{s_h}^{\pm}$ by $\overline{E}$ weakens each inequality. The actual dimension and prime-power constants, the level lower bound, and the fixed-σ threshold are in (7.40).

The superscript AA below marks errors from this linear-sieve theorem. It does not denote a separate family of profiles. For row j, sum over its actual prefix set $\mathcal{H}_j$. Let $\mathcal{F}_j$ be F on the upper rows and f on the lower rows. The definitions used in the signed equalities are

$$\begin{aligned}
m_j&=\sum_{h\in\mathcal{H}_j}X_hV_N(z_h)\mathcal{F}_j(s_h),\\
\mathfrak{M}_j&=\sum_{h\in\mathcal{H}_j}X_hV_N(z_h),\\
\mathcal{A}_j^{\mathrm{AA}}&=\overline{E}\,\mathfrak{M}_j,\\
r_j&=\sum_{h\in\mathcal{H}_j}\{R_1^{\nu_j}(h)+R^{\nu_j}(h)\},\qquad \nu_j=+\text{ on upper rows},\quad \nu_j=-\text{ on lower rows},\\
z_j&=m_j+\mathcal{A}_j^{\mathrm{AA}}+r_j-T_j\quad\text{on upper rows},\\
z_j&=T_j-m_j+\mathcal{A}_j^{\mathrm{AA}}+r_j\quad\text{on lower row } j=1,\\
z_3&=R_3-m_3+\mathcal{A}_3^{\mathrm{AA}}+r_3.
\end{aligned} \tag{7.23}$$

Summing (7.22) over prefixes proves $z_j\ge 0$. Also $\mathfrak{M}_j\ge 0$, so $\mathcal{A}_j^{\mathrm{AA}}\ge 0$. These nonnegativity statements are proved at source level for the exact signed remainders in (7.21). No sign of these remainders is assumed. Write the directed equalities as

$$\begin{aligned}
T_0&=m_0+\mathcal{A}_0^{\mathrm{AA}}+r_0-z_0,\quad T_1=m_1-\mathcal{A}_1^{\mathrm{AA}}-r_1+z_1,\\
T_2&=m_2+\mathcal{A}_2^{\mathrm{AA}}+r_2-z_2,\quad R_3=m_3-\mathcal{A}_3^{\mathrm{AA}}-r_3+z_3,\qquad \mathcal{A}_j^{\mathrm{AA}},z_j\ge 0.
\end{aligned} \tag{7.24}$$

Lemma 7.1 (Signed row identity)**.** *With $M_K=m_0-m_1+m_2-m_3$,*

$$\Delta_{\mathrm{arith}}:=T_0-T_1+T_2-R_3-M_K=\sum_{j=0}^{3}\mathcal{A}_j^{\mathrm{AA}}+\sum_{j=0}^{3}r_j-\sum_{j=0}^{3}z_j,$$

and every $\mathcal{A}_j^{\mathrm{AA}}$ occurs with coefficient $+1$.

Proof. Substitute (7.24). The lower rows carry the row sign $-$, which turns $-\mathcal{A}_j^{\mathrm{AA}}-r_j+z_j$ into $+\mathcal{A}_j^{\mathrm{AA}}+r_j-z_j$. □

For rows $j=0,1,2$, the total moduli in these remainders fit the level of (7.5). A tail tuple satisfies $\prod_i D_i<D_h$. In the parity ending in a defining cubic condition, remove two copies of the last label. In the other parity, use the preceding cubic condition and the ordering of the last two labels. The empty and one-label cases follow from the tuple definitions. Each p_i lies in the interval $D_i\le p_i<\min(D_i^{\rho},z_h)$, and $d<D_h^{\sigma}$. Thus, with the same S,θ as above,

$$q=hd\prod_i p_i\le hD_h^{\sigma+\rho}\le N^{\rho S+(\sigma+\rho)(\theta-\rho S)}=N^{\theta+(\sigma+\tau)(\theta-\rho S)}\le N^{\theta+(\sigma+\tau)\theta}<N^{1/2-\eta_L/2}.$$

This power buffer is below $N^{1/2}L^{-B}$ for every fixed B, once N is large enough.

The factors of such a modulus that are at least z_h are exactly the ordered prefix primes. They determine h and its clipped cell. Factoring the remaining part at $D_h^{\sigma^2}$ determines the squarefree smooth divisor and the tail-prime multiset. The number of tail labels is at most $J_\sigma=\lceil\sigma^{-2}\rceil$. Equal-label permutations contribute at most $J_\sigma!$. Each

AA coefficient has modulus at most one. The number of label vectors is less than $\exp(8\sigma^{-3})$. Hence the total coefficient at each modulus is at most $C_{\mathrm{K2},\sigma} = 1 + \exp(8\sigma^{-3})J_\sigma!$, independent of N. Apply (7.5) with target $A = 4$. Divide by $\mathcal{N}_K = (1-\varepsilon)C(N)N/L^2$; here $C(N) \ge C_2 > 0$ by Paper II, (1.2). This proves

$$\frac{|r_0| + |r_1| + |r_2|}{\mathcal{N}_K} \ll_{b,\sigma} L^{-2}. \tag{7.25}$$

The low-row bound uses Fact 3.3, the prime Siegel–Walfisz theorem, the prime number theorem and the Paper II convolution inputs. It gives no estimate for the principal masses or row three.

The m_j are the directed main terms. The r_j are signed arithmetic remainders. Thus, with $M_K = m_0 - m_1 + m_2 - m_3$,

$$\begin{aligned} B_b &\le E_{\mathrm{top}} + P_3 + M_K + \sum_{j=0}^{3} \mathcal{A}_j^{\mathrm{AA}} + \sum_{j=0}^{3} r_j - \sum_{j=0}^{3} z_j \\ &\le E_{\mathrm{top}} + P_3 + M_K + \sum_{j=0}^{3} \mathcal{A}_j^{\mathrm{AA}} + \sum_{j=0}^{3} |r_j|. \end{aligned} \tag{7.26}$$

This step is finite algebra. The directed source comparisons are proved separately.

The canonical row masses are explicit. For a prefix cell $\mathfrak{c}$, put

$$H_j(\mathfrak{c}) = \sum_{\substack{a_1 > \cdots > a_j \\ a_i \text{ in the cell}}} \frac{1}{\varphi(a_1 \cdots a_j)}.$$

Since the prefix primes exceed every sieve prime, the product over sieve primes at each row is $V_N(z_h)$. With the ordered cell sets D_j from (7.31),

$$\begin{aligned} m_0 &= X_b V_N(N^u) F(s_0), \\ m_1 &= X_b V_N(N^u) \sum_{\mathfrak{c} \in D_1} H_1(\mathfrak{c}) f(s_1(\mathfrak{c})), \\ m_2 &= X_b V_N(N^u) \sum_{\mathfrak{c} \in D_2} H_2(\mathfrak{c}) F(s_2(\mathfrak{c})), \\ m_3 &= X_b \sum_{\mathfrak{c} \in D_3} \chi_{\mathfrak{c}} V_N(N^{t_3}) H_3(\mathfrak{c}) f(\ell_{\mathfrak{c}}/t_3). \end{aligned} \tag{7.27}$$

The masses $\mathfrak{M}_j$ are these sums with F and f replaced by 1. In particular, the row-three retention indicator $\chi_{\mathfrak{c}}$ appears in both sums.

7.3. **The principal functional.** We display the numerical functional used for M_K. Set

$$\begin{aligned} v_1(s) &= \min\left\{ \frac{1+s}{2}, \frac{32-34s}{39}, \frac{57}{64} - \frac{3}{2}s \right\}, \\ v(s) &= v_1(s) - u, \qquad \psi(x,y) = x 1_{x \ge y}, \\ \Theta_*(x,y,z) &= \max\{v(x), v(y), v(x+y), v(x+y+z), \\ &\qquad \psi(v(x+z), x+2y+z), \psi(v(y+z), 2x+y+z)\}. \end{aligned} \tag{7.28}$$

The subtraction of u precedes both guards. This is Paper II, (4.70)–(4.72) [27, (4.70)–(4.72)]. Put

$$\begin{aligned} H_{\log} &= \log(\mu_*/u), \\ R_{\mathrm{up}} &= 500(F(30)-1) + 500 H_{\log}(1 - f(30)) + 250 H_{\log}^2 (F(30)-1), \\ J_* &= \int_u^{\mu_*} \frac{dx}{x} \int_u^x \frac{dy}{y} \int_u^y \frac{1 - f((\Theta_*(x,y,z) - x - y - z)/z)}{z^2}\, dz, \\ G_* &= \frac{1}{\mu_*} + R_{\mathrm{up}} + J_*. \end{aligned} \tag{7.29}$$

We keep the true delay-function tail. Paper II, (4.73)–(4.75), gives the constant cancellation

$$\frac{1 - H_{\log} + H_{\log}^2/2}{u} - \int_u^{\mu_*} \frac{dx}{x} \int_u^x \frac{dy}{y} \int_u^y \frac{dz}{z^2} = \frac{1}{\mu_*}. \tag{7.30}$$

The finite-parent comparison also needs its direction. Fix a finite clipped partition Π of $[u, \mu_*)$. Its cells have endpoint ratios at most ρ. Each clipped cell is a product of half-open exponent intervals $[t_i, t_i')$ with $t_i' \le \rho t_i$. Here t_i is the lower endpoint, so $N^{t_i} \le a_i < N^{\rho t_i}$, i.e. $t_i \le x_i < \rho t_i$. The closure $\bar{\mathfrak{c}}$ is used only for infima and lower

bounds. The cutoffs N^{t_3} in R_3, P_3 and the factors $\rho t_1, \rho(t_1+t_2)$ in s_1, s_2 use this convention. For a clipped cell $\mathfrak{c}$, let $w_{\mathfrak{c}} = \int_{\mathfrak{c}} \prod_i dx_i/x_i$. Define

$$\begin{aligned} s_0 &= (1/2-\eta_L)/u, \\ s_1(\mathfrak{c}) &= (1/2-\eta_L-\rho t_1)/u, \\ s_2(\mathfrak{c}) &= (1/2-\eta_L-\rho(t_1+t_2))/u, \\ E_\Pi &= \frac{F(s_0)-1}{u} + \sum_{\mathfrak{c}\in D_1} \frac{w_{\mathfrak{c}}}{u}(1-f(s_1(\mathfrak{c}))) \\ &\quad + \sum_{\mathfrak{c}\in D_2} \frac{w_{\mathfrak{c}}}{u}(F(s_2(\mathfrak{c}))-1), \\ P_\Pi &= \frac{1-H_{\log}+H_{\log}^2/2}{u} + E_\Pi - \sum_{\mathfrak{c}\in D_3} w_{\mathfrak{c}} 1_{\ell_{\mathfrak{c}}\ge 2\rho t_3} \frac{f(\ell_{\mathfrak{c}}/t_3)}{t_3}. \end{aligned} \tag{7.31}$$

Here $D_j = \{u \le x_j < \cdots < x_1 < \mu_*\}$. Each cell is intersected with this ordered domain. Its weight includes the actual ordered multiplicity. The required comparisons are

$$0 \le E_\Pi \le R_{\mathrm{up}}, \qquad P_\Pi \le \frac{1}{\mu_*} + E_\Pi + J_* \le G_*. \tag{7.32}$$

The last line is an upper comparison. It asserts no equality between finite sums and integrals.

Compare the six canonical branches of Θ_* with their matching branches among the eight parent branches of the packing function H in (3.29). Adding u to an unguarded canonical branch gives the matching parent branch. For an active canonical guarded branch, the parent guard is also active, and adding u again gives the matching parent branch. An inactive canonical guarded branch is zero and cannot maximize the canonical ceiling, since its first unguarded branch is positive on the ordered domain. The two extra parent branches can only increase H. Hence $H(x,y,z) \ge \Theta_*(x,y,z)+u$ pointwise on that domain. On a closed anchored cell, each coordinate moves by at most $\mu_*\tau$. Fix a point and a maximizing canonical branch there, and use its matching parent branch. If it is guarded, its guard has reserve $u - 7\mu_*\tau > 0$ throughout the cell. This selected branch changes by at most $(9/2)\mu_*\tau$ across the cell. Therefore

$$\begin{aligned} \Theta_{\mathfrak{c}} &\ge \Theta_*(x,y,z) + u - (9/2)\mu_*\tau, \\ \ell_{\mathfrak{c}} &\ge \Theta_*(x,y,z) - x - y - z + g, \\ g &= u - \eta_L - (9/2)\mu_*\tau > 2\mu_*\tau. \end{aligned} \tag{7.33}$$

Monotonicity of $f(s)$, together with this reserve, gives the directed kernel comparison

$$\chi_{\mathfrak{c}} \frac{f(\ell_{\mathfrak{c}}/t_3)}{t_3} \ge \frac{f((\Theta_*(x,y,z)-x-y-z)/z)}{z}. \tag{7.34}$$

Summing over cells gives the upper comparison in (7.32). It does not identify a finite sum with the integral.

Put $[x]_+ = \max\{x, 0\}$. We need the arithmetic-to-integral estimate

$$\frac{[M_K - \mathcal{N}_K\, 2e^{-\gamma} I(b) G_*]_+}{\mathcal{N}_K} = o_{b,\sigma}(1). \tag{7.35}$$

The quantity M_K contains only the canonical values of F and f. The linear-sieve errors $E_{s_h}^{\pm}$ and the remainders r_j lie outside it. They are paid in (7.38) and (7.47). Fix (b,σ). Three inputs enter. The first is the two-orientation arithmetic-mass estimate for the source (7.2). The second is its four signed canonical AA main masses (7.27). The third is the finite functional P_Π in (7.31). For all sufficiently large even N they give

$$M_K/\mathcal{N}_K \le 2e^{-\gamma} I(b) G_* + o_{b,\sigma}(1).$$

Here the error tends to zero with b, σ and the cell count fixed, as N tends to infinity. This estimate is for the finite functional in (7.31). It gives (7.35) directly, so $N_{\mathrm{mass}}(b,\sigma)$ can be the threshold of this fixed-(b,σ) statement. This bound controls only the principal-mass term. It does not control the AA remainders, the optional numerical certificate, the complete exterior comparison or the endpoint terms. No sharper arithmetic-mass estimate is used here.

For the exterior comparison we use the common G_0 coefficient

$$\begin{aligned} G_0 &= \frac{6088440363}{10^9}, \qquad G_* \le G_0, \\ 2e^{-\gamma} G_0 &\le c_{K2,G0} := \frac{6836825163875517756286846321650754372931094689}{10^{45}}. \end{aligned} \tag{7.36}$$

For $G_* \le G_0$, use the tight endpoint $J_* \le 0.967120$ of Paper II, (A.10a). It holds for the exact profile (7.29) [27, (4.73), (4.75), (A.10a)]. Its weaker displayed endpoint is not used. The Taylor-model delay certificate is used for this coefficient comparison and is not an input to the shifted replay itself. It gives the explicit bounds

$$0 \le F(30) - 1 < 3 \cdot 10^{-49}, \qquad 0 \le 1 - f(30) < 3 \cdot 10^{-49}.$$

Since $0 < H_{\log} = \log(3125/32) < 5$, (7.29) then gives

$$R_{\mathrm{up}} < (500 + 500 \cdot 5 + 250 \cdot 25)3 \cdot 10^{-49} < 10^{-44}, \qquad G_* < \frac{128}{25} + \frac{967120}{10^6} + 10^{-44} < \frac{6088440363}{10^9} = G_0.$$

The finite delay calculation and this rational budget do not replace the source-to-profile comparison or its fixed-parameter hypotheses. For the scalar inequality, define

$$\begin{gathered} \gamma_- := \frac{57721566490153286060651209008240243104215933593992}{10^{50}}, \\ \gamma_+ := \frac{57721566490153286060651209008240243104215933593993}{10^{50}}, \\ \gamma_- < \gamma < \gamma_+, \\ c_{K2,G0} - 2G_0 \sum_{j=0}^{60} \frac{(-\gamma_-)^j}{j!} > \frac{75307}{10^{50}} > 0. \end{gathered} \tag{7.37}$$

The rational enclosure follows from Euler–Maclaurin for H_n at $n = 10^5$, through B_{16}. The next-term remainder is bounded by $|B_{18}|/(18n^{18})$. Both $\log n$ and H_n are enclosed outwards by the certified exact-input record. For $0 < x < 1$, the even Taylor partial sum of e^{-x} is an upper bound. Thus the exact rational comparison in (7.37) proves $2e^{-\gamma}G_0 < c_{K2,G0}$.

The switched comparison uses the same coefficient. At $b = 2999993/7000000$, the common final-margin inequality (9.24) has slack $0.000228954311267289\ldots > 1/5000$ on the stated route $g_6 := 0$. Both comparisons use the same G_0 bound. Scalar positivity does not establish the required source estimates.

7.4. **The complete error bound.** The analytic hypotheses are stated below. The sums are over the original applications. The four principal linear-sieve errors satisfy

$$\begin{gathered} \frac{\sum_{j=0}^{3} \mathcal{A}_j^{\mathrm{AA}}}{\mathcal{N}_K} \le A_K(b) C_{\mathrm{AA}} \sigma, \\ A_K(b) = 2e^{-\gamma} I(b) W_{\mathrm{AA}}, \qquad W_{\mathrm{AA}} = 500 \left(1 + H_{\log} + \frac{H_{\log}^2}{2} + \frac{H_{\log}^3}{6}\right). \end{gathered} \tag{7.38}$$

Let C_{AA37} be the absolute constant in Paper I, Theorem 2.6, (2.31), and define

$$C_{\mathrm{AA}} = 4C_{\mathrm{AA37}}. \tag{7.39}$$

The coefficient four is the factor two in (7.43) times the factor two in the common bound $|E_s^{\pm}| \le 2C_{\mathrm{AA37}}\sigma$. The sum over the four rows is already included in W_{AA}. It is independent of σ, N. It is unrelated to the global tensor constant in (6.40). All four actual levels satisfy $D \ge N^{1/250}$. We use (2.31) with the actual fixed dimension constant $K_{\dim}$. For the squarefree density in (7.15), the left side of (2.27) is zero. Hence the actual prime-power constant is $L_\omega = 2$. This value is used in the exponential below.

$$|E_s^{\pm}| \le C_{\mathrm{AA37}}\{\sigma + \sigma^{-8} e^{K_{\dim}+2} (\log D)^{-1/3}\} \le 2C_{\mathrm{AA37}}\sigma \tag{7.40}$$

once $L \ge 250 \exp(3(K_{\dim} + 2))\sigma^{-27}$. Name the corresponding even threshold

$$N_{\mathrm{AA,dec}}(\sigma) := 2 \left\lceil \frac{1}{2} \exp\bigl(250 \exp(3(K_{\dim} + 2))\sigma^{-27}\bigr) \right\rceil.$$

It is chosen at fixed σ. Every $N \ge N_{\mathrm{AA,dec}}(\sigma)$ satisfies the displayed decay bound. The four ordered harmonic row masses, before the common Euler and source factor, are bounded by

$$\begin{gathered} \frac{1}{u}, \qquad \frac{H_{\log}}{u}, \qquad \frac{H_{\log}^2}{2u}, \\ \text{row-3 mass} \le \rho \int_u^{\mu_*} \frac{dx}{x} \int_u^x \frac{dy}{y} \int_u^y \frac{dz}{z^2} \le \frac{\rho H_{\log}^2}{2u} < \frac{H_{\log}^3}{6u}. \end{gathered} \tag{7.41}$$

For the last inequality use $3\rho < H_{\log} = \log(3125/32)$. The factor ρ allows the anchored-bin endpoint enlargement. The first three identities are the volumes of ordered logarithmic simplexes. For the fourth, bound $\int_u^y dz/z^2 \le 1/u$ and integrate over the ordered (x, y) simplex. The sum of the four displayed upper envelopes is W_{AA}, the four-row

envelope used here. It is distinct from Paper II's coarser global sieve-error bound. Let $H = H_{\log} = \log(3125/32)$. PNT and Mertens' theorems, uniformly over the finitely many fixed-σ cells, give

$$V_N(N^a) = \frac{2e^{-\gamma}C(N)}{aL}(1+o(1)), \qquad H_j(\mathfrak{c}) = w_{\mathfrak{c}} + o(1).$$

For the retained row, $1/t_3 \le \rho/x_3$ on each cell. Dropping the retention restriction only increases its mass. Consequently

$$\begin{aligned} \frac{\sum_{j=0}^{3}\mathfrak{M}_j}{\mathcal{N}_K} &\le 2e^{-\gamma}I(b)\left\{\frac{1}{u}+\frac{H}{u}+\frac{H^2}{2u}+\rho J_4\right\}+o(1), \\ J_4 &= \frac{H^2/2-H+1}{u}-\frac{1}{\mu_*} \le \frac{H^2}{2u}. \end{aligned} \tag{7.42}$$

The bracket is smaller than $W_{\mathrm{AA}} = u^{-1}(1+H+H^2/2+H^3/6)$, with relative gap greater than 0.2842 for $0 < \sigma \le 10^{-10}$. The coarser bound $\rho J_4 \le \rho H^2/(2u)$ still leaves relative gap greater than 0.1723. Consequently, for each fixed b, σ, there is a threshold $N_{\mathrm{AA,dens}}(b,\sigma)$ such that, for every even $N \ge N_{\mathrm{AA,dens}}(b,\sigma)$, $\sum_j \mathfrak{M}_j \le \mathcal{N}_K\, 2e^{-\gamma}I(b)W_{\mathrm{AA}}$. Define

$$N_{\mathrm{AA}}(b,\sigma) := \max\{N_{\mathrm{AA,dens}}(b,\sigma), N_{\mathrm{AA,dec}}(\sigma)\}.$$

Thus both the density estimate and (7.40) hold above $N_{\mathrm{AA}}(b,\sigma)$, as required for (7.38). The maxima defining T_K and T_r below include this combined threshold. The displayed source-density bound below is weaker by a factor of two. Let $\mathfrak{M}_j$ be the nonnegative principal mass multiplying the relative sieve error in row j. The resulting bound is

$$\sum_{j=0}^{3}\mathfrak{M}_j \le 2\mathcal{N}_K\, 2e^{-\gamma}I(b)W_{\mathrm{AA}}. \tag{7.43}$$

Multiplying it by $2C_{\mathrm{AA}37}\sigma$ gives (7.38) with (7.39). The Euler factors and the actual-source mass identification are included above. The secondary decay is paid once.

The positive peel is bounded by the two-form argument of Paper II, [27, (4.46)–(4.56)], applied to the original source. Put $\mathcal{B}_m = \{e : \ m \mid N-e\}$ with multiplicity $A_b(e)$. For $m = a_1a_2a_3r$ in P_3, put $E = N^{t_3}$ and $w = N^u \le E$. Write $A_b = \sum_{h=1}^{2} 1_{\{A_b \ge h\}}$. Every e in the support is a product of two primes at least N^b. Hence e is odd and w-rough. Write $N - e = mk$. Then k is odd. A prime $p < w$ dividing k and N would divide e, which is impossible. Hence both forms k and $N - mk$ are free of all primes below w. Therefore $S_b(m;E)$ is at most the two-form Selberg majorant of [27, (4.48)–(4.53)] on $1 \le k \le N/m$, once for each layer. All prime factors of m are at least $E \ge w$ and lie in $\mathcal{P}_N$. Hence $(m,N) = 1$, and $\nu_N(p)$ is as in [27, (4.48)]. Retention gives, as in [27, (4.55)],

$$\log_N m \le \rho(t_1+t_2+2t_3) \le \rho(\Theta_{\mathfrak{c}} - \eta_L) < \tfrac{153}{256} + 2\tau, \qquad \log_N(w^2m) < 1 - 0.0116, \qquad \log_N w = u \ge \tfrac{2}{1001}. \tag{7.44}$$

Consequently [27, (4.47), (4.53), (4.54)] give, eventually and uniformly in m,

$$S_b(m;E) \le 2\cdot\frac{4.002}{u^2}\cdot\frac{U_C}{m} \le \frac{2008008\,U_C}{m}, \qquad \max_h M_h \le 2\tau, \qquad \sum_h M_h \le \frac{459}{100}, \tag{7.45}$$

since one layer is at most $(N/m)/\mathcal{G}_N(w) + O(w^2\log^5 w)$ with $\mathcal{G}_N(w) = (1+o(1))u^2L^2/(4C(N))$, and $4.002/u^2 = 1000500 < 1004004 = 2\cdot 502002$. The bin sums are [27, (4.56)]: $\log(\mu_*/u) = \log(3125/32) < 4.59$, and a_3, r lie in one parent bin. Therefore

$$\frac{P_3}{U_C} \le 2008008\cdot 2\left(\frac{459}{100}\right)^3\tau = \frac{24272444031579}{62500}\tau < 388400000\,\sigma^9, \tag{7.46}$$

and $P_3/\mathcal{N}_K \le P_K\sigma^9$ with $P_K = 388400000/(1-\varepsilon) = 3884000000000/9999$, for $N \ge N_{\mathrm{peel}}(\sigma)$. The coefficient 2008008 includes the two layers. No orientation factor is added.

The remaining estimates are

$$\begin{aligned} \frac{|r_0|+|r_1|+|r_2|}{\mathcal{N}_K} &\ll_{b,\sigma} L^{-2}, \\ \frac{|r_3|}{\mathcal{N}_K} &\ll_{b,\sigma} N^{-\ell_0\sigma^2}L^3 + N^{-b/4}L^7 + L^{-260} + L^{-120} + L^{-2} + N^{-c_\sigma}L^{4024+C} = O_{b,\sigma}(L^{-2}). \end{aligned} \tag{7.47}$$

The first line is (7.25). The second is Proposition 7.3 below. Its displayed terms come, in order, from the squareful mask, proper powers, logarithm freezing and the boundary. The last two are the combined logarithmic Type-II and Type-I-2 bound and the Type-II power-saving term.

7.5. **The row-three remainder.** We bound the complete third-row remainder of the actual source (7.2) by Paper I, Theorem 3.2. Every other contribution is bounded separately. Fix b, σ. Write $\Xi_3^\sharp$ for the retained cells $\mathfrak{c} \in D_3$ with $\chi_{\mathfrak{c}} = 1$. Its size is $O_\sigma(1)$. For $\mathfrak{c} \in \Xi_3^\sharp$ put $D_{\mathfrak{c}} = N^{\ell_{\mathfrak{c}}}$, $z_{\mathfrak{c}} = N^{t_3}$ and $u_{\mathfrak{c}} = D_{\mathfrak{c}}^{\sigma^2}$. Let I_i be the actual half-open exponent interval of coordinate i of the clipped cell. Then $\mathfrak{c} \subset I_1 \times I_2 \times I_3$. Put $\mathcal{P}_i = \{a \in \mathcal{P}_N : \log_N a \in I_i\}$. For every $a_i \in \mathcal{P}_i$, $N^{t_i} \le a_i < N^{\rho t_i}$ and $a_i < N^{\mu_*}$, also for a clipped interval. Write $t_i' \le \min(\rho t_i, \mu_*)$ for the upper endpoint of I_i. On $\overline{\mathfrak{c}}$ the actual coordinates satisfy $x_1 \ge x_2 \ge x_3 \ge u$. Two of the intervals are equal or disjoint.

Exact occurrence map. By (7.21) and Paper I, (2.30), (2.32)–(2.33), applied to $\mathcal{A}_h$ at level $D_{\mathfrak{c}}$, cutoff $z_{\mathfrak{c}} \le D_{\mathfrak{c}}^{1/2}$ and $\epsilon_s = \sigma$,

$$r_3 = \sum_{\mathfrak{c} \in \Xi_3^\sharp} \sum_{k \in \mathcal{K}_{\mathfrak{c}}} \sum_q O_{\mathfrak{c},k}(q)\, \Delta_{A_b}(q, N), \qquad \mathcal{K}_{\mathfrak{c}} = \{\mathrm{R1}\} \cup \mathcal{D}^-(D_{\mathfrak{c}}), \tag{7.48}$$

$$O_{\mathfrak{c},k}(q) = \sum_{\substack{a_1 > a_2 \ge a_3 \\ a_i \in \mathcal{P}_i}} \sum_{\substack{d < D_{\mathfrak{c}}^\sigma \\ d \mid P_N(u_{\mathfrak{c}})}} \lambda_d^{(k)} \sum_{\substack{(p_j)_{j \le \rho_k} \\ p_j \in \mathcal{Q}_{k,j} \\ p_i \ne p_j \ ((i,j) \in I(\mathbf{D}))}} 1_{q = a_1 a_2 a_3 d\, p_1 \cdots p_{\rho_k}}.$$

Here $\lambda_d^{(\mathrm{R1})} = \phi_d^-$ with no tail ($\rho_{\mathrm{R1}} = 0$). For $k = \mathbf{D}$, $\lambda_d^{(k)} = \lambda_d^-(\mathbf{D})$ and $\mathcal{Q}_{k,j} = \mathcal{P}_N \cap [D_j, \min(D_j^{1+\tau}, z_{\mathfrak{c}}))$. All coefficients have modulus at most one and depend only on $(\mathfrak{c}, k)$. The identity is exact at every supported modulus. Let $M_{\mathfrak{c}} \in \{1, 2, 6\}$ be the number of orderings of the equal intervals among I_1, I_2, I_3. Let $\Gamma_{\mathfrak{c},k}$ be $M_{\mathfrak{c}}^{-1}$ times the same sum, with $a_i \in \mathcal{P}_i$ independent and no tail distinctness. Then

$$O_{\mathfrak{c},k} = \Gamma_{\mathfrak{c},k} - \Xi_{T,\mathfrak{c},k} - \Xi_{P,\mathfrak{c},k}, \tag{7.49}$$

where Ξ_T collects unrestricted terms with $p_i = p_j$ for some forbidden pair $(i,j) \in I(\mathbf{D})$. Ξ_P collects the remaining unrestricted terms with $a_i = a_j$ for an equal-interval pair $I_i = I_j$. A prefix prime is at least $z_{\mathfrak{c}}$, and every tail prime is below it. The primes of d are below $u_{\mathfrak{c}} \le D_j$. Unequal grid intervals are disjoint. Hence every modulus in the support of Ξ_T or Ξ_P is divisible by p^2 for a prime $p \ge u_{\mathfrak{c}} \ge u_0$. Here $u_0 = N^{\ell_0 \sigma^2}$.

Support, cutoffs and constants. Every supported modulus is coprime to $2N$. The cells are clipped to $[u, \mu_*)$, and $v_{\mathrm{pk}}(x) = (1+x)/2$ on $[0, \mu_*]$. Thus $\Theta_{\mathfrak{c}} \ge (1+t_1)/2$. The actual coordinate bound $x_1 \ge x_3 \ge u$ on $\overline{\mathfrak{c}}$ also gives $\Theta_{\mathfrak{c}} \ge (1+u)/2$. Hence

$$\ell_{\mathfrak{c}} \ge \max\{A_3 - t_3, 2\rho t_3\} \ge \tfrac{2}{3} A_3 = \tfrac{33109}{240000}, \qquad A_3 = \tfrac{1}{2} - \eta_L - \tfrac{3}{2}\mu_*, \tag{7.50}$$

and $\ell_{\mathfrak{c}} - \ell_0 \ge 231463/24720000$. By the cubic tail conditions, $\sum_j s_j < \ell_{\mathfrak{c}}$ for $s_j = \log_N D_j$. Thus every supported modulus satisfies

$$q < N^{\rho(t_1+t_2+t_3) + \sigma\ell_{\mathfrak{c}} + \rho\ell_{\mathfrak{c}}} \le N^{e_Q}, \qquad e_Q = (\rho+\sigma)(H_{\max} - \eta_L) < \tfrac{3}{5}, \qquad H_{\max} = \tfrac{153}{256}. \tag{7.51}$$

Since the cell ceiling gives $0 < \ell_{\mathfrak{c}} \le H_{\max} = 153/256 < 1$ and $0 < \sigma \le 10^{-10}$, $\ell_{\mathfrak{c}}\sigma^2 < \sigma^2 < u = 1/500$. Thus $u_{\mathfrak{c}} = N^{\ell_{\mathfrak{c}}\sigma^2} < N^u$. The cutoff comparison uses only this exponent inequality.

Every labelled prime is at least $u_{\mathfrak{c}}$, so the number of labels is below $3/(5\ell_0\sigma^2)$. With $M_\sigma = \lceil 3/(5\ell_0\sigma^2) \rceil$ and the vector count $\exp(8\sigma^{-3})$, one has $|\Gamma_{\mathfrak{c},k}|, |\Xi_T|, |\Xi_P| \le M_\sigma!$, and $\sum_{\mathfrak{c},k} \big(|\Gamma_{\mathfrak{c},k}| + |\Xi_T| + |\Xi_P|\big) \le C'_{\mathrm{K2},\sigma}$.

The squareful terms. Since $|\Delta_{A_b}(q,N)| \le 2(N/q+1) + 2N/\varphi(q)$, Paper I, (B.5), with $Y = u_0$ and $Q_{\mathrm{supp}} = N^{e_Q}$ gives

$$\sum_{\mathfrak{c},k} \sum_q |\Xi_{T,\mathfrak{c},k}(q) + \Xi_{P,\mathfrak{c},k}(q)|\, |\Delta_{A_b}(q,N)| \ll_\sigma N^{1-\ell_0\sigma^2} L. \tag{7.52}$$

Prime detection and the two Heath–Brown expansions. Put $z_1 = N^{1/(\log L)^3}$, so $z_1 \ge z_0(X)$ for every occurring box scale $\varepsilon N/2 \le X \le N$. Both source primes exceed z_1. Apply Paper I, (B.7), to $1_{\mathbb{P}}(m)$ and to $1_{\mathbb{P}}(p')$. Apply Paper I, (B.9), with roughness z_1, to $\Lambda_{z_1}(m)$ with cap $U_1 = 2N^{1/6}$ and to $\Lambda_{z_1}(p')$ with cap $U_2 = 2N^{(1-b)/3}$. The identities apply since $m \le N^{1/2} < U_1^3$ and $p' \le N^{1-b} < U_2^3$. For patterns $j, j' \in \{1,2,3\}$ the tested index is a labelled tuple $\mathbf{v} = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) \in \mathcal{R}_{z_1}^{2j+2j'}$, at most twelve coordinates, with

$$m = \prod \mathbf{v}^{(1)}, \quad p' = \prod \mathbf{v}^{(2)}, \quad e = mp', \quad N^b \le m \le N^{1/2}, \quad \varepsilon N < e \le N',$$

the Möbius coordinates of $\mathbf{v}^{(i)}$ at most U_i, and coefficient $c_j(\mathbf{v}^{(1)}) c_{j'}(\mathbf{v}^{(2)})/(\log m \log p')$. Here c_j is the coefficient after (B.10). For a fixed product e, put

$$\mathcal{S}_e(q;N) := \Delta_{\delta_e}(q,N) = 1_{e \equiv N \ (q)} - \frac{1_{(e,q)=1}}{\varphi(q)}. \tag{7.53}$$

Thus the Heath–Brown tuple map uses the point-mass discrepancy $\mathcal{S}_e(q;N)$. The exact weighted point-mass expansion is

$$\Delta_{A_b}(q,N) = \sum_e A_b(e) \mathcal{S}_e(q;N).$$

The weight $A_b(e)$ keeps both eligible ordered orientations, as in (7.2). A prime square is counted once. The proper-power remainder $\mathcal{E}_{\mathrm{pp}}$ consists of the pairs in which m or p' is a proper prime power. For every fixed $(\mathfrak{c}, k)$,

$$\sum_q \Gamma_{\mathfrak{c},k}(q)\Delta_{A_b}(q,N) = \mathcal{E}_{\mathrm{pp}} + \sum_{j,j'}\sum_{\mathbf{v}} \frac{c_j(\mathbf{v}^{(1)})c_{j'}(\mathbf{v}^{(2)})}{\log m \log p'} \sum_q \Gamma_{\mathfrak{c},k}(q)\mathcal{S}_e(q;N), \tag{7.54}$$

exactly. Each tuple keeps the natural discrepancy of its own product. The per-cell coefficient bound is the factorial bound above; the sum over cells uses the separate aggregate bound defined there. Since $e < N$, $|c_j c_{j'}|/(\log m \log p') \le 9$ and $\sum_q |\Gamma(q)||\mathcal{S}_e(q;N)| \le C_{\mathrm{K2},\sigma} F(e)$ with $F(e) = \tau(N-e) + 2(1+L)$. The tuple multiplicity is at most $\tau_{12}(e)$. The divisor bounds $\tau_{12}^2 \le \tau_{144}$, $\tau^4 \le \tau_{16}$ and Cauchy–Schwarz give

$$\sum_{e<N} \tau_{12}(e)F(e) \ll NL^{73}, \qquad \sum_{e<N} \tau_{12}(e)F(e)^2 \ll NL^{79}. \tag{7.55}$$

There are $\ll N^{1-b/2}L$ proper-power pairs, and $\sum_{e<N}\tau(e)F(e)^2 \ll NL^9$. Therefore

$$|\mathcal{E}_{\mathrm{pp}}| \ll_\sigma (N^{1-b/2}L)^{1/2}(NL^9)^{1/2} = N^{1-b/4}L^5. \tag{7.56}$$

Boxes, logarithm freezing and faces (H6). Put $\delta = L^{-334}$. Use the equal-ratio mesh of Paper I, (B.12), with δ_X replaced by δ, on every ambient coordinate axis. Keep $\{1\}$ as an atom. A box $\mathcal{B}$ is a Cartesian product of cells. Its lower endpoints are V_i, and $X = \prod_i V_i$. Let λ_1, λ_2 be the sums of $\log V_i$ over the two sides. Then $\varepsilon N/2 \le X \le N$ on every box meeting the source, so $K_{\mathrm{scale}} = 1$. Replacing $1/(\log m \log p')$ by $1/(\lambda_1\lambda_2)$ changes the weight by $O(L^{-335})$, since $\lambda_1 \ge bL/2$. By (7.55),

$$|E_{\mathrm{LOG}}| \ll_\sigma NL^{72-334}. \tag{7.57}$$

The faces are $m = N^b$, $m = N^{1/2}$, $e = \varepsilon N$, $e = N'$, and the cap faces $v_i = U_1$, $v_i = U_2$ on Möbius coordinates. All are singletons. On each, an active coordinate is at least $N^{1/20}$. Paper I, Lemma (B.14), with $s \le 12$, $K_\gamma = 1$ and $\eta_{\mathrm{dist}} \le 12\delta$, gives at most $\ll NL^{11-334}$ boundary tuples. Cauchy–Schwarz with (7.55) gives

$$|E_{\mathrm{bdry}}| \ll_\sigma NL^{45-334/2}. \tag{7.58}$$

Thus (H6) holds with $c_{\log} = 72$, $c_R = 45$. On a nonboundary box every source cut and cap is constant. The coefficient is the outer scalar $c_{\mathcal{B}} = (-1)^{j+j'}\binom{3}{j}\binom{3}{j'}(\log X)^2/(\lambda_1\lambda_2)$, times $w(\mathbf{v}) = \prod_i w_i(v_i)$ with $w_i \in \{\mu_{z_1}, 1_{\mathcal{R}_{z_1}}, (\log v/\log X)1_{\mathcal{R}_{z_1}}\}$. Each side carries exactly one logarithmic coordinate, and $(\log X)^2/(\lambda_1\lambda_2) \le 1/(b(1-b)) + o(1) \le 4.5 + o(1)$, hence the scalar norm bound $\|c_{\mathcal{B}}\|_\infty \le 45$. Each coordinate meets at most $2L^{335}$ cells, so

$$\sum_{\mathfrak{c},k}\sum_{j,j'}\sum_{\mathcal{B}} |c_{\mathcal{B}}| \ll_\sigma L^{4020} \le L^{4022}, \qquad K_0 = 4022 \le 24C_{\mathrm{box}} + 26 = 8042. \tag{7.59}$$

Classification. Let $e_i = \log_N V_i$, $\mathcal{W} = [60\sigma, 1/3 + 14\sigma]$, and $\kappa = (\log 2 + 12\log(1+\delta))/L \le \sigma/202$. A box is Type II if some $e_i \in \mathcal{W}$. Select the largest such e_i, breaking ties by the fixed coordinate order. Otherwise every Möbius coordinate is small, since $1/6 + \kappa$ and $(1-b)/3 + \kappa$ are below $1/3 + 14\sigma$. On the m-side, $\sum e_i \ge b - \kappa > 360\sigma$. On the p'-side, $\sum e_i \ge 1/2 - \kappa - \log(1/\varepsilon)/L > 360\sigma$. These are the only side bounds needed to rule out an all-small side. Together with the global Heath–Brown sum bound $\sum e_i < 1 + \kappa$, the threshold $1/3 + 14\sigma$ permits at most two large variables. Hence each side has exactly one large smooth coordinate. Every other box has exactly two large smooth coordinates $N_1 \ge N_2 \ge X^{1/3}$. It has at most ten small coordinates, of product M, with

$$\log_X M \le \frac{(600 + 1/202)\sigma}{1 - \sigma/101} < 6.02 \cdot 10^{-8}, \qquad N_1N_2M \le 2^{30}X, \qquad q < X^{0.5977}. \tag{7.60}$$

There are no Type-I-1 boxes.

Type-II boxes: Paper I, (H1)–(H5) and LEN.. Fix $(\mathfrak{c}, k)$ and a Type-II box. Let its selected lower endpoint be T, put $\xi = \log_N T$, and let the other lower endpoints have product $M = X/T$.

- **H1.:** $C_{\mathcal{B}} = \Gamma_{\mathfrak{c},k}$ has the form (3.4). Here J is the three prefix labels and the tail labels. The prime sets are $\mathcal{P}_i$ and $\mathcal{Q}_{k,j}$, $u_L = u_{\mathfrak{c}} \ge u_0$, and $d \mid P_N(u_{\mathfrak{c}})$ is squarefree. Repeated primes are kept. We have $q \le X^{0.6}$ by (7.51) once $(3/5 - e_Q)L \ge (3/5)\log(2/\varepsilon)$. Also $(q, 2N) = 1$. The selector $\chi_{\mathfrak{c}}$ is constant on the block. The removed corrections are Ξ_T, Ξ_P and $\mathcal{E}_{\mathrm{pp}}$. There is no comparator correction, since the comparator is natural on every tuple.
- **H2.:** Take the upper exponents $t_1' \ge t_2' \ge t_3'$ and $\min(\rho s_j, t_3)$. For $d_* = \rho(\Theta_{\mathfrak{c}} - \eta_L)$, the merged list satisfies $\mathrm{WF}(d_*)$. The reserves are $\rho(1/2 - \eta_L - 3\mu_*/2) = 33109\rho/160000$, $\rho(1/2 - \eta_L - 5\mu_*/2) = 1859\rho/160000$, $\rho(\ell_{\mathfrak{c}} - t_3) \ge \rho(2\rho - 1)t_3$. For a tail rank, the reserve is ρ times the strict inequality $\sum_{i<j} s_i + 2s_j < \ell_{\mathfrak{c}}$ from the $\mathcal{D}^-$ conditions. Put $h = d_* + 60\sigma$, $\delta_{\mathrm{pk}} = 30\sigma$, $g = \xi - 30\sigma$. Then $h - 1/2 \ge 9 \cdot 10^{-4}$, $\Theta_{\mathfrak{c}} - h > 0.0000999939$, $h \le H_{\max}$,

$$30\sigma \le g \le 1/3 - 16\sigma, \qquad \delta_{\mathrm{pk}} = 30\sigma, \qquad g \le 1/3 - \delta_{\mathrm{pk}}/2.$$

The upper requirement has reserve at least σ, since $(1/3-15\sigma)-(1/3-16\sigma)=\sigma$. Also $h \le \Theta_{\mathfrak{c}} \le \widetilde{H}(t_1',t_2',t_3')$ because $(t_1',t_2',t_3') \in \bar{\mathfrak{c}}$ and $t_1' \le \mu_*$. Paper I, Theorem 4.1, gives $J = J_1 \sqcup J_2 \sqcup J_3$. Put d in slot one, with γ_s as in Paper I, (3.5). The tuple map gives $\Gamma_{\mathfrak{c},k} = M_{\mathfrak{c}}^{-1} C_{\mathrm{slot}}(\gamma_1 * \gamma_2 * \gamma_3)$, $C_{\mathrm{slot}} \le M_\sigma!$, $|\gamma_s| \le 1$. Slots two and three are u_0-rough, and empty slots carry the atom $1_{v=1}$. The scalar $M_{\mathfrak{c}}^{-1}$ is an outer coefficient.

H3.: Let Q, R, S be the scales of $\gamma_1, \gamma_3, \gamma_2$. The loads (4.3), the slot-one load $\sigma\ell_{\mathfrak{c}} \le \sigma H_{\max}$ of d, and $A_0 = X^{1+\sigma/20}$ give reductions $(7527/256,\ 60,\ 114421/1280,\ 26247/256)\sigma$. They are, in turn, in Q/T, T^2R^2S, $A_0^{7/32}TQR^2S^5$, $(TR/S)^{7/16}TQR^2S^5$. The required reductions are $29, 8, 15, 15$, with losses at most 2ω, $\omega = \log_N(N/X) \le \sigma/202$. We also have $\log_N(QRS) \le e_Q$, so $QRS < M_I X^{-89\sigma}$ with reserve > 0.0691. Next, $N_I \ge QX^{\sigma/3}$ with reserve 29.069σ, and $M_I \ge X^{1/2}$. Thus (3.6)–(3.7) hold. The selected coefficient is $\alpha_n = w_{\mathrm{sel}}(n) 1_{I_T}(n) \in \{\mu(n), 1, \log n/\log X\} 1_{P^-(n)\ge z_1} 1_{I_T}(n)$, with $\zeta = z_1 \ge z_0(X)$ and $I_T = (T, (1+\Delta)T]$. The complementary coefficient $\beta_{m^\sharp}$, the product of the other at most eleven weights, satisfies $|\beta_{m^\sharp}| \le \tau_{11}(m^\sharp) \le \tau_{23}(m^\sharp)$. Here $\tau_{23} = d_{23}$ is the 23-fold divisor function, with $d_{23}(n) = \sum_{n=n_1\cdots n_{23}} 1$. As in Paper I, (B.22)–(B.23), with at most eleven complementary cells, $N_I \ge T/(6L^{334})$, $M_I = M((1+\Delta)^r - 1) \asymp ML^{-334}$, $m_0 = M(1+(1+\Delta)^r)/2 \le 2X/T$, so $C_{\mathrm{ctr}} = 2$. On the doubled support, $mn \le X(1+\Delta)(1+\frac{3}{2}((1+\Delta)^{11}-1)) \le N'(1+20\delta)$, because $X \le e \le N'$ for every tested product of the box. Hence $N - mn \ge NL^{-334}(L-20) \ge 1$ and $|mn - N| \le N < X^3$, which is (3.10). The integer convention of I_T is that of Paper I, Lemma B.22, since the mesh and the roughness are the same. LEN follows from the condition

$$\sigma L \ge 2^{130}(335)(\log L + 1). \tag{7.61}$$

It gives $T^{2^{-126}} \ge 12L^{334}$ and $N_I \ge (2T)^{1-2^{-126}}$.

H4.: $a = N$, $0 < N \le XL \le A_0$, and every supported modulus is coprime to $2N$.

H5.: $N^{60\sigma} \le T \le N^{1/3+14\sigma}$, and (3.99) gives $T \le X^{1/3+15\sigma}$ and $T < X^{1/2}$.

The selected-factor Siegel–Walfisz bound is Paper I, (3.25). It is proved for these three coefficient classes with any $\zeta \ge z_0$. In the large-norm case of Paper I, (3.48), the zero mode uses Lemma 6.5(ii) in place of [3, Theorem 0(a)]. Its premises (3.1), (3.2), (3.6), (3.8) with LEN, (3.11) and (3.25)–(3.27) are those verified above. Theorem 2.8 of Paper I enters only inside the proof of Theorem 3.2, with the dictionary

$$r_T = z, \quad s_T = p_0, \quad n_T = p_0 m, \quad a_T = \varepsilon_y N, \quad N_T = 8p_0 H_* SF, \quad A_T = A_0, \tag{7.62}$$

both signs, all nine smooth weights and the full bracket. Paper I, Theorem 3.2, (3.13), gives, uniformly in the box, the block and the cell,

$$|S_{\mathcal{B}}^{\mathrm{PSDT}}| \ll_{\sigma,A'} XL^{-A'} + X^{1-c_\sigma}L^C, \qquad S_{\mathcal{B}}^{\mathrm{PSDT}} = \sum_q \Gamma_{\mathfrak{c},k}(q) \sum_{\mathbf{v}\in\mathcal{B}} w(\mathbf{v}) \mathcal{S}_e(q;N). \tag{7.63}$$

With $A = 4$, $C_{\mathrm{box}} = 334$, $A' = A + 24C_{\mathrm{box}} + 31 = 8051$ and (7.59), Paper I, (3.92), gives

$$E_{\mathrm{II}} = \sum |c_{\mathcal{B}}|\, |S_{\mathcal{B}}^{\mathrm{PSDT}}| \ll_{b,\sigma} NL^{4022-8051} + N^{1-c_\sigma}L^{4022+C}. \tag{7.64}$$

Here $S_{\mathcal{B}}^{\mathrm{PSDT}}$ already contains $\Gamma_{\mathfrak{c},k}$, hence its factor $M_{\mathfrak{c}}^{-1}$ from (7.49). The coefficient norm (7.59) sums only $|c_{\mathcal{B}}|$, so the orbit factor is paid exactly once.

Lemma 7.2 (Relaxed Type-I-2 estimate)**.** *The source hypotheses are the Type-I-2 conditions (SH1)–(SH5) stated in Paper I:*

For a direct shell $[X, 2X)$, assume the following source conditions (SH1)–(SH5).

(SH1) *The coefficient $\gamma(q)$ is independent of tested variables, $(q, 2N) = 1$, $|\gamma(q)| \le C_\sigma$, $q \le N^{e_Q}$, $e_Q < 153/256$, and $(a, q) = 1$ on its support.*

(SH2) *The tested geometric source and cap restrictions are constant on each nonboundary box; the roughness sieve is separate as in (SH3). Smooth factor weights have* $\sup|w| + V(w) \ll 1$.

(SH3) *All HB factors are rough at the threshold $z^* = z_0(X)$ or $z_0(N)$.*

(SH4) *The discrepancy is exactly (3.3).*

(SH5) *The assignment priority is proper prime powers, boundary boxes, Type II, Type I-1, then Type I-2.*

Assume (SH1)–(SH5) for a nonboundary K2 box. Let the two large smooth variables have nonnegative weights w_1, w_2 and $N_1, N_2 \ge X^{1/3}$. Let the remaining small variables, at most ten, have product M. Assume

$$M \le X^{6.02\cdot 10^{-8}}, \qquad N_1N_2M \le 2^{30}X, \qquad q < X^{0.5977},$$

and $|\alpha_m| \ll \tau_{10}(m) \le \tau(m)^9$. Then, for every fixed $A > 0$, uniformly in the supported unit residues,

$$\sum_{q\asymp Q} |\gamma(q)| \left| \sum_m \alpha_m \sum_{n_i\in I_i,\ P^-(n_i)\ge z^*} w_1(n_1)w_2(n_2)\mathcal{S}_{mn_1n_2}(q;a) \right| \ll_{\sigma,A} XL^{-A}.$$

The same statement holds for a fixed finite sum of signed weights. Split each bounded real weight into its two nonnegative parts.

Proof. Restrict first to $(m,q)=1$, and write $\Delta = C - M_q$ for the positive congruence and comparator functionals. In the K2 application the inner product is $e = mn_1n_2$, the residue is N, and $(q,N)=1$. If $(m,q)>1$, then $(e,q)>1$, so both $1_{e\equiv N\ (q)}$ and the comparator $1_{(e,q)=1}/\varphi(q)$ vanish. The omitted range contributes zero. Apply the Brun–Bonferroni sandwich of Paper I, (3.102)–(3.104), at $D_f = X^{\epsilon_I/2}$, $\epsilon_I = 7.5\sigma$. For a requested fixed logarithmic precision A_{sv}, put

$$s_f = \frac{\log D_f}{\log z^*}, \quad E = \sum_{p<z^*} \frac{1}{p}, \quad J = 2\lfloor s_f/2 \rfloor,$$

and increase the common threshold until $J \ge \max\{2, e^2E, A_{\mathrm{sv}} \log L\}$. Since $s_f \asymp \sigma(\log\log N)^3$ and $E = O(\log\log N)$, this is eventual at fixed σ, and $E^J/J! \le L^{-A_{\mathrm{sv}}}$.

For the two large factors put $R_i = 1_{P^-(n_i)\ge z^*}$ and $G_i = \Psi_i^+ - \Psi_i^- \ge 0$. Positivity gives

$$\Psi_1^+\Psi_2^+ - G_1\Psi_2^+ - \Psi_1^+G_2 \le R_1R_2 \le \Psi_1^+\Psi_2^+,$$

so the three signed gaps and the two nonnegative comparator gaps bound $|\Delta(R_1R_2)|$, as in Paper I, (3.113). Expand the three gaps over $d_i \le D_f$. The completed hyperbola estimate (3.109) gives $O_{\epsilon_X}(D_f^2 q^{1/2} X^{\epsilon_X})$. Here we fix $\epsilon_X = 10^{-4}$. Squareful moduli are included by the full-modulus Weil–Estermann bound.

The two one-dimensional sieve gaps satisfy

$$\sum_{n\in I_i} G_i(n) w_i(n) \ll_{A_{\mathrm{sv}}} N_i L^{-A_{\mathrm{sv}}} + D_f, \qquad \sum_{n\in I_i} \Psi_i^+(n) w_i(n) \ll N_i + D_f.$$

Thus the total is bounded by

$$MD_f^2 Q^{3/2} X^{\epsilon_X} L^C + XL^{C-A_{\mathrm{sv}}} + MD_f(N_1+N_2)L^C + MD_f^2 L^C. \tag{I-2'}$$

The ten-variable check uses the following upper bound. The expression is increasing throughout $0 < \sigma \le 10^{-10}$, so its maximum is evaluated at $\sigma = 10^{-10}$:

$$\frac{600\sigma+\kappa}{1-\omega} \le \frac{(600+1/202)\sigma}{1-\sigma/101} \le \frac{(600+1/202)10^{-10}}{1-10^{-10}/101} = \frac{121201}{2019999999998} \approx 6.0000495049564356926\cdot 10^{-8} < 6.02\cdot 10^{-8},$$

where $\omega, \kappa \le \sigma/202$ absorb all fixed dyadic factors, so $1-\omega \ge 1-\sigma/202 \ge 1-\sigma/101$, consistently with (7.60). Eleven small variables would give $\frac{133321}{2019999999998} \approx 6.600049504957030 \cdot 10^{-8}$, which exceeds the allowed bound. With $q < X^{0.5977}$, the first exponent in (I-2') is at most

$$6.02\cdot 10^{-8} + 7.5\cdot 10^{-10} + 1.5(0.5977) + \epsilon_X = 0.89665006095 < 0.897,$$

leaving slack 0.00034993905. Also, using $N_1N_2M \le 2^{30}X$, $N_1 \ge N_2 \ge X^{1/3}$, and $D_f = X^{3.75\sigma}$,

$$MD_f(N_1+N_2) \le 2^{31} D_f X/N_2 \ll X^{2/3+3.75\cdot 10^{-10}} < X^{0.668}, \qquad MD_f^2 \le X^{6.02\cdot 10^{-8}+7.5\cdot 10^{-10}} < X^{0.668}.$$

The coefficient bound uses $\tau_{10}(m) \le \tau(m)^9$, so $B_0 = 9$ is fixed. Finally, a family of at most $O_\sigma(L^K)$ boxes is absorbed by requesting $A_{\mathrm{sv}} \ge K + A + 5$, after all fixed logarithmic losses. This proves the lemma. □

Type-I-2 boxes: the complementary estimate. Lemma 7.2 is a relaxed form of Paper I, (3.111), for this source. Below we check each displayed hypothesis against the actual K2 source.

For the present K2 Type-I-2 box, the nontrivial SH2 check is the display

$$\begin{aligned} &1 \le v \le X, \qquad w(v) \in \left\{1, \frac{\log v}{\log X}\right\}, \qquad P^-(n_i) \ge z^* \text{ is a separate tested condition,} \\ &\sup|1| + V(1) = 1, \qquad \sup\left|\frac{\log v}{\log X}\right| + V\left(\frac{\log v}{\log X}\right) \le 1 + \frac{2\log(1+L^{-334})}{\log X} = O(1). \end{aligned} \tag{7.65}$$

Thus the two tested large factors have the required nonnegative weights and bounded supremum plus variation. The roughness condition is not hidden inside the weight.

For the Type-I-2 boxes above, (SH1) has $\gamma = \Gamma_{\mathfrak{c},k}$. The support and bound are (7.51) and (7.49), $e_Q < 153/256$, and the residue is the unit $a = N$. The partition construction proves the geometric constancy in (SH2) on each nonboundary box. In this two-large-variable case the two tested large factors are the selected smooth factors. The roughness restriction is kept as the separate condition $P^-(n_i) \ge z^*$ in the inner sum. Their smooth weights are then individually 1 or $\log v/\log X$, not μ_{z_1}. Hence they are nonnegative with $\sup|w| + V(w) \ll 1$ as required by (SH2). The μ_{z_1} factor occurs only among the complementary small factors. Condition (SH3) is $z_1 \ge z_0(X)$. Condition (SH4) is the natural full-product discrepancy fixed above. Condition (SH5) is the priority in the preceding boundary and Type-II classification.

On a large-coordinate cell $(V,(1+\delta)V]$, $\delta = L^{-334}$, one has $1 \le V \le X$. The smooth weight $w(v) = 1$ has supremum plus variation equal to 1. The alternative $w(v) = \log v/\log X$ is nonnegative and increasing, with

$$\sup|w| + V(w) \le 1 + \frac{2\log(1+\delta)}{\log X} = O(1).$$

The roughness restriction remains a separate tested condition under (SH3) and inside the sieve, and it is not claimed to be constant on a box.

Thus the relaxed estimate of this chapter applies with $z^* = z_1$, the two large smooth weights nonnegative, and

$$|\alpha_{m^\sharp}| \le \tau_{10}(m^\sharp) \le \tau(m^\sharp)^9.$$

Each of the at most ten small labels contributes one factor. Normalized logarithmic weights $\log v/\log X$ are at most one on the support. Thus the coefficient hypothesis uses the fixed divisor exponent $B_0 = 9$, independent of N. The first term of (I-2') uses the following values. The slack is 0.00034993905 at exponent 0.89665006095. The two final terms have the bound $2/3 + 3.75 \cdot 10^{-10}$. The absorption is $\omega, \kappa \le \sigma/202$, and the divisor bound is τ_{10} with $B_0 = 9$. By (7.60), Lemma 7.2 applies to every K2 Type-I-2 box. Requesting $A_I = K + A + 5$ for the fixed $O_\sigma(L^K)$ box family (here $K = 4022$) gives

$$E_{\mathrm{I2}} \ll_{b,\sigma} NL^{-A} = NL^{-4}. \tag{7.66}$$

Proposition 7.3 (Row-three remainder)**.** *Fix $b \in (1/3, 1/2)$ and $0 < \sigma \le 10^{-10}$. Under Paper I, Theorems 2.8 and 3.2, with the relaxed Type-I-2 Lemma 7.2 and Lemma 6.5(ii) at (3.48),*

$$\frac{|r_3|}{\mathcal{N}_K} \ll_{b,\sigma} N^{-\ell_0\sigma^2}L^3 + N^{-b/4}L^7 + L^{-260} + L^{-120} + L^{-2} + N^{-c_\sigma}L^{4024+C}.$$

Proof. By (7.48) and (7.49), r_3 is the Γ-sum minus the squareful sum (7.52). By (7.54), the Γ-sum is $\mathcal{E}_{\mathrm{pp}}$ plus the tuple sum. Every tuple lies in exactly one box. The boundary tuples give (7.58). On the other boxes logarithm freezing costs (7.57). Each box is of Type II or Type I-2. Sum (7.52), (7.56), (7.57), (7.58), (7.64) and (7.66), with $O_\sigma(1)$ pairs $(\mathfrak{c}, k)$. Dividing by $\mathcal{N}_K = (1-\varepsilon)C(N)N/L^2 \ge (1-\varepsilon)C_2N/L^2 \gg_b N/L^2$ gives the claim. □

Proposition 7.4 (Exterior source estimates)**.** *Fix b, σ before N. For the source* (7.2)*:*

1. (7.32) *and* (7.35) *hold for the two-orientation source and four signed AA main terms in* (7.27) *and* (7.31)*;*
2. (7.43) *and* (7.38) *hold for $N \ge N_{\mathrm{AA}}(b,\sigma)$;*
3. (7.46) *holds for $N \ge N_{\mathrm{peel}}(\sigma)$;*
4. (7.47) *holds.*

Proof. Item 1 is (7.33)–(7.34) and the derivation after (7.35). Item 2 is (7.40)–(7.42). Item 3 is (7.44)–(7.46). Item 4 is (7.25) and Proposition 7.3. □

Proposition 7.5 (Exterior comparison)**.** *For fixed b, σ there is a nonnegative $r_K(N; b, \sigma) \to 0$ such that, for $N \ge T_r(b,\sigma)$ with T_r as defined below,*

$$\begin{aligned} \frac{4B_b(N)}{W(N)} &\le c_{K2,G0}I(b) + A_K(b)C_{\mathrm{AA}}\sigma + P_K\sigma^9 + r_K(N;b,\sigma), \\ P_K &= \frac{388400000}{1-\varepsilon} = \frac{3884000000000}{9999}. \end{aligned} \tag{7.67}$$

Here $T_K = \max\{N_{\mathrm{mass}}(b,\sigma), N_{\mathrm{AA}}(b,\sigma), N_{\mathrm{peel}}(\sigma), T_{\mathrm{len}}\}$. The threshold N_{mass} is from the fixed-parameter estimate at (7.35)*. N_{AA} is from item 2 of Proposition 7.4. Both exist with b, σ fixed. Finally, T_{len} is the named length and application threshold of Section 2. In addition,*

$$r_K = \frac{E_{\mathrm{top}} + [M_K - \mathcal{N}_K 2e^{-\gamma}I(b)G_*]_+ + \sum_{j=0}^3 |r_j|}{\mathcal{N}_K} = o_{b,\sigma}(1). \tag{7.68}$$

In particular, let $\widehat{T}_r$ be the least even threshold above which $r_K \le \delta_{\mathrm{err}}/10$, and set $T_r = \max\{N_{\mathrm{mass}}(b,\sigma), N_{\mathrm{AA}}(b,\sigma), T_K, \widehat{T}_r\}$. These thresholds exist at fixed b, σ.

Remark 7.6 (Single payment of each error). In [26, Theorem 2.6, (2.30)] the three summands XVE_s, R_1^ν and R^ν are distinct. The first is paid only by (7.38), the other two only by (7.47). The decaying part of (7.40) and the $o(1)$ in (7.42) are absorbed by thresholds at fixed σ. The peel term P_3 contains no linear-sieve application. Thus

$$r_K = \frac{E_{\mathrm{top}} + \sum_j |r_j| + [M_K - \mathcal{N}_K 2e^{-\gamma}I(b)G_*]_+}{\mathcal{N}_K},$$

and no error is counted twice.

Proof. Use (7.26) and divide by $\mathcal{N}_K = W/4$. The principal term is bounded by $\mathcal{N}_K 2e^{-\gamma}I(b)G_*$ plus its positive part. Also $2e^{-\gamma}G_* \le c_{K2,G0}$ by (7.36)–(7.37). Proposition 7.4(2),(3) pay the sieve and peel terms above T_r. By definition, T_r includes $N_{\mathrm{mass}}(b,\sigma)$, $N_{\mathrm{AA}}(b,\sigma)$ and T_K. The remaining terms form r_K. First, $E_{\mathrm{top}}/\mathcal{N}_K \ll L^{-331}$. The positive part contributes $o_{b,\sigma}(1)$ by (7.35). The row remainders are $O_{b,\sigma}(L^{-2})$ by (7.47). Hence $r_K = o_{b,\sigma}(1)$. The convergence is ineffective through Siegel–Walfisz, so T_r exists but is not computed. □

8. The switched sum

We estimate the single term $-S_{\mathrm{sw}}/W$. The normalization is (2.2). Fix σ as in (2.4). Put

$$\eta_L = 10^{-4}, \quad \varepsilon = 10^{-4}, \qquad H_{\max} = \frac{153}{256}, \qquad \ell_0 = \frac{264897}{2060000}, \qquad \ell_1 = \frac{109}{1000}, \tag{8.1}$$
$$N_{\mathrm{cut}} = N(1 - L^{-333}), \qquad \mathcal{P}_N = \{p \in \mathbb{P} : p \nmid 2N\}.$$

All limits in this section fix σ first.

8.1. **A common nonnegative source.** We use the prime-coordinate cells of Paper II [27, (5.29a)–(5.31)]. Their rules are

$$I_1 = [\alpha, \beta), \quad I_2 = [\beta, \lambda), \quad I_3 = [\lambda, 1/3), \quad I_4 = [1/3, 1],$$
$$C_{\mathrm{L3}} = \frac{57}{64} - u - \frac{11\alpha}{5} = \frac{6408241}{9192000}, \qquad v_\beta = \frac{2(C_{\mathrm{L3}} - \beta)}{5},$$
$$k_1 + k_2 + k_3 + k_4 = k, \quad 5 \le k \le 11, \quad k_1 \ge m_1 \ge \cdots \ge m_{k_2} \ge 0,$$
$$v_j \le v_\beta \implies m_j = k_1, \tag{8.2}$$
$$v_j > v_\beta \implies x_{m_j} < C_{\mathrm{L3}} - \frac{5}{2} v_j \le x_{m_j+1},$$
$$w_C = 3 + 1_{k_1=0} - 2k_1 - 2k_2 - k_3 + \binom{k_1}{2} + \sum_{j=1}^{k_2} m_j.$$

The x_i and v_j increase in I_1 and I_2. The outer sentinels are infinite. Let $\mathcal{C}_+$ contain the cells with $w_C > 0$. For such a cell, put

$$a_C(e) = 1_{e=p_1\cdots p_k,\ N^\alpha \le p_1 < \cdots < p_k,\ (\log_N p_1, \ldots, \log_N p_k) \in C},$$
$$A(e) = \sum_{C \in \mathcal{C}_+} w_C a_C(e) 1_{\varepsilon N < e < N_{\mathrm{cut}}}, \quad 0 \le A(e) \le 36,$$
$$X_A = \sum_e A(e), \qquad Y_A(q) = \sum_{(e,q)=1} A(e), \tag{8.3}$$
$$C_A(q; N) = \sum_e A(e) 1_{q \mid N-e}.$$

Since $\sum_j m_j \le k_1 k_2$, we have

$$w_C \le 3 + 1_{k_1=0} - 2k_1 - 2k_2 - k_3 + \frac{k_1(k_1-1)}{2} + k_1 k_2 \le 36.$$

For $k_1 \ge 2$, maximize at $k_3 = 0$, $k_2 = 11 - k_1$; the resulting concave quadratic has maximum 36 at $k_1 = 10, 11$. The cases $k_1 = 0, 1$ give at most $4, 1$. The strict cells are disjoint, so this proves the bound for A. Every prime is permitted in A. In particular, A has no factor $1_{(e,2N)=1}$. Keep the selector $1_{\Omega(e)\ge 3}$ in the switched source. The required pointwise comparison is

$$1_{\Omega(e)\ge 3}(W_{13}(e))_+ \le A(e) \quad \begin{pmatrix} \mu_{\mathrm{Mob}}(e)^2 = 1, \quad P^-(e) \ge N^\alpha, \\ \varepsilon N < e < N_{\mathrm{cut}} \end{pmatrix}. \tag{8.4}$$

Here μ_{Mob} denotes the Möbius function. Paper II gives zero positive part when $\Omega(e) = 3, 4$ [27, (1.7), (4.9), (5.29a)–(5.31)].

Proof of (8.4). Let e satisfy the side conditions and write $e = p_1 \cdots p_k$, $p_1 < \cdots < p_k$, $x_i = \log_N p_i$. If $(e, N) > 1$, every occurrence in (4.3) vanishes, because it requires $(d, N) = 1$ and $(n/d, Nd) = 1$ [27, (1.7)]. Then $W_{13}(e) = 0$. Assume $(e, N) = 1$. Since $N^{k\alpha} \le e < N$ and $12\alpha = 1200/1149 > 1$, we have $k \le 11$. If $k \le 2$, the selector vanishes. If $k \in \{3, 4\}$, Paper II, (4.9), gives $W_{13}(e) \le 0$. Its hypotheses are exactly squarefree e, $(e, N) = 1$, $\varepsilon N < e < N$ and $P^-(e) \ge N^\alpha$ [27, (4.9)]. Let $5 \le k \le 11$. The table after Paper II, (5.29a), shows that the three negative pair counts vanish, so

$$W_{13}(e) = 3 + 1_{k_1=0} - 2k_1 - 2k_2 - k_3 + \binom{k_1}{2} + \sum_{j=1}^{k_2} m_B(v_j),$$

where $m_B(v)$ counts first-band coordinates below $B_{\mathrm{O1}}(v)$ [27, (5.29a), (5.30)]. The bins determine $k_1, \ldots, k_4$. If $v_j \le v_\beta$, then $B_{\mathrm{O1}}(v_j) = \beta$ exceeds every first-band coordinate, so $m_B(v_j) = k_1$. If $v_j > v_\beta$, then $B_{\mathrm{O1}}(v_j) = C_{\mathrm{L3}} - \frac{5}{2} v_j$, and $m_B(v_j)$ is the unique m_j with $x_{m_j} < C_{\mathrm{L3}} - \frac{5}{2} v_j \le x_{m_j+1}$ (infinite sentinels). It is nonincreasing in j. Thus x lies in exactly one cell C of (8.2), and $W_{13}(e) = w_C$ [27, (5.31)]. If $w_C \le 0$, the left side of (8.4) is zero. Otherwise

$C \in \mathcal{C}_+$ and $a_C(e) = 1$, so the left side equals $w_C = A(e)$. This proves (8.4). The same argument, without the upper cut, gives

$$1_{\Omega(e)\ge 3}(W_{13}(e))_+ \le A^0(e) := \sum_{C\in\mathcal{C}_+} w_C a_C(e) \le 36 \tag{8.5}$$

for squarefree e with $P^-(e) \ge N^\alpha$ and $\varepsilon N < e < N$.

Put

$$b_A(n) = A(N-n), \qquad P_N(z) = \prod_{\substack{p<z\\ p\in\mathcal{P}_N}} p,$$

$$S_v(z) = \sum_{\substack{n\ge 1\\ v\mid n}} b_A(n) 1_{(n,P_N(z))=1}, \qquad S_A = S_1(N^{\mu_*}), \tag{8.6}$$

$$E_{\rm src} = (S_{\rm sw} - S_A)_+, \qquad Z_{\rm src} = (S_A - S_{\rm sw})_+.$$

The definitions give the finite identity

$$S_{\rm sw} = S_A + E_{\rm src} - Z_{\rm src}, \qquad E_{\rm src}, Z_{\rm src} \ge 0. \tag{8.7}$$

Under (8.4), the required bound is

$$E_{\rm src} \le 910 N^{1-\alpha} + 36(NL^{-333}+1) + 36N^{\mu_*}. \tag{8.8}$$

There are at most $2N^{1-\alpha}$ rough squareful integers. Each costs at most 455, by the pointwise weight bound of Paper II, proof of Lemma 1.1, immediately after (1.11) [27, Lemma 1.1, proof after (1.11)]. The last two terms cover the top strip and small primes.

Proof of (8.8). Write $e = N - p$ for $p \in \mathbb{P}$, $p < (1-\varepsilon)N$. Every prime $p \ge N^{\mu_*}$ has $(p, P_N(N^{\mu_*})) = 1$, and $A \ge 0$. Hence $S_A \ge \sum_{p\ge N^{\mu_*}} A(N-p)$, and

$$S_{\rm sw} - S_A \le \sum_{p\ge N^{\mu_*}} \big(s_{\rm sw}(e) - A(e)\big)_+ + \sum_{p<N^{\mu_*}} s_{\rm sw}(e).$$

Here $s_{\rm sw}(e) = 0$ unless $(e,N) = 1$ and $P^-(e) \ge N^\alpha$, and always $0 \le s_{\rm sw}(e) \le |W_{13}(e)| \le 455$ [27, Lemma 1.1, proof after (1.11)]. Split the summands into three classes; overlaps only enlarge the bound. (i) Non-squarefree $e < N$ with $P^-(e) \ge N^\alpha$ number at most $\sum_{\ell\ge N^\alpha} N/\ell^2 \le 2N^{1-\alpha}$; they cost at most $455 \cdot 2N^{1-\alpha} = 910N^{1-\alpha}$. (ii) Squarefree e with $N_{\rm cut} \le e < N$ number at most $NL^{-333}+1$; by (8.5) they cost at most $36(NL^{-333}+1)$. (iii) The primes $p < N^{\mu_*}$ with squarefree e cost at most $36N^{\mu_*}$, again by (8.5). For the remaining squarefree e with $\varepsilon N < e < N_{\rm cut}$ and $p \ge N^{\mu_*}$, (8.4) gives a zero summand. This proves (8.8). Since $W \gg N/L^2$, $E_{\rm src}/W \ll L^{-331}$.

8.2. **Four rows and the lower-sieve cutoff.** Use the clipped intervals $[N^t, N^{\rho t}) \cap [N^u, N^{\mu_*})$. Let $\mathcal{A}_j$ comprise the ordered prefixes

$$N^u \le a_j < \cdots < a_1 < N^{\mu_*}, \qquad a_i \in \mathcal{P}_N.$$

Write $v = a_1 \cdots a_j$. The empty prefix has $v = 1$. Define

$$T_0 = S_1(N^u), \quad T_1 = \sum_{\mathcal{A}_1} S_{a_1}(N^u), \quad T_2 = \sum_{\mathcal{A}_2} S_{a_1a_2}(N^u),$$
$$T_3^{\rm dyn} = \sum_{\mathcal{A}_3} S_{a_1a_2a_3}(a_3), \qquad S_A = T_0 - T_1 + T_2 - T_3^{\rm dyn}. \tag{8.9}$$

This is Buchstab's finite identity [27, (4.18)]. Its coefficient is $1 - k + \binom{k}{2} - \binom{k-1}{2} = 0$ for $k \ge 1$. At $k = 0$ it equals 1.

At depth three, let Θ_c be the infimum of $\widetilde{H}$ on the closed clipped parent. The underlying prime cells stay half-open. The closure is used only for this infimum. Use the whole anatomy function before choosing a branch [27, (2.17d)–(2.17e), (4.19)]. Set

$$\ell_c = \Theta_c - \eta_L - t_1 - t_2 - t_3, \quad D_c = N^{\ell_c}, \quad z_c = N^{t_3}, \quad \chi_c = 1_{\ell_c \ge 2\rho t_3}. \tag{8.10}$$

Retained prefixes have $\chi_c = 1$. For these prefixes define

$$B_3 = \sum_{\mathcal{A}_3:\chi_c=1} S_v(z_c), \qquad D_- = \sum_{\mathcal{A}_3:\chi_c=0} S_v(a_3),$$
$$D_+ = \sum_{\mathcal{A}_3:\chi_c=1} \{S_v(z_c) - S_v(a_3)\}, \quad P_3 = \sum_{\mathcal{A}_3:\chi_c=1} \sum_{\substack{r\in\mathcal{P}_N\\ z_c\le r<a_3}} S_{vr}(z_c), \tag{8.11}$$
$$Z_{\rm peel} = D_- + P_3 - D_+ \ge 0.$$

The union bound gives $D_+ \le P_3$. Therefore

$$S_A = T_0 - T_1 + T_2 - B_3 + P_3 - Z_{\mathrm{peel}}. \tag{8.12}$$

The favorable term enters exactly once [27, (4.45)].

8.3. **Capping the first three levels.** At depths $j = 0, 1, 2$, replace the relative level by

$$T_c = \sum_{i=1}^{j} t_i, \quad \ell_c = \frac{1}{2} - \eta_L - \rho T_c, \quad D_c = N^{\ell_c}, \quad z_{j,c} = N^u. \tag{8.13}$$

Here $T_c = 0$ at depth zero. Depth three keeps (8.10), with $z_{3,c} = z_c$. The elementary bounds are

$$\ell_c \ge \frac{1}{2} - \eta_L - j\rho\mu_* \ge \ell_1, \qquad s_c = \ell_c/u \ge 54.6375 - 10^{-87} > 30,$$

$$\text{continuum reference values for depths } 0,1,2 \text{ (the first three rows) at } (\eta_L, \rho) = (0,1) : 250,\ \frac{4875}{32},\ \frac{875}{16}, \tag{8.14}$$

$$q < N^{\rho T_c} D_c^{\rho+\sigma} \le N^{1/2-\eta_L+(\sigma+\tau)/2} < N^{1/2-9\cdot 10^{-5}}.$$

The modulus includes the smooth divisor and every tail prime. The exponent $\rho + \sigma$ is kept. The cap changes no row-three level.

The lower floor changes the repeated-prime argument. Put

$$u_1 = N^{\ell_1\sigma^2}, \qquad M_\sigma = \left\lceil \frac{3}{5\ell_0\sigma^2} \right\rceil, \qquad C_{\mathrm{L3},\sigma} = M_\sigma!. \tag{8.15}$$

(P2.8b) Rough-coordinate and repeated-prime bounds. Assume the support and rough-coordinate hypotheses stated below. These are the capped version of Paper II's Lemmas B.1–B.4. The coefficients and masks are defined in (8.30). Every rough coordinate is at least $u_c = D_c^{\sigma^2} \ge u_1$. On an active block, count these coordinates by m_c. The capped modulus bound gives

$$u_1^{m_c} \le q < N^{1/2}, \qquad m_c < \frac{500}{109\sigma^2} \le \frac{3}{5\ell_0\sigma^2}, \qquad \frac{m_c!}{M_c} \le C_{\mathrm{L3},\sigma}. \tag{8.16}$$

The prefix orbit size is $M_c \in \{1, 2, 6\}$. The small-prime part of q fixes the smooth divisor. The rough multiset fills at most $m_c!$ labelled positions. Thus

$$\frac{|C_{\mathfrak{b}}(q)|}{M_c} \le C_{\mathrm{L3},\sigma}. \tag{8.17}$$

Define $\mathcal{H}_1 = \{q < N^{1/2} : p^2 \mid q \text{ for some } p \ge u_1\}$. The forbidden and allowed-repeat masks satisfy

$$|\Xi_{U,\mathfrak{b}}(q)| + |\Xi_{A,\mathfrak{b}}(q)| \le C_{\mathrm{L3},\sigma} 1_{\mathcal{H}_1}(q). \tag{8.18}$$

Writing $q = vp^2$ and overcounting gives

$$\sum_{q\in\mathcal{H}_1} \frac{1}{q} \le \frac{2(1+L)}{u_1}, \qquad \#\mathcal{H}_1 \le \frac{2N^{1/2}}{u_1}. \tag{8.19}$$

The tested measure has $0 \le A \le 36$ and support below N. For $N \ge N_{\mathrm{dom}}$, $q/\varphi(q) \le 1 + L$ throughout $q < N^{1/2}$. Consequently

$$|\Delta_P(q)|,\ |d_A(q)| \le 36\left(\frac{N}{q} + 1 + \frac{N}{\varphi(q)}\right) \le 36\left(\frac{N(2+L)}{q} + 1\right). \tag{8.20}$$

Per block, these bounds give

$$\sum_q \big(|\Xi_U(q)\Delta_P(q)| + |\Xi_A(q)d_A(q)|\big) \le \frac{216 C_{\mathrm{L3},\sigma} N(1+L)^2}{u_1}. \tag{8.21}$$

There are finitely many blocks at fixed σ. Division by $W \gg N/L^2$ proves the conditional estimate

$$|E_{\mathrm{sq}}|/W = O_\sigma(N^{-\ell_1\sigma^2} L^4). \tag{8.22}$$

These are Paper II, Lemmas B.1–B.4 and (B.3)–(B.7), with u_0 replaced by u_1 on rows $0, 1, 2$ [27, Appendix B.1, Lemmas B.1–B.4, (B.3)–(B.7)]. Row three keeps $u_0 = N^{\ell_0\sigma^2}$, since its level floor in (8.43) exceeds ℓ_0.

Verification for the capped rows. Fix a block in row $j \le 2$. Prefix primes lie in $[N^{t_i}, N^{\rho t_i}) \cap [N^u, N^{\mu_*})$, so their product is below $N^{\rho T_c}$. The tail tuple satisfies $\prod_i D_i < D_c$ by its last cubic constraint, and the tail primes are below D_i^ρ; also $d < D_c^\sigma$. Thus

$$\log_N q < \rho T_c + (\rho+\sigma)\ell_c = (\rho+\sigma)(\tfrac12 - \eta_L) - \rho(\rho+\sigma-1)T_c \le (\rho+\sigma)(\tfrac12-\eta_L) < \tfrac12 - 9\cdot 10^{-5}.$$

Every prime of q lies in $\mathcal{P}_N$ and $d \mid P_N(u_c)$, so $(q, 2N) = 1$. Tail primes are at least $D_i \ge D_c^{\sigma^2} = u_c$. Prefix primes are at least $N^u > u_c$, because $\ell_c\sigma^2 < u$. Also $u_c = N^{\ell_c\sigma^2} \ge N^{\ell_1\sigma^2} = u_1$. For an active block choose a nonzero realization. Then $u_1^{m_c} \le q < N^{1/2}$, so $m_c < 1/(2\ell_1\sigma^2) = 500/(109\sigma^2)$, and $1/(2\ell_1) \le 3/(5\ell_0)$. This is (8.16). The primes of q below u_c determine d. The rough multiset fills the m_c labelled positions in at most $m_c!$ ways, and $|\lambda_d| \le 1$. This gives (8.17). It also gives (8.18). A forbidden tuple repeats a prime $\ge u_c$. An allowed non-squarefree tuple has $p^2 \mid q$ with $p \ge u_c$, since d is squarefree and coprime to the rough part. The grid count of Paper II, Lemma B.1(4), gives finitely many blocks at fixed σ. These are Paper II, Lemmas B.1–B.2, with $(u_0, N^{3/5})$ replaced by $(u_1, N^{1/2})$ [27, Appendix B.1, Lemmas B.1–B.2]. Lemma B.4 with u_1 and $Q = N^{1/2}$ gives (8.19). Lemma B.3, with $A \le 36 < A_{\max}$, support in $[1, N)$ and $q/\varphi(q) \le 1 + \log_2 q \le 1 + L$, gives (8.20). Then $\sum_{q\in\mathcal{H}_1}(N(2+L)/q + 1) \le 6N(1+L)^2/u_1$ gives (8.21). Row three is Paper II, Lemmas B.1–B.4 as printed: its modulus bound $q < N^{e_Q} < N^{3/5}$ and floor $\ell_c > \ell_0$ are (8.43). Hence (8.22) holds for all four rows.

8.4. **The signed sieve identity.** Put $\nu_j = (-1)^j$. For all four rows, define

$$s_c = \frac{\ell_c}{\log_N z_{j,c}} = \begin{cases} \ell_c/u, & j = 0,1,2, \\ \ell_c/t_3, & j = 3. \end{cases} \tag{8.23}$$

Let $K_j = F$ for even j, and $K_j = f$ for odd j. For a prefix v, set

$$\begin{aligned} &\mathcal{X}_v = X_A/\varphi(v), \qquad \omega_v(p) = \frac{p}{p-1}1_{p\in\mathcal{P}_N}, \qquad \omega_v(p^h) = 0 \quad (h \ge 2), \\ &V_N(z) = \prod_{\substack{p<z \\ p\in\mathcal{P}_N}} \left(1 - \frac{1}{p-1}\right), \qquad \Delta_0(q) = C_A(q;N) - \frac{X_A}{\varphi(q)}. \end{aligned} \tag{8.24}$$

All cutoff primes lie below the prefix primes. Thus the density product on the sieve support is V_N. The squarefree remainder is $\Delta_0(vd)$. Apply Fact 3.17 with $\epsilon_s = \sigma$, $\delta_s = \tau$, and $L_\omega = 2$. Choose a common nonnegative majorant

$$E_{s,c} := C_E\{\sigma + \sigma^{-8}e^{K_{\mathrm{M}}+2}(\ell_c L)^{-1/3}\}. \tag{8.25}$$

The signed sieve error satisfies $|E_s| \le E_{s,c}$.
Verification of the sieve data. Fix a row, a cell and a prefix v, and sieve the finite multiset $\{n : v \mid n\}$ with multiplicity $b_A(n) \ge 0$ by $\mathcal{P}_N$ below $z = z_{j,c}$. Every prefix prime is at least z: on rows $0, 1, 2$ it is at least $N^u = z$, and on row three $a_3 \ge N^{t_3} = z_c$. Hence $(d, v) = 1$ for $d \mid P_N(z)$. In particular $\omega_v(d) = \prod_{p\mid d} p/(p-1)$ for these d; the primes $p \mid v$ are all $\ge z$ and are never sifted. For squarefree such d,

$$|\mathcal{A}_d| = \sum_e A(e)1_{vd\mid N-e} = C_A(vd;N), \qquad \frac{\omega_v(d)}{d}\mathcal{X}_v = \frac{X_A}{\varphi(v)\varphi(d)} = \frac{X_A}{\varphi(vd)},$$

so $r(\mathcal{A}, d) = \Delta_0(vd)$. Each remainder modulus in (8.26) is squarefree. Here d uses primes below $D_c^{\sigma^2}$. Tail primes are at least $D_c^{\sigma^2}$ and distinct (equal labels by the mask, unequal grid intervals disjoint). Prefix primes are at least z. Next, $0 < \omega_v(p) = p/(p-1) < p$ for every $p \in \mathcal{P}_N$ (all such p are odd), $\omega_v = 0$ off $\mathcal{P}_N$, and $\omega_v(p^h) = 0$ for $h \ge 2$. Hence the second condition in (3.19) holds with $L_\omega = 2$. For the first, $(1 - \omega_v(p)/p)^{-1} = (1 - 1/(p-1))^{-1}$, and Paper II, (5.5), gives

$$\prod_{\substack{w\le p<z \\ p\in\mathcal{P}_N}} \left(1 - \frac{1}{p-1}\right)^{-1} \le \frac{\log z}{\log w}\left(1 + \frac{K}{\log w}\right) \qquad (2 \le w < z)$$

with one absolute K, uniformly in N [27, (5.5)]. Take K_{M} in (8.25) to be the maximum of this K and the constant of (3.25). It remains absolute and is fixed before σ. The product $V(z)$ of Fact 3.17 is then $V_N(z)$. The levels satisfy $z \le D_c$ on rows $0, 2$ ($s_c > 30$) and $z \le D_c^{1/2}$ on rows $1, 3$ ($s_c > 30$, resp. $\chi_c = 1$, i.e. $\ell_c \ge 2\rho t_3$). Fact 3.17 with $\epsilon_s = \sigma$, $\delta_s = \tau$, $D = D_c$ therefore applies to every actual sequence, and (8.28) follows with $E_s \le E_{s,c}$, $\log D_c = \ell_c L$.

For each cell fix one coefficient construction. Use it for every prefix realization in that cell. Write its remainders as

$$\begin{aligned} R_{1,v}^{\nu} &= \sum_{\substack{d<D_c^{\sigma} \\ d\mid P_N(D_c^{\sigma^2})}} \phi_d^{\nu}\Delta_0(vd), \\ R_v^{\nu} &= \sum_{\mathbf{D}\in\mathcal{D}^{\nu}(D_c)} \sum_{\substack{d<D_c^{\sigma} \\ d\mid P_N(D_c^{\sigma^2})}} \lambda_d^{\nu}(\mathbf{D}) \sum_{\substack{D_i\le p_i<\min(D_i^{\rho},z_{j,c}) \\ p_i\in\mathcal{P}_N,\ \text{equal-label entries distinct}}} \Delta_0(vd\textstyle\prod_i p_i). \end{aligned} \tag{8.26}$$

The coefficients satisfy $|\phi_d^{\nu}|, |\lambda_d^{\nu}(\mathbf{D})| \le 1$ [26, Theorem 2.6, (2.29)–(2.33b)]. Define the finite quantities

$$\begin{aligned} M_j &= \sum_c \sum_{v\in c} \frac{X_A}{\varphi(v)} V_N(z_{j,c}) K_j(s_c), \\ E_j &= \sum_c \sum_{v\in c} \frac{X_A}{\varphi(v)} V_N(z_{j,c}) E_{s,c}, \qquad R_j = \sum_c \sum_{v\in c} (R_{1,v}^{\nu_j} + R_v^{\nu_j}), \\ Q_j &= M_j + \nu_j E_j + \nu_j R_j, \qquad (S_0,S_1,S_2,S_3) = (T_0,T_1,T_2,B_3), \\ Z_{\mathrm{AA}} &= \sum_{j=0}^{3} \nu_j (Q_j - S_j), \qquad M_{\mathrm{prin}} = \sum_{j=0}^{3} \nu_j M_j, \\ M_{\mathrm{err}} &= \sum_{j=0}^{3} E_j, \qquad R_{\mathrm{adv}} = \sum_{j=0}^{3} R_j. \end{aligned} \tag{8.27}$$

The sums keep strict prefix order. Under the stated sieve hypotheses,

$$Q_0 \ge S_0, \quad Q_1 \le S_1, \quad Q_2 \ge S_2, \quad Q_3 \le S_3, \quad Z_{\mathrm{AA}} \ge 0. \tag{8.28}$$

Multiplying the signs gives

$$\sum_{j=0}^{3} \nu_j S_j = M_{\mathrm{prin}} + M_{\mathrm{err}} + R_{\mathrm{adv}} - Z_{\mathrm{AA}}. \tag{8.29}$$

Every remainder in R_{adv} has sign $+1$.

8.5. **The natural coprime comparator.** A block $\mathfrak{b} = (j, c, f, \mathbf{D})$ fixes a row, a cell, and either remainder type f. The tail vector is empty for R_1. Prefix realizations stay inside the coefficient. Let K_P, K_T impose prefix and tail distinctness. Let $C_{\mathfrak{b}}$ be the unrestricted labelled coefficient. Define

$$\begin{aligned} C_{\mathfrak{b}}(q) &= \sum_{\text{full labelled tuples}} \widetilde{\lambda}_d 1_{q=d\prod p_i}, \\ O_{\mathfrak{b}}(q) &= \frac{1}{M_c} \sum_{\text{full labelled tuples}} \widetilde{\lambda}_d K_P K_T 1_{q=d\prod p_i}, \\ \Xi_{U,\mathfrak{b}}(q) &= \frac{1}{M_c} \sum_{\text{full labelled tuples}} \widetilde{\lambda}_d (1-K_P K_T) 1_{q=d\prod p_i}, \\ O_{\mathfrak{b}} &= C_{\mathfrak{b}}/M_c - \Xi_{U,\mathfrak{b}}, \qquad \Xi_{A,\mathfrak{b}} = O_{\mathfrak{b}} 1_{q \text{ not squarefree}}. \end{aligned} \tag{8.30}$$

The two comparator discrepancies are

$$\Delta_P(q) = C_A(q;N) - \frac{Y_A(q)}{\varphi(q)}, \qquad d_A(q) = \Delta_0(q) - \Delta_P(q) = -\frac{X_A - Y_A(q)}{\varphi(q)}. \tag{8.31}$$

For $\langle f, g\rangle = \sum_q f(q)g(q)$, put

$$\begin{aligned} E_{\mathrm{disp},f} &= \sum_{\mathfrak{b}:\mathrm{type}=f} \frac{\langle C_{\mathfrak{b}}, \Delta_P\rangle}{M_c} \quad (f=1,2), \\ E_{\mathrm{sq}} &= \sum_{\mathfrak{b}} \{-\langle \Xi_{U,\mathfrak{b}}, \Delta_P\rangle + \langle \Xi_{A,\mathfrak{b}}, d_A\rangle\}, \\ E_{\mathrm{nc}} &= \sum_{\mathfrak{b}} \sum_{q \text{ squarefree}} O_{\mathfrak{b}}(q) d_A(q), \\ R_{\mathrm{adv}} &= E_{\mathrm{disp},1} + E_{\mathrm{disp},2} + E_{\mathrm{sq}} + E_{\mathrm{nc}}. \end{aligned} \tag{8.32}$$

This is a finite coefficient identity [27, (4.22)–(4.26), Appendix B.1–B.4]. All moduli satisfy $(q, 2N) = 1$. That restriction does not puncture A.

8.6. **The full-support bilinear application.** Split each prime coordinate into relative-width L^{-334} intervals. An interior box has constant order and descriptor predicates. For $B=\prod_i[A_i,B_i)$, put

$$T=\lceil A_1\rceil-1,\quad U_J=\lceil B_1\rceil-1,\quad I_T=(T,U_J],\quad X=\prod_i A_i,$$

$$a_n=1_{\mathbb{P}}(n)1_{I_T}(n),\qquad b_m=\sum_{\substack{p_2<\cdots<p_k\\ p_i\in[A_i,B_i)}}1_{m=p_2\cdots p_k},\tag{8.33}$$

$$b_m\in\{0,1\},\qquad N^{\alpha}/2\le T<N^{1/5},\qquad \varepsilon N\le X\le N.$$

The box contributes exactly $w_C a_n b_m$. For $T>2^{50}$ and $U_J\le 2T$,

$$a_n=1_{P^-(n)\ge T^{51/100}}1_{I_T}(n).\tag{8.34}$$

A composite $n\le 2T$ has a smaller prime divisor. This is Paper I, (3.8), option 1 [26, (3.8)–(3.9)]. Its logarithmic replacement error is zero.

Define the full-support discrepancy

$$\Delta_B(q,N)=\sum_{nm\equiv N\ (q)}a_nb_m-\frac{1}{\varphi(q)}\sum_{(nm,q)=1}a_nb_m,$$
$$\mathcal{S}_B(\mathfrak{b})=\sum_q C_{\mathfrak{b}}(q)\Delta_B(q,N).\tag{8.35}$$

There is no condition $nm\le X$ in these sums. The exact interior decomposition is

$$D_{\mathrm{int}}^{(j)}=\sum_{\mathfrak{b}:\mathrm{row}=j}\frac{1}{M_c}\sum_C\sum_{B\subset C\ \mathrm{interior}}w_C\mathcal{S}_B(\mathfrak{b}),$$
$$E_{\mathrm{disp},1}+E_{\mathrm{disp},2}=\sum_{j=0}^{3}D_{\mathrm{int}}^{(j)}+D_{\mathrm{bdry}}.\tag{8.36}$$

For rows $0,1,2$, apply Fact 3.3. Its Siegel–Walfisz sequence must be the selected prime sequence. The dictionary with Fact 3.3 is $b_n^{\mathrm{F2}}:=a_n$, $a_m^{\mathrm{F2}}:=b_m$, and $Y:=T$. Thus the prime sequence carries the Siegel–Walfisz hypothesis. We need Paper I, (3.25), in the form

$$\sum_{\substack{n\in I\\ n\equiv l\ (k)\\ (n,d)=1}}1_{\mathbb{P}}(n)-\frac{1}{\varphi(k)}\sum_{\substack{n\in I\\ (n,dk)=1}}1_{\mathbb{P}}(n)\ll_{A_{\mathrm{SW}}}T\tau(d)L^{-A_{\mathrm{SW}}},\qquad (l,k)=1,\quad I\subset I_T.\tag{8.37}$$

The normalized hypothesis also needs

$$\|a\|_2T^{1/2}\gg TL^{-168}.\tag{8.38}$$

Use Fact 3.1, specifically (3.2). Its endpoint estimate gives, for every fixed $A>335$,

$$\pi(U_J)-\pi(T)=\mathrm{Li}(U_J)-\mathrm{Li}(T)+O_A(TL^{-A}).\tag{8.39}$$

Full mesh intervals satisfy $U_J-T\gg TL^{-334}$. Thus $\#(I_T\cap\mathbb{P})\gg TL^{-335}$ uniformly. This implies (8.38). The $k=1$ discrepancy in (8.37) is identically zero. The counting input is the endpoint estimate just displayed.

Proof of (8.37) and (8.38). Let $N^{\alpha}/2\le T<N^{1/5}$, $U_J\le 2T$, $(l,k)=1$, and let $I=(y_1,y_2]\subset I_T$ be an arbitrary subinterval, so $T\le y_1\le y_2\le U_J$. (Paper I, (3.25), is quantified over all such I [26, (3.25)].) The condition $(n,d)=1$ removes at most $\omega(d)$ primes from each sum. If $k\le L^{A+2}$, apply (3.2) at $y=y_2$ and $y=y_1$, with $C=A+3$. Since $T\le y\le 2T$, $\log y\asymp L$ and the error is $O_A(TL^{-A})$ at each endpoint. Hence $\pi(y_2;k,l)-\pi(y_1;k,l)=(\mathrm{Li}(y_2)-\mathrm{Li}(y_1))/\varphi(k)+O_A(TL^{-A})$. The same holds for the unrestricted count divided by $\varphi(k)$, up to the $\omega(k)$ primes dividing k. If $k>L^{A+2}$, since $y_2-y_1\le T$ both terms are at most $T/k+1+T/\varphi(k)\ll TL^{-A}$, using $\varphi(k)\gg k/\log\log k$. This proves (8.37) with $\tau(d)$, ineffectively, uniformly in $I\subset I_T$. For the norm, $U_J-T\ge A_1L^{-334}-1\ge TL^{-334}/2$ on a full mesh interval, and (8.39) with $A=336$ gives $\#(I_T\cap\mathbb{P})\ge TL^{-335}/4$ eventually. Hence $\|a\|_2T^{1/2}\ge TL^{-335/2}/2\ge TL^{-168}$. Both bounds are uniform over all interior boxes, since only T enters.

Assume (8.37) and (8.38) uniformly. Taking $A_{\mathrm{SW}}=A_0+168$ verifies (3.4) with divisor exponent one. Both factor lengths exceed $x_B^{1/12}$ eventually. Split each of the $k-1$ complementary prime intervals at any dyadic boundary. This gives at most 2^{k-1} pieces. Localize each resulting product on $m\sim M_B$. Since $(1+L^{-334})^{k-1}<2$ eventually, there are at most two product dyadic intervals per piece. This bounded splitting changes no logarithmic exponent. Set

$x_B = TM_B \asymp N$. There are at most T selected integers in $(T, 2T]$ and at most $2M_B$ integers in a product dyadic interval. Unique factorization gives $\|a\|_2\|b\|_2 \le \sqrt{2x_B}$. The required application is

$$\begin{aligned} \frac{|C_{\mathfrak{b}}(q)|}{M_c} &\le C_{\mathrm{L3},\sigma}, \qquad Q < N^{1/2-9\cdot 10^{-5}} \le x_B^{1/2}(\log x_B)^{-B_{\mathrm{BV}}}, \\ \|a\|_2\|b\|_2 &\le \sqrt{2}\, x_B^{1/2}, \qquad |\mathcal{S}_B(\mathfrak{b})|/M_c \ll_\sigma NL^{-3692}. \end{aligned} \tag{8.40}$$

Use $A = 3692$, $B = 1$, $\epsilon = 1/12$ and $B_{\mathrm{BV}} = B_{\mathrm{BV}}(3692, 1, 1/12)$ in (3.5). The source has $O_\sigma(L^{3689})$ weighted boxes. There are finitely many blocks at fixed σ. Since $W \gg N/L^2$, summation gives

$$\frac{|D_{\mathrm{int}}^{(0)}| + |D_{\mathrm{int}}^{(1)}| + |D_{\mathrm{int}}^{(2)}|}{W} \ll_\sigma L^{-1}. \tag{8.41}$$

Verification of (8.40). The coefficient bound is (8.17), proved above, and the modulus range is (8.14). Every supported q is prime to $2N$, so $a = N$ is a reduced class and the maximum in (3.5) covers it. The selected sequence a_n is supported in $(T, 2T]$, and b_m, after at most two dyadic pieces, in $(M_B, 2M_B]$. For $x_B = TM_B$, $x_B \in [\varepsilon N/8, 2N]$ and $\log T/\log x_B \in [\alpha - o(1), 1/5 + o(1)] \subset [1/12, 11/12]$, because $\alpha - 1/12 = 17/4596 > 0$. The hypothesis (3.4) holds with $B = 1$ by (8.37), (8.38) and $A_{\mathrm{SW}} = A_0 + 168$. Interior boxes have constant order predicates, so $nm = e$ has one labelled representation and $b_m \in \{0, 1\}$. Thus $\|a\|_2\|b\|_2 \le (2x_B)^{1/2}$. On each interior box, $\mathcal{S}_B(\mathfrak{b}) = \sum_q C_{\mathfrak{b}}(q)\Delta_B(q, N)$ with $\Delta_B = \sum_{\mathrm{pieces}} \Delta_{\mathrm{fs}}$, and Fact 3.3 gives $|\mathcal{S}_B(\mathfrak{b})|/M_c \ll_\sigma NL^{-3692}$. Fact 3.3 is the full-support bilinear estimate derived from Bombieri–Friedlander–Iwaniec, Theorem 0(b). We verified its hypotheses above for this source.

8.7. **The source-level row-three estimate.** Row three keeps the full anatomy level. Its factorization keeps every labelled occurrence. The smooth divisor stays in the first factor. For $d_* = \rho(\Theta_c - \eta_L)$ and $e_i = \rho t_i$, the prefix inequalities are

$$\begin{aligned} d_* - 2e_1 &\ge \frac{33109}{160000}\rho, \\ d_* - e_1 - 2e_2 &\ge \frac{1859}{160000}\rho, \\ d_* - e_1 - e_2 - 2e_3 &\ge \frac{33109}{480000}\rho \quad (\chi_c = 1). \end{aligned} \tag{8.42}$$

The anatomy branch gives $\Theta_c \ge (1 + t_1)/2$. The condition $\chi_c = 1$ gives the last inequality. In particular, with $A_3 = 33109/160000$,

$$\begin{aligned} \ell_c \ge \max\{A_3 - t_3, 2\rho t_3\} \ge \frac{33109}{240000}, \qquad \ell_c - \ell_0 \ge \frac{231463}{24720000} = 0.0093633899676\ldots, \\ \Theta_c \le H_{\max} = \frac{153}{256}, \qquad e_Q = (\rho + \sigma)(153/256 - \eta_L) < \frac{3}{5}, \\ \frac{3}{5} - e_Q \ge \frac{391}{160000} - \frac{95609}{160000}(\sigma + \tau) > 0.0024437499. \end{aligned} \tag{8.43}$$

Here e_Q bounds the full modulus exponent $\rho(t_1 + t_2 + t_3) + (\rho + \sigma)\ell_c$. The inherited anatomy and rough-factor estimates are Paper II, (4.20) and Appendix B, Lemma B.1(1)–(2) [27, (4.20), Appendix B, Lemma B.1(1)–(2)]. For the tail exponents s_i, the cubic conditions give

$$\sum_{i<l} s_i + 2s_l < \ell_c, \qquad h = d_* + 60\sigma, \qquad \Theta_c - h \ge \eta_L - \tau H_{\max} - 60\sigma > 0. \tag{8.44}$$

Theorem 4.1 of Paper I then supplies the load surpluses

$$\left(\frac{7527}{256},\ 60,\ \frac{114421}{1280},\ \frac{26247}{256}\right)\sigma. \tag{8.45}$$

The required length data are

$$\begin{aligned} 1 \le |mn - N| \le N < X^3, \quad N_I \ge \frac{T}{6L^{334}} \ge (2T)^{1-2^{-126}}, \\ X^{60\sigma} \le T \le X^{1/3+15\sigma}, \quad QX^{\sigma/3} \le N_I, \quad QRS < M_I X^{-89\sigma}, \quad M_I \ge X^{1/2}. \end{aligned} \tag{8.46}$$

Here N_I is the selected integer interval length. The complement has length scale M_I. The letters Q, R, S are the three factor scales. Each condition holds only above its fixed-σ application threshold. The floor $2\lceil e^{2^{150}}/2\rceil$ is imposed separately. See Paper I, (3.6)–(3.9) and (4.9) [26, (3.6)–(3.9), Theorem 4.1, (4.9)]. We verify them for this source below.

The remaining analytic estimate is

$$|\mathcal{S}_B(\mathfrak{b})| \ll_{\sigma, A_{\mathrm{loc}}} XL^{-A_{\mathrm{loc}}} + X^{1-\ell_0\sigma^2/4}L^{C_{\mathrm{loc}}} \quad (\mathfrak{b} \text{ in row 3}). \tag{8.47}$$

It is uniform over the permitted boxes and blocks. Here $A_{\mathrm{loc}} > 0$ is arbitrary; C_{loc} is fixed by the local theorem. Paper I, Theorem 3.2, (3.13), supplies the local dispersion theorem. Its Theorem 2.8 enters only in the inner spectral stage. There we use the form (P1.7), Fact 3.22, with the full bracket (3.38). The inner substitution is

$$r_T = z, \quad s_T = p_0, \quad n_T = p_0 m, \quad a_T = \epsilon_y N \quad (\epsilon_y = \pm 1), \quad A_T = X^{1+\sigma/20}. \tag{8.48}$$

The auxiliary z and p_0 are those of Paper I's dispersion proof. Both signs and the mixed cusp pair are kept. Let χ be smooth, zero below zero and one above one. For the fixed bump ψ_0, define

$$\begin{aligned} \psi_{0,1}(v) &= \psi_0(v)\{1 - \chi((v-0.8)/0.1)\}, \\ \psi_{0,2}(v) &= \psi_0(v)\chi((v-0.8)/0.1)\{1 - \chi((v-1.4)/0.2)\}, \\ \psi_{0,3}(v) &= \psi_0(v)\chi((v-1.4)/0.2). \end{aligned} \tag{8.49}$$

They sum to ψ_0. Their strict dyadic windows are $(0.45, 0.90]$, $(0.80, 1.60]$, and $(1.40, 2.80]$. The two smooth modulus coordinates use all nine products $\psi_{0,i}(b/D)\psi_{0,j}(c/C)$, $1 \le i, j \le 3$. Multiply each by the fixed cutoffs on the supported n_T, z, p_0. These cutoffs equal one on the actual coefficient support. The derivative hypothesis is (3.36) with $\epsilon_0 = \sigma/2000$ and $\epsilon = \sigma/100$.

Write $\theta_{\mathrm{sp}} = 7/32$ and $V_{\mathrm{sp}} = T^4 R^4 S^6$. The coefficient in (P1.7) is zero outside the actual indices $n_T = p_0 m$, $r_T = z$, $s_T = p_0$. Use its supported norm

$$\begin{aligned} &\sum_{n'' \mid N^\infty} (n'')^{\theta_{\mathrm{sp}}} \|\widetilde{b}(n'')\|_2^2, \\ &\|\widetilde{b}(n'')\|_2^2 = \sum_{\substack{n', r, s \\ (n', N) = 1}} |b_{n' n'', r, s}|^2. \end{aligned} \tag{8.50}$$

Here $b_{n,r,s}$ is the coefficient of (P1.7). It is distinct from the complementary sequence b_m. The norm includes no coefficients outside that support. After substitution, the four bracket contributions divided by V_{sp} are bounded by

$$\begin{aligned} K_A &= A_T^{\theta_{\mathrm{sp}}} \frac{TQR^2S^5}{X^2}, & K_2 &= \frac{A_T}{X} K_A, \\ K_R &= \left(\frac{TR}{S}\right)^{2\theta_{\mathrm{sp}}} \frac{TQR^2S^5}{X^2}, & K_4 &= \frac{TQR^2S^5}{X^2} \frac{T^2R^2}{X}. \end{aligned} \tag{8.51}$$

Only uniform $X^{o(1)}$ factors and powers of p_0 are suppressed. These are precisely Paper I, (3.81), (3.85), and (3.87) [26, (3.69), (3.81)–(3.87)]. Each term carries the actual-use hypotheses.

8.8. **The row-three local dispersion estimate.** The source-wide estimate also has boundary and comparator contributions. The local argument gives only its interior row-three block.

We treat one row-three block with $\chi_c = 1$ and one full interior prime-coordinate mesh box. We do not assert the source-wide boundary, comparator, or row-zero-to-two estimates. Fix $\sigma > 0$, $\tau = \sigma^9$, $\rho = 1 + \tau$, and a retained closed clipped parent c, used only for the displayed infimum and endpoint checks. The actual prime cells are half-open. Keep its inherited labels $t_1 \ge t_2 \ge t_3$, $\Theta_c = \inf_c \widetilde{H}$, $\ell_c = \Theta_c - \eta_L - t_1 - t_2 - t_3$, $D_c = N^{\ell_c}$, $z_c = N^{t_3}$ and $u_c = D_c^{\sigma^2}$. On the whole block $\chi_c = 1$. The inequalities (8.42)–(8.43) apply with the inherited labels, including clipped endpoints. Here an interior source box is a full relative-width $x = (\log N)^{-334}$ mesh box whose closure lies on one side of every source, order and descriptor face. A clipped terminal mesh fragment belongs to the boundary contribution. LEN is not applied to it.

The prefix constants can be checked without an unclipped replacement. Put $A_3 = 33109/160000$. Since $t_1, t_2 \le 25/128$ and $\Theta_c \ge (1 + t_1)/2$, the first two gaps are at least $\rho(1/2 - 3(25/128)/2 - \eta_L) = \rho A_3$ and $\rho(1/2 - 5(25/128)/2 - \eta_L) = \rho\, 1859/160000$. For the third, $\ell_c - t_3 \ge A_3 - 2t_3$ and retention gives $\ell_c - t_3 \ge (2\rho - 1)t_3$; their maximum is at least $A_3/3$. Likewise $\ell_c \ge \max\{A_3 - t_3, 2\rho t_3\} \ge 2A_3/3 > \ell_0$. Thus all three prefix WF inequalities have strict room. The whole anatomy infimum is taken before this branch bound is used.

Exact coefficient and labelled factorization (H1–H2). Write $\mathfrak{b} = (3, c, f, \mathbf{D})$. Give the three prefix occurrences their independent prime sets $[N^{t_i}, N^{\rho t_i}) \cap [N^u, N^{\mu_*}) \cap \mathcal{P}_N$. For $f = 2$, append each tail occurrence, independently, with prime set $[D_j, \min(D_j^\rho, z_c)) \cap \mathcal{P}_N$. For $f = 1$ the tail list is empty. In both cases put J equal to this complete labelled list, and set $\lambda_d = \lambda_d^-(\mathbf{D})$ or ϕ_d^-, respectively. Thus, exactly,

$$C_{\mathfrak{b}}(q) = \sum_{\substack{d < D_c^\sigma \\ d \mid P_N(u_c)}} \lambda_d \sum_{(p_j)_{j \in J}} 1_{q = d \prod_{j \in J} p_j}, \qquad |\lambda_d| \le 1.$$

Neither distinctness mask is inserted here. The orbit divisor M_c, the source weight w_C, and the two distinctness corrections remain outside this coefficient, as in (8.30)–(8.32). In particular its supports include repeated-prime moduli, and $(q, 2N) = 1$. The prime sets are the actual clipped sets, not their unclipped parents. All labelled primes are at least $u_c \ge u_0$ with $u_0 = N^{\ell_0 \sigma^2}$, since $\ell_c > \ell_0$ and $\ell_c \sigma^2 < u$.

For packing use the actual upper exponents of the clipped prime sets. The first three are at most ρt_i; all tail upper exponents are at most t_3. The prefix masks in (8.42) and the tail masks $\sum_{i<j} s_i + 2s_j < \ell_c$ show that their merged decreasing list satisfies $\mathrm{WF}(d_*)$, where $d_* = \rho(\Theta_c - \eta_L)$. For a tail rank, this follows after adding at most $\rho(t_1 + t_2 + t_3)$ to its tail inequality. The top three upper exponents are the prefix endpoints, belonging to the closed clipped parent. Hence their anatomy value is at least Θ_c. Put $h = d_* + 60\sigma$, $\delta = 30\sigma$ and $g = \log_N T - \delta$. Every point $\mathbf{x}$ of the closed clipped parent satisfies $u \le x_1 \le \mu_* = 25/128$. On this interval the $v_{\mathrm{pk}}(x_1)$ branch of Paper I gives $\widetilde{H}(\mathbf{x}) \ge v_{\mathrm{pk}}(x_1) = (1 + x_1)/2$; hence $\Theta_c \ge (1 + u)/2$. Therefore $h \ge \Theta_c - \eta_L + 60\sigma \ge \frac{1+u}{2} - \eta_L > \frac{1}{2}$; also $h < \Theta_c \le 153/256$, since $\Theta_c - h \ge \eta_L - \tau H_{\max} - 60\sigma > 0$. Theorem 4.1 of Paper I applies to this same labelled list, with $\delta \le g \le 1/3 - \delta/2$. Let its partition be $J = J_1 \sqcup J_2 \sqcup J_3$, with the smooth divisor attached to slot one. Define, using precisely the preceding prime sets,

$$\gamma_1(v) = \frac{1}{|J_1|!} \sum_{d,(p_j)_{J_1}} \lambda_d 1_{v=d \prod_{J_1} p_j}, \qquad \gamma_s(v) = \frac{1}{|J_s|!} \sum_{(p_j)_{J_s}} 1_{v=\prod_{J_s} p_j} \quad (s = 2, 3).$$

Empty slots two and three mean the atom $1_{v=1}$. Slot one always keeps the d sum, even if J_1 is empty. The finite tuple map gives

$$C_{\mathfrak{b}} = C_{\mathrm{slot}}(\gamma_1 * \gamma_2 * \gamma_3), \qquad C_{\mathrm{slot}} = \prod_s |J_s|!, \qquad |\gamma_s| \le 1.$$

The primes below u_c determine d uniquely. At most $|J_s|!$ assignments have a fixed slot product. The count allows repeated primes. The rough coordinate count is less than $3/(5\ell_0\sigma^2)$, so $C_{\mathrm{slot}} = O_\sigma(1)$ and $|C_{\mathfrak{b}}(q)|/M_c \le M_\sigma!$, with $M_\sigma = \lceil 3/(5\ell_0\sigma^2) \rceil$. For the tested boxes below, take $C_{\mathrm{ctr}} = 2$. Uniformly for sufficiently large N,

$$\frac{N_I}{TL^{-334}} \in [1/2, 2], \qquad \frac{M_I}{(X/T)L^{-334}} \in [2, 20].$$

The same tested box (H3–H5). Use exactly (8.33) and (8.35): $X = \prod_i A_i$, $T = \lceil A_1 \rceil - 1$, $U_J = \lceil B_1 \rceil - 1$, and $\alpha_n = 1_{\mathbb{P}}(n) 1_{T<n\le U_J}$. No product truncation is added to Δ_B. Take $C_{\mathrm{box}} = 334$; $\log X = \log N + O(1)$ throughout this family. For sufficiently large N, $T > 2^{50}$ and $U_J \le 2T$, so

$$\alpha_n = 1_{P^-(n) \ge T^{51/100}} 1_{T<n\le U_J}.$$

Thus (3.8), option 1, holds with $\zeta = T^{51/100} \ge z_0(X)$. The complementary sequence is the actual $b_m \in \{0, 1\}$ of (8.33); in particular $|b_m| \le \tau_{23}(m)$. There is no Heath–Brown expansion, logarithm freezing, or proper-power replacement error for this selected direct-prime representation. The selected-factor Siegel–Walfisz bound is Paper I (3.25), uniform in this stronger cutoff and every endpoint and auxiliary coprimality. Its relative-norm alternative is kept as in (3.48). We assume no lower bound on any coprimality-truncated selected norm. In the large-norm case of (3.48), use Lemma 6.5(ii). Its premises (3.1), (3.2), (3.6), (3.8) with LEN, (3.11), (3.25), (3.26) and (3.27) are the data verified in this subsection.

Let $M = \prod_{i=2}^k A_i$, $M_+ = \prod_{i=2}^k B_i$, $M_I = M_+ - M$, $m_0 = (M + M_+)/2$, and $\psi_I(m) = \phi_0((m - m_0)/M_I)$, with the fixed bump of Paper I (3.9). Then $\psi_I = 1$ on the full complementary product interval. Since $5 \le k \le 11$ and the mesh boxes are full, $M_I \asymp ML^{-334}$, $m_0 \le 2X/T$, and $N_I = U_J - T \ge T/(6L^{334})$, where $L = \log N$. For fixed σ enlarge the threshold so that $\sigma L \ge 2^{130}(\log L + 1)$; then

$$N_I \ge (2T)^{1-2^{-126}}, \quad X^{60\sigma} \le T \le X^{1/3+15\sigma}, \quad M_I \ge X^{1/2}.$$

These follow from $N^\alpha/2 \le T < N^{1/5}$ and $\varepsilon N \le X \le N$. The integer convention is $(T, U_J]$; no continuous interval length replaces N_I.

The doubled support lies in $[M - M_I/2, M_+ + M_I/2]$. Since $k - 1 \le 10$, $(1 + x)^{k-1} - 1 \le 11x$ eventually. On this entire support and for every selected integer, $mn \le N_{\mathrm{cut}} + 25xN$. Consequently

$$1 \le N - mn, \qquad N - mn \ge (L - 25)NL^{-334} > 0, \qquad |mn - N| \le N < X^3.$$

This verifies nonresonance on the full doubled rectangle, including both Cauchy–Schwarz copies. Checking only where $b_m \ne 0$ would not suffice.

After localization, Q, R, S denote the scales of $\gamma_1, \gamma_3, \gamma_2$, respectively. Theorem 4.1 gives, before fixed dyadic rounding and base conversion, reductions

$$(7527/256,\ 60,\ 114421/1280,\ 26247/256)\sigma$$

in Q/T, T^2R^2S, $A_0^{7/32}TQR^2S^5$ and $(TR/S)^{7/16}TQR^2S^5$, respectively, relative to their bounds $1, N, N^2, N^2$. Here $A_0 = X^{1+\sigma/20}$. The first slot includes the additional load $\sigma\ell_c \le \sigma H_{\max}$. Each listed reduction strictly exceeds the required losses $29, 8, 15, 15$. Hence fixed dyadic constants and $\log_N(N/X) = O(1/L)$ are absorbed for fixed σ. In particular the exact requirements (3.6)–(3.7) hold:

$$Q < TX^{-29\sigma}, \quad N_I \ge QX^{\sigma/3}, \quad QRS < M_I X^{-89\sigma}, \quad M_I \ge X^{1/2},$$
$$T^2R^2S < X^{1-8\sigma}, \quad A_0^{7/32}TQR^2S^5 < X^{2-15\sigma}, \quad (TR/S)^{7/16}TQR^2S^5 < X^{2-15\sigma}.$$

For $QRS < M_I X^{-89\sigma}$ use the positive fixed gap $4/5 - 153/256$ and $M_I \gg N^{4/5}L^{-334}$. The complete modulus, including d and all labelled primes, obeys

$$q < N^{e_Q}, \qquad e_Q = (\rho+\sigma)(153/256 - \eta_L) < 3/5.$$

Because $X \ge \varepsilon N$, impose $(3/5 - e_Q)\log N \ge (3/5)\log(1/\varepsilon)$ to obtain $q \le X^{3/5}$. Take the moving frequency $a = N$ and $K_{\mathrm{scale}} = 1$: eventually $0 < N \le X\log N \le A_0$ and every supported modulus is prime to $2a$. This proves H1–H5 with the support relaxation of Paper I Theorem 3.2.

Application and restoration. Apply Paper I Theorem 3.2, (3.13), to this full unrestricted coefficient and this full-support natural-coprime discrepancy. For every fixed $A_{\mathrm{loc}} > 0$ it gives exactly

$$|\mathcal{S}_B(\mathfrak{b})| \ll_{\sigma, A_{\mathrm{loc}}} X(\log X)^{-A_{\mathrm{loc}}} + X^{1-\ell_0\sigma^2/4}(\log X)^{C_{\mathrm{loc}}}.$$

Constants are uniform in these boxes and in blocks with $\chi_c = 1$ after fixing σ and source constants. The threshold is ineffective. Theorem 3.1's smaller abstract support class is not used. The proof includes all local restorations (3.32)–(3.50): with $\epsilon = \ell_0\sigma^2/8$ and $A_{\mathrm{work}} = A_{\mathrm{loc}} + 4$,

$$XL^{-A_{\mathrm{work}}} + XL^C z_0^{-1/2} + XL^C(Q/N_I)^{1/2} + XX^{\epsilon/2}L^C\{u_0^{-1/2} + (Q/N_I)^{1/2}\}.$$

Keep both terms in braces. They come from restoring the five cross-copy gcd failures of (3.30), the close-pair set, and the zero-mode main-term cancellation. The spectral estimate alone does not give them. The z_0 term joins the arbitrary logarithmic saving, and the remaining terms have savings exceeding $\ell_0\sigma^2/4$.

Theorem 2.8 enters only at Paper I (3.69)–(3.90). In that inner stage, with $p_0 = \min\{p : p \nmid 2N\}$, set $r_T = z$, $s_T = p_0$, $n_T = p_0 m$, $a_T = \varepsilon_y N$, $N_T = 8p_0 H_* SF$ and $A_T = A_0$. Both signs are kept. The coefficient is exactly $\beta_{z,m}^{\chi,\varepsilon_y}$ of (3.68) on these indices and zero elsewhere. In particular

$$\sum_{n''|N^\infty} (n'')^{7/32}\|\widetilde{b}(n'')\|_2^2 \ll X^{o(1)}(H_*S)^2F, \quad \|\widetilde{b}(n'')\|_2^2 = \sum_{\substack{n',r,s\\(n',N)=1}} |b_{n'n'',r,s}|^2,$$

with no coefficients added outside that support. The unweighted norm has the same bound. These are the endpoint norm bounds (3.70)–(3.81). When slot two is a unit, $S = 1/2$ and every nonzero-frequency block is empty. Otherwise (3.82)–(3.85) identify the exact phase, permit $p_0 \mid c$, and use all nine dyadic weights from (8.49), with $\epsilon_0 = \sigma/2000$ and $\epsilon = \sigma/100$. The full bracket of (3.38), including its moving-frequency and mixed-cusp terms, yields all four quantities in (8.51). No bracket term is discarded. The full bounds (3.87)–(3.90) then control the inner spectral term. After that, the restoration terms above and C_{slot} are put back.

Finally, this estimate concerns $C_{\mathfrak{b}}\Delta_B$. The outer identity is still

$$R_{\mathrm{adv}} = E_{\mathrm{disp},1} + E_{\mathrm{disp},2} + E_{\mathrm{sq}} + E_{\mathrm{nc}}.$$

Thus M_c^{-1} and w_C enter once. The forbidden and allowed-repeat masks remain in E_{sq}. The natural-to-X_A comparator remains in E_{nc}. The source cutoff, peel, boundary and terminal-mesh restorations, capped rows zero to two, and complete-family coefficient counts require their own estimates. This local estimate alone gives neither the complete L3 bound nor the RSW bound.

The outer application of Theorem 3.2 above supplies (8.47). The scoped spectral implications in Section 3 do not by themselves give it.

8.9. **The complete box family.** The required complete-family bounds are

$$\#\mathcal{E}_{\mathrm{bdry}} \ll NL^{-334}, \qquad D_{\mathrm{bdry}} \ll_\sigma NL^{-331/2}, \qquad \sum_B |w_B| \ll_\sigma L^{3689}. \tag{8.52}$$

There are at most 122 face families per k. Their integer normals have maximum coefficient five. Their coefficient sums are at most eleven.

Proof of the complete-family bounds. Use the unclipped geometric mesh $[A^{(j)}, A^{(j+1)})$, $A^{(j)} = (N^\alpha/2)(1 + L^{-334})^j$, $j \ge 0$, in every coordinate. Boxes with a coordinate $\ge N^{1-4\alpha}$ carry no source mass, since $k \ge 5$ and every prime is at least N^α. So there are no terminal fragments, and every interior box is a full mesh box. *Box count.* Each coordinate has at most $((1-\alpha)L + 1)(L^{334} + 1) + 1 \le L^{335}$ mesh intervals for $L \ge 24$. With $k \le 11$, finitely many cells and $w_C \le 36$, $\sum_B |w_B| \le 36|\mathcal{C}_+|\sum_{k=5}^{11} L^{335k} \ll L^{3685} \le L^{3689}$. *Boundary count.* Let $h' = \log(1 + L^{-334})/L \le L^{-335}$. A source integer e in a boundary box has its vector within h' in each coordinate of a face. A radial face forces e into $[y(1 + L^{-334})^{-11}, y(1 + L^{-334})^{11}]$ with $y \in \{\varepsilon N, N_{\mathrm{cut}}\}$. There are at most $2(23NL^{-334} + 1)$ such e. Every other face is $c_i x_i + c_j x_j = \theta$ with integers $|c_i|, |c_j| \le 5$, $c_i \ne 0$ [27, (5.29a)–(5.31) and the facet discussion after (5.40)]. Write $e = p_i m'$. For fixed m', p_i lies in an interval of relative width at most $21L^{-334}$ below N/m', which contains at most $21L^{-334}N/m' + 1$ integers. Here m' is a product of at most ten primes in $[N^\alpha, N)$, and $m' < N^{1-\alpha}$. By Mertens, $\sum_{m'} 1/m' \le \exp(\sum_{N^\alpha \le p < N} 1/p) \le e/\alpha < 32$ eventually. Hence one face family contributes at most $700NL^{-334} + N^{1-\alpha}$. The number of face families is finite (≤ 122 for each k). Thus $\#\mathcal{E}_{\mathrm{bdry}} \le C_{\mathrm{bd}}NL^{-334}$ with an absolute C_{bd}. No numerical value is required below. *Boundary discrepancy.* Let A_{bd} be A restricted to boundary

boxes. Then $D_{\mathrm{bdry}} = \sum_{\mathfrak{b}} M_c^{-1}\langle C_{\mathfrak{b}}, \Delta_{P,\mathrm{bd}}\rangle$. By $|C_{\mathfrak{b}}|/M_c \le C_{\mathrm{L3},\sigma}$, $\sum_q 1_{q|N-e} \le \tau(N-e)$, Cauchy–Schwarz, $\tau^2 \le \tau_4$, $\sum_{n\le N}\tau_4(n) \le N(1+L)^3$ and $\sum_{q\le N} 1/\varphi(q) \le 2(1+L)$,

$$|D_{\mathrm{bdry}}| \ll_\sigma 36(\#\mathcal{E}_{\mathrm{bdry}})^{1/2}(N(1+L)^3)^{1/2} + 36\,\#\mathcal{E}_{\mathrm{bdry}}(1+L) \ll_\sigma NL^{-331/2},$$

with finitely many blocks at fixed σ. This proves (8.52), with the first bound in the form $\#\mathcal{E}_{\mathrm{bdry}} \ll NL^{-334}$. Under (8.47), take $A_{\mathrm{loc}} \ge 3692$. The L^{3689} box count and $W \gg N/L^2$ then leave L^{-1}. Equations (8.41) and (8.52) then give

$$(R_{\mathrm{adv}})_+/W = o_\sigma(1), \tag{8.53}$$

provided (8.22) and the coprime-comparator bound hold.

8.10. **The principal term and its price.**
(P2.8d) The source-independent profile comparison. We use Paper II, (4.70)–(4.75), in the following form. Put

$$\begin{aligned} v_1(s) &= \min\left\{\frac{1+s}{2}, \frac{32-34s}{39}, \frac{57}{64} - \frac{3}{2}s\right\}, \\ v_*(s) &= v_1(s) - u, \qquad \psi_*(x,y) = \begin{cases} x, & x \ge y, \\ -\infty, & x < y, \end{cases} \\ \Theta_*(x,y,z) &= \max\{v_*(x), v_*(y), v_*(x+y), v_*(x+y+z), \\ &\qquad \psi_*(v_*(x+z), x+2y+z), \psi_*(v_*(y+z), 2x+y+z)\}. \end{aligned} \tag{8.54}$$

The subtraction of u precedes both guards. This is the same guard convention as equation B.14. It agrees with Paper II's zero guard because an unguarded branch is positive, as explained immediately after that display. Define

$$\begin{aligned} H_{\log} &= \log(\mu_*/u), \qquad a_3(\mathbf{t}) = \frac{\Theta_*(\mathbf{t}) - t_1 - t_2 - t_3}{t_3}, \\ J_* &= \int_u^{\mu_*} \frac{dt_1}{t_1}\int_u^{t_1}\frac{dt_2}{t_2}\int_u^{t_2}\frac{1-f(a_3(\mathbf{t}))}{t_3^2}\,dt_3, \\ R_{\mathrm{up}} &= 500(F(30)-1) + 500H_{\log}(1-f(30)) + 250H_{\log}^2(F(30)-1), \\ G_* &= \frac{1}{\mu_*} + R_{\mathrm{up}} + J_*. \end{aligned} \tag{8.55}$$

Extend f by zero below two. These are Paper II, (4.70)–(4.75) [27, (4.70)–(4.75)]. Their source-independent comparison uses

$$g = (u - \eta_L) - \frac{9}{2}\mu_*\tau > 0.001899 > 2\mu_*\tau. \tag{8.56}$$

Hence a positive canonical lower term lies in a cell with $\chi_c = 1$. The capped rows still satisfy $s_c > 30$. Their residuals are bounded by the same R_{up}. Thus the cap changes neither G_* nor its price.

Let $\mathcal{P}_\Pi$ be the signed prefix profile:

$$\begin{aligned} \mathcal{P}_\Pi = \frac{1}{u}F(s_0) - \frac{1}{u}\sum_{c:j=1}\mathrm{vol}_c\, f(s_c) + \frac{1}{u}\sum_{c:j=2}\mathrm{vol}_c\, F(s_c) \\ - \sum_{c:j=3,\chi_c=1}\frac{\mathrm{vol}_c}{t_3}f(\ell_c/t_3), \qquad \mathrm{vol}_c = \int_c\prod_i\frac{dt_i}{t_i}. \end{aligned} \tag{8.57}$$

The required mass and comparison statements are

$$\begin{aligned} X_A &\le \frac{N}{L}\{(1-\varepsilon)I_{\mathrm{HL}} + O(L^{-1})\}, \\ M_{\mathrm{prin}} &= X_A\frac{2e^{-\gamma}C(N)}{L}(\mathcal{P}_\Pi + o_\sigma(1)), \qquad \mathcal{P}_\Pi \le G_* + o_\sigma(1). \end{aligned} \tag{8.58}$$

The first bound is Paper II, (5.34)–(5.42). The last comparison uses (4.72)–(4.75) [27, (4.72)–(4.75), (5.34)–(5.42)]. Proof of (8.58). *Mass.* The cells $\mathcal{C}_+$ are Paper II's cells with $w_C \ge 1$, and the radial range $\varepsilon N < e < N$ is Paper II's. The outer-box majorant f^+ of Paper II, (5.34), counts labelled factorizations into all primes in $[N^\alpha, N)$, and $c(\mathbf{j}) \ge w_C$ for the box $\mathbf{j}$ containing the ordered factor vector of e, which lies in C_N. Hence $A \le f^+$ pointwise directly (no comparison with the intermediate g is used). Paper II, (5.34)–(5.42), then gives $X_A \le X^+ = \frac{N}{L}\{(1-\varepsilon)I_{\mathrm{HL}} + O(L^{-1})\}$ [27, (5.32), (5.34), (5.42)]. In particular $X_A \le N/L$ eventually, since $I_{\mathrm{HL}} \le \overline{H} < 1$. *Principal term.* As in Paper II, Lemma B.9, $V_N(z) = (1+o(1))2e^{-\gamma}C(N)/\log z$ uniformly for $z \ge N^h$, $h = 1/1000$ [27, (B.9) and its proof]. The proof of (B.9) gives an exact factorization. Mertens' theorem gives the asymptotic. The primes $p \mid N$ with $p \ge z$ change the product by $1 + O(\omega(N)/z)$. Every actual cutoff is at least $N^{u/\rho^3} > N^h$ [27, Lemma B.8]. At fixed σ the cells are finitely many, with fixed exponent boxes. Mertens gives $\sum_{v\in c} 1/\varphi(v) = \mathrm{vol}_c + o(1)$, the primes dividing

$2N$ costing $O(\omega(N)N^{-u})$. Also $K_j(s_c)$ is bounded. Summing $\nu_j M_j$ gives $M_{\mathrm{prin}} = X_A \frac{2e^{-\gamma}C(N)}{L}(\mathcal{P}_\Pi + o_\sigma(1))$, with $\log_N z_{3,c} = t_3$ supplying the factor $1/t_3$. *Profile comparison.* Write $F = 1 + (F-1)$, $f = 1 - (1-f)$. On rows $0, 1, 2$, $s_c > 30$, and F decreases and f increases. With $\sum_{j=1} \mathrm{vol}_c = H_{\log}$ and $\sum_{j=2} \mathrm{vol}_c = H_{\log}^2/2$, the residuals are at most R_{up} [27, (4.73)]. For row three, if $f(a_3(\mathbf{x})) > 0$ at an actual point $\mathbf{x}$ of a cell, then (8.56) gives $\chi_c = 1$ and $\ell_c \geq \Theta_*(\mathbf{x}) - x_1 - x_2 - x_3$ [27, (4.72) and the two paragraphs after it]. For $x_3 \in [t_3, \rho t_3]$, monotonicity of $sf(s)$ at fixed numerator ℓ_c gives $f(\ell_c/t_3)/t_3 \geq f(\ell_c/x_3)/x_3 \geq f(a_3(\mathbf{x}))/x_3$. Integrating over each retained cell, $\sum_{j=3,\chi_c=1} \mathrm{vol}_c\, f(\ell_c/t_3)/t_3 \geq \iiint f(a_3)\, dt_1 dt_2 dt_3/(t_1 t_2 t_3^2)$. The constant parts give $1/\mu_*$ by [27, (4.74)]. Hence $\mathcal{P}_\Pi \leq 1/\mu_* + R_{\mathrm{up}} + J_* + o_\sigma(1) = G_* + o_\sigma(1)$, which is [27, (4.75)] for this source.

Use the same coefficient bound as the exterior comparison:

$$G_* \leq G_0 = \frac{6088440363}{10^9}, \qquad 2e^{-\gamma}G_0 \leq c_{K2,G0} = \frac{6836825163875517756286846321650754372931094689}{10^{45}}, \tag{8.59}$$
$$I_{\mathrm{HL}} \leq \overline{H} = \frac{389253}{5000000}.$$

The functional bound $G_* \leq G_0$ is Paper II, [27, (4.73), (4.75), (4.79), (A.10a)], for the identical canonical functional (8.55). Its scalar rounding is the same as (7.36). The mass bound is Paper II, (A.11) [27, (A.11)]. The optional unshifted certificate in the unshifted-comparison subsection of Appendix B is outside this proof.

The constant c_{O1} is optional, non-certified appendix material outside the L3 proof. In the optional scalar comparisons below, c denotes the rational from (1.6). No functional comparison with c_{O1} is needed here. Its rational definition is

$$c_{\mathrm{O1}} = \frac{\sum_{i=0}^{7} n_i B^i}{\sum_{i=0}^{7} d_i B^i}, \qquad B = 10^{25}, \tag{8.60}$$

with the integers in Table 8.1. The exact rational comparison of c and c_{O1} is in Section 9, (9.5). It uses no exponential

TABLE 8.1. Exact certificate constant

i	n_i	d_i
0	4790995369589327356095267	5033118760359274030825472
1	6963256824825772577191322	8266496527677353149933429
2	8996423827217290029713216	9138360880901719297497815
3	4916164251930472277375479	4671424972330288696699300
4	3295726346285190737123 85	9395901084673313239101 41
5	9317769528669888650546075	5634060699913223990647374
6	8280830054071768648850166	2330146450712536415611653
7	389762593301325772	57030507720277440

comparison. It is independent of the coarse enclosure for U_0. The mass-integral bound is Paper II, (A.11) [27, (A.11)]. Put

$$m_{\mathrm{sw}} = \frac{e^{-\gamma}}{2} I_{\mathrm{HL}} G_*, \qquad \Delta_{\mathrm{cert}} = \frac{c_{K2,G0}}{4}\overline{H} - m_{\mathrm{sw}} \geq 0,$$
$$C_{\mathrm{sw}} = \frac{650000000}{1111} C_E, \qquad P_{\mathrm{sw}} = \frac{48544888063158}{27775}, \tag{8.61}$$
$$a_{\mathrm{sw}} = C_{\mathrm{sw}}\sigma, \qquad p_{\mathrm{sw}} = P_{\mathrm{sw}}\sigma^9.$$

The remaining required estimates are

$$(M_{\mathrm{prin}}/W - m_{\mathrm{sw}})_+ = o_\sigma(1),$$
$$M_{\mathrm{err}}/W \leq C_{\mathrm{sw}}\sigma + o_\sigma(1), \qquad P_3/W \leq P_{\mathrm{sw}}\sigma^9 + o_\sigma(1), \tag{8.62}$$
$$|E_{\mathrm{nc}}|/W = O_\sigma(N^{-\alpha}L^4).$$

The exact numerical multipliers are

$$\frac{4000 \cdot 585}{4(1-10^{-4})} = \frac{650000000}{1111}, \qquad \frac{36 \cdot 502002(459/100)^3}{1-10^{-4}} = \frac{48544888063158}{27775}. \tag{8.63}$$

(P2.8c) Density product and prefix sums. Put $h = 1/1000$ and $H_N = \sum_{N^h \leq p \leq N} (p-1)^{-1}$. The two bounds used in the first multiplier are

$$V_N(z) \leq \frac{4C(N)}{\log z} \leq \frac{4000C(N)}{L} \quad (z \geq N^h),$$
$$\sum_{\text{actual prefixes } v} \frac{1}{\varphi(v)} \leq \sum_{j=0}^{3} H_N^j \leq 1 + 8 + 64 + 512 = 585. \tag{8.64}$$

These are Paper II, Appendix B, Lemmas B.9 and B.11, (B.9) and (B.11) [27, Appendix B, Lemmas B.9, B.11, (B.9), (B.11)]. Each half-open prefix cell is counted once. Using $X_A \le N/L$ from (P2.8d) now gives

$$M_{\mathrm{err}} \le \frac{X_A 4000C(N)}{L}\, 585\{C_E\sigma + o_\sigma(1)\}. \tag{8.65}$$

Division by $W = 4(1-\varepsilon)C(N)N/L^2$ gives C_{sw}.
(P2.8e) The upper mass and the peel. The source mass is (8.58), Paper II, (5.34)–(5.42). The native lower-sieve cutoff uses the finite peel (8.11) and the estimate

$$\frac{P_3}{W} \le \frac{36\cdot 502002(459/100)^3}{1-\varepsilon}\tau + o_\sigma(1). \tag{8.66}$$

For the uniform two-form calculation, put $m = vr$ and $w = \min(z_c, N^\alpha)$. The required ranges are

$$\begin{gathered}\log_N m \le 153/256 + 2\tau, \qquad \log_N w \ge 2/1001,\\ w^2 m \le N^{253/256+(153/64)\tau} < N^{1-0.0116}.\end{gathered} \tag{8.67}$$

For each sieve prime $p < w$, every prime factor of $m = vr$ is at least $z_c \ge w$, so $p \nmid m$. The local two-form root count is then $\nu_N(p) = 1$ for $p \mid N$, and $\nu_N(p) = 2$ otherwise. For squarefree d, define $g_N(d)$ multiplicatively by $g_N(p) = \nu_N(p)/(p - \nu_N(p))$. The denominator and one-layer bound are

$$\begin{gathered}\mathcal{G}_N(w) = \sum_{\substack{d\le w\\ d\mid P(w)}} g_N(d) = \frac{\log^2 w}{4C(N)}(1+o(1)),\\ S_m^{(1)}(w) \le \frac{N/m}{\mathcal{G}_N(w)} + O(w^2\log^5 w) \le 502002\,\frac{2C(N)N}{mL^2}.\end{gathered} \tag{8.68}$$

Here $P(w)$ is the product of all primes below w. The last step uses $2.001/(2/1001)^2 = 501251.00025 < 502002$. The exponent saving in (8.67) absorbs its remainder. There are at most 36 source layers. For the parent-bin prime sums M_h, the needed estimates are

$$M_h = \sum_{p \text{ in bin } h} \frac{1}{p} \le 2\tau, \qquad \sum_h M_h \le \frac{459}{100}, \qquad \sum_{a_1,a_2,a_3,r} \frac{1}{a_1a_2a_3r} \le 2\left(\frac{459}{100}\right)^3\tau. \tag{8.69}$$

The pair a_3, r belongs to one parent bin. Multiplication of (8.68) by 36 and this sum, then division by W, gives (8.66). These are the exact steps of Paper II, (4.45)–(4.56), with the layer ceiling changed from three to 36 [27, (4.45)–(4.56), (5.34)–(5.42)]. The uniform two-form estimates are proved here under the stated ranges.
Proof of (8.62). *Principal.* Since $\mathcal{P}_\Pi \le G_* + o_\sigma(1)$ and $G_* > 0$, the mass bound gives $M_{\mathrm{prin}}/W \le \frac{(1-\varepsilon)I_{\mathrm{HL}}2e^{-\gamma}(G_*+o(1))}{4(1-\varepsilon)} = m_{\mathrm{sw}} + o_\sigma(1)$. *Secondary sieve.* $E_{s,c} \le C_E\sigma + o_\sigma(1)$ uniformly, since $\ell_c \ge \ell_1$. Lemma B.9, Lemma B.11 and $X_A \le N/L$ give (8.65). Every prefix lies in one row, so the four depths together contribute at most 585 [27, (B.9), (B.11)]. *Peel.* For $m = vr$ in P_3, all primes of m are at least z_c, $(m,N) = 1$ and $z_c \in [N^{u/\rho^3}, N^{\rho\mu_*}]$. Since $\chi_c = 1$, $\log_N m \le \rho(t_1+t_2+2t_3) \le \rho(153/256 - \eta_L) < 153/256 + 2\tau$ [27, (4.55)]. Write $A = \sum_{h=1}^{36} 1_{\{A\ge h\}}$. Each layer consists of N^α-rough e. With $N - e = mk$, both k and $N - mk$ are w-rough, $w = \min(z_c, N^\alpha)$. A prime $p < w$ dividing k and N would divide e, and k is odd. The two-form Selberg bound [27, (4.48)–(4.54)] gives (8.68) for each layer, with remainder absorbed by (8.67). The prime sums (8.69) are [27, (4.56)], since r and a_3 lie in one parent bin. Multiplying by 36 gives (8.66). *Comparator.* For squarefree q, $|d_A(q)| = \varphi(q)^{-1}\sum_{(e,q)>1} A(e)$. A common prime ℓ of e and q is at least N^α, so with $q = \ell r$,

$$\sum_{q\ \mathrm{sf}} |O_{\mathfrak{b}}(q)d_A(q)| \le C_{\mathrm{L3},\sigma} \sum_{\ell\ge N^\alpha} \frac{36N}{\ell(\ell-1)} \sum_{r\le N} \frac{1}{\varphi(r)} \ll_\sigma N^{1-\alpha}L.$$

Summing finitely many blocks and dividing by W gives $|E_{\mathrm{nc}}|/W = O_\sigma(N^{-\alpha}L^3)$, which implies the stated $O_\sigma(N^{-\alpha}L^4)$. This is the coprime-comparator bound used in (8.53).

8.11. **The finite signed identity and its consequence.**
(P2.8a) Finite signed identities. For the finite source (8.3) and stated sieve data, combine (8.7), (8.12) and (8.29). Their conditional signed identity is

$$S_{\mathrm{sw}} = E_{\mathrm{src}} + P_3 + M_{\mathrm{prin}} + M_{\mathrm{err}} + R_{\mathrm{adv}} - (Z_{\mathrm{src}} + Z_{\mathrm{AA}} + Z_{\mathrm{peel}}). \tag{8.70}$$

Define

$$\begin{gathered}\delta_P = P_3/W - p_{\mathrm{sw}}, \qquad \delta_A = M_{\mathrm{err}}/W - a_{\mathrm{sw}}, \qquad \delta_M = M_{\mathrm{prin}}/W - m_{\mathrm{sw}},\\ R_{\mathrm{sw}} = E_{\mathrm{src}}/W + \delta_P + \delta_A + \delta_M + R_{\mathrm{adv}}/W,\\ V_{\mathrm{sw}} = (R_{\mathrm{sw}})_+ \le E_{\mathrm{src}}/W + (\delta_P)_+ + (\delta_A)_+ + (\delta_M)_+ + (R_{\mathrm{adv}})_+/W.\end{gathered} \tag{8.71}$$

Under the stated sieve hypotheses, all three Z terms are nonnegative. Consequently

$$\begin{aligned} -\frac{S_{\mathrm{sw}}}{W} &= -m_{\mathrm{sw}} - a_{\mathrm{sw}} - p_{\mathrm{sw}} - R_{\mathrm{sw}} + \frac{Z_{\mathrm{src}} + Z_{\mathrm{AA}} + Z_{\mathrm{peel}}}{W} \\ &\geq -\frac{c_{K2,G0}}{4}\overline{H} - C_{\mathrm{sw}}\sigma - P_{\mathrm{sw}}\sigma^9 - V_{\mathrm{sw}}. \end{aligned} \tag{8.72}$$

The favorable Z_{peel} is used once. Every adverse remainder remains in (8.71).

Assume all estimates stated above for the same A. Then (8.8), (8.53) and (8.62) imply

$$V_{\mathrm{sw}} = o_\sigma(1). \tag{8.73}$$

Proof of (8.73). All estimates above are proved for the same A, the same cells and the same blocks. By (8.8), $E_{\mathrm{src}}/W \ll L^{-331}$. By (8.62), $(\delta_M)_+, (\delta_A)_+, (\delta_P)_+$ are $o_\sigma(1)$. For R_{adv}, (8.32) and (8.36) give

$$\frac{|R_{\mathrm{adv}}|}{W} \leq \frac{|D_{\mathrm{int}}^{(0)}| + |D_{\mathrm{int}}^{(1)}| + |D_{\mathrm{int}}^{(2)}| + |D_{\mathrm{int}}^{(3)}| + |D_{\mathrm{bdry}}| + |E_{\mathrm{sq}}| + |E_{\mathrm{nc}}|}{W}.$$

The terms are $O_\sigma(L^{-1})$ by (8.41); $O_\sigma(L^{-1})$ by (8.47) with $A_{\mathrm{loc}} = 3692$, $|w_C|/M_c \leq 36$ and $\sum_B |w_B| \leq L^{3689}$; $O_\sigma(L^{-327/2})$ by (8.52); and $O_\sigma(N^{-\ell_1\sigma^2}L^4)$, $O_\sigma(N^{-\alpha}L^4)$. Hence $V_{\mathrm{sw}} = o_\sigma(1)$. After fixing σ_*, choose the eventual threshold so that

$$V_{\mathrm{sw}} \leq \frac{1}{200000}. \tag{8.74}$$

That threshold may be ineffective. It also covers

$$\begin{aligned} N_{\mathrm{dom}} &= 2\left\lceil \frac{1}{2}\exp\bigl(\max\{10^4, 2\log 2/(\ell_1\sigma_*^2)\}\bigr)\right\rceil, \\ N_{z,0} &= 2\left\lceil \frac{1}{2}\exp\Bigl(2\exp((\ell_1\sigma_*^2)^{-1/3})\Bigr)\right\rceil, \qquad N_{\mathrm{LEN,app}}(\sigma_*). \end{aligned} \tag{8.75}$$

These structural thresholds alone do not imply (8.74).

Relative to the comparison $-S_{\mathrm{sw}}/W \geq -2\overline{H}$, the constant difference is

$$\begin{aligned} \Delta D_{L3} &= \Bigl(2 - \frac{c_{K2,G0}}{4}\Bigr)\overline{H} \\ &= 0.022638464724298154340603\ldots. \end{aligned} \tag{8.76}$$

The exact cost of using the common coefficient instead of the optional coefficient is

$$\frac{\overline{H}}{4}(c_{K2,G0} - c) = 0.000049489563023753677508253183146688036059673516909.$$

This is scalar arithmetic. Its use in the final count requires (8.72) and the common error bounds. The capped-row application is (8.40)–(8.41). The row-three estimate is (8.47), proved in the row-three subsection. All errors use the common source (8.3). The inputs are Paper I, Theorem 2.8; Fact 3.3; Lemma 6.5(ii) for (3.48); and the prime estimates of Fact 3.1. Hence (8.72) holds at the threshold of (8.74). The constant (8.76) is a gross coefficient difference. It enters the final count only together with the switched error $C_{\mathrm{sw}}\sigma_* + P_{\mathrm{sw}}\sigma_*^9 + V_{\mathrm{sw}}$ of (8.72). In the final count this error is bounded by the caps (9.14) ($C_{\mathrm{sw}}\sigma_* \leq 10^{-5}$, $P_{\mathrm{sw}}\sigma_*^9 \leq 10^{-5}$) and by (8.74), in total at most $1/40000$.

9. The final bound

We keep the notation of Sections 1 and 2. All error terms below are nonnegative. The analytic inputs are (A1)–(A5) of Table 1.2.

9.1. **Addition of the comparisons.** Put $\delta_{\mathrm{err}} = 10^{-4}$. The constant of (1.14) is

$$\begin{aligned} D_B = D_{\mathrm{II}} &+ \frac{5719}{10^6} + M_{5,\mathrm{internal}} + g_{\mathrm{RA,lo}} \\ &- \delta_{\mathrm{err}} + \Bigl(2 - \frac{c_{K2,G0}}{4}\Bigr)\overline{H}, \qquad \overline{H} = \frac{389253}{5000000}. \end{aligned} \tag{9.1}$$

Here D_{II} is Paper II's original lower constant [27, (6.3a)]. The six endpoint gains g_6 of (5.18) are not used. The two profile gains enter separately. In particular,

$$\begin{aligned} D_{\mathrm{II}} &= \frac{45849916338009170593}{25000000000000000000}, \\ M_{5,\mathrm{internal}} &= -K_{\mathrm{hi}}\frac{6073714261}{10^9} - L_{G1}, \\ g_{\mathrm{RA,lo}} &= \frac{989598353}{10^{10}} = 0.0989598353. \end{aligned} \tag{9.2}$$

The last line is the conservative rational floor for the integral gain $\mathfrak{g}_{\mathrm{RA}}$. It is not a central estimate. Section 6 separates it from the integral gain. The certified integral enclosure is $[0.0989598353353, 0.0989598353355]$. We take K_{hi} and L_{G1} as defined in [27, (A.10d), (4.84)]. Their exact values are

$$K_{\mathrm{hi}} = \frac{39894807910705784663904430098309337537484429474781101122 23}{10^{58}},$$
$$L_{G1} = -\frac{24395042826519465625420692193994776286887136187440955209509 7}{10043362776618689222137263077132266265763768711142455220633 6}. \tag{9.3}$$

Thus $M_{5,\mathrm{internal}} = M_n/M_d$ exactly, where

$$M_n = \sum_{i=0}^{5} m_i 10^{20i}, \qquad M_d = \sum_{i=0}^{5} d_i 10^{20i}. \tag{9.4}$$

The coefficients are printed in Table 9.1.

Table 9.1: Exact shifted-profile gain

i	m_i	d_i
0	66922148532305907989	0
1	34893223604885136721	0
2	3973350097562772509	0
3	87266878234046404033	486353642291200000 00
4	28374567121249055735	733841876926926749 21
5	3998	680564

The exact rational c_{O1} is defined in (8.60) and Table 8.1. It records only the coefficient from the historical, non-certified sharper comparison. The main count uses $c_{K2,G0}$ in both comparisons. The common functional comparison is (8.59). For comparison with the optional sharper coefficient, exact rational arithmetic gives

$$0 < c - c_{\mathrm{O1}} < 4.081 \cdot 10^{-46}, \qquad 0 < \frac{c - c_{\mathrm{O1}}}{4}\overline{H} < 7.943 \cdot 10^{-48}. \tag{9.5}$$

This is a rational comparison only. It gives no bound for $2e^{-\gamma}U_0$. The further cost of the common coefficient relative to c is exactly

$$\frac{c_{K2,G0} - c}{4}\overline{H} = 0.00004948956302375367750825318314668803605967351690 9.$$

The resulting main switched gain is exactly

$$\Delta D_{L3} = 0.02263846472429815434060381043792394540367262995113415.$$

The optional sharper comparisons are not required for (9.1). Each gain is proved on the original arithmetic source of its comparison. The gains $5719/10^6$ and $M_{5,\mathrm{internal}}$ come from Proposition 5.10, through Propositions B.1 and B.2. The gain $g_{\mathrm{RA,lo}}$ comes from Proposition 6.13, and ΔD_{L3} from (8.72). Table 9.4 lists the bindings.

Table 9.2 itemizes the constant. Its decimal column gives approximations only. The defining formulas determine the exact sum.

Table 9.2: The six terms in D_B

Term	Value
D_{II}	1.83399665352036682372
$5719/10^6$	0.005719
$M_{5,\mathrm{internal}}$	0.005874949945026357…
$g_{\mathrm{RA,lo}}$	0.0989598353
$-\delta_{\mathrm{err}}$	−0.0001
$\Delta D_{L3} = (2 - c_{K2,G0}/4)\overline{H}$	0.022638464724298154…
D_B	1.967088903489691335…

Adding (1.10) and (1.13) cancels the sixth endpoint:

$$\overline{J}_6 - \overline{J}_6 = 0, \qquad 2\overline{H} - \frac{c_{K2,G0}}{4}\overline{H} = \Big(2 - \frac{c_{K2,G0}}{4}\Big)\overline{H}. \tag{9.6}$$

Thus their conditional consequence is

$$\frac{4G_b(N)}{W(N)} \geq D_B + \delta_{\mathrm{err}} - c_{K2,G0} I(b) - \mathcal{E}_B(N; b, \sigma). \tag{9.7}$$

This uses the complete S_2 of Paper II. The finite identities keep their original signs and multiplicities. The fixed errors in Paper II are subtracted once [27, (6.4)–(6.6)], and not again in (9.1).

9.2. **Ten error allocations.** Use the exterior error from (9.8). For the table below, a conservative majorant is convenient:

$$\begin{aligned} E_K(N;b,\sigma) &= A_K(b)C_{\mathrm{AA}}\sigma + \frac{3884000000000}{9999}\sigma^9 + r_K(N;b,\sigma) \\ &\le 4A_K(b)C_{\mathrm{AA}}\sigma + \frac{3884000000000}{9999}\sigma^9 + r_K(N;b,\sigma), \\ A_K(b) &= 2e^{-\gamma}I(b)W_{\mathrm{AA},K}, \qquad P_K = \frac{3884000000000}{9999}, \\ r_K(N;b,\sigma) &= o_{b,\sigma}(1). \end{aligned} \tag{9.8}$$

Here $W_{\mathrm{AA},K}$ is the exterior remainder-weight bound. It belongs to the source in Section 7. The extra factor four only enlarges this nonnegative error. It leaves the principal constant unchanged. Similarly, the switched comparison requires

$$\begin{aligned} E_{\mathrm{sw}}(N;\sigma) &\le \frac{650000000}{1111}C_E\sigma + \frac{48544888063158}{27775}\sigma^9 + V_{\mathrm{sw}}(N;\sigma), \\ V_{\mathrm{sw}}(N;\sigma) &= o_\sigma(1). \end{aligned} \tag{9.9}$$

Here V_{sw} is the positive part in (8.71). The conditional limit is (8.73). These two limits concern the actual sums stated there. Both majorants are the error terms of (7.67) under (A3) and of (8.72) under (A4), with C_{sw} from (8.61). By (A5), C_{AA} and C_E do not depend on σ. The exterior weight $W_{\mathrm{AA},K}$ is W_{AA} of (7.38):

$$\begin{aligned} A_K(b) = 2e^{-\gamma}I(b)W_{\mathrm{AA},K}, \qquad W_{\mathrm{AA},K} &= 500\Big(1 + H_{\log} + \frac{H_{\log}^2}{2} + \frac{H_{\log}^3}{6}\Big), \\ H_{\log} = \log\frac{\mu_*}{u} &= \log\frac{3125}{32} > 0. \end{aligned} \tag{9.10}$$

Since $I(b) > 0$, this gives $0 < A_K(b) < \infty$. In particular $A_K(b)C_{\mathrm{AA}}\sigma \le 4A_K(b)C_{\mathrm{AA}}\sigma$.

The retained and Root–A comparisons require

$$E'(N;\sigma) = o_\sigma(1), \qquad E_{\mathrm{RA}}(N;\sigma) = o_\sigma(1). \tag{9.11}$$

For Root–A, use $E_{\mathrm{RA}}(N;\sigma)$ from (6.72). The first includes every conversion and the saved-array contribution $\sum_{j\in\mathcal{H}}|\Lambda^0_{j,N}|$ in (1.12); that contribution is not charged again. The pointwise error may use the coarser bound

$$E_{\mathrm{exc}}(N) = \frac{4E_{\mathrm{pt}}(N)}{W(N)} \le \frac{2197N^{1-\alpha}}{W(N)} = o(1). \tag{9.12}$$

Here E_{pt} is defined in (4.12). The explicit pointwise argument in Section 4 gives $E_{\mathrm{pt}} = 0$ for $N \ge N_{\mathrm{pt}}$. The coarser majorant retains the same error bound. The inequality follows from (4.13), using both the pointwise bound and the exceptional-set cardinality. Since $C(N) \ge C_2 > 0$ and $\alpha > 0$, the coarser majorant tends to zero: $N^{-\alpha}(\log N)^2 \to 0$. No sharper exceptional coefficient is needed. Proposition 5.10 gives $E'(N;\sigma) \to 0$, and Proposition 6.13 gives $E_{\mathrm{RA}}(N;\sigma) \to 0$ under (A2), each at fixed σ. Lemma 4.1 gives $E_{\mathrm{exc}}(N) = 0$ for $N \ge N_{\mathrm{pt}}$.

Fix b, all absolute constants, and the source geometry first. Choose positive absolute constants C_E^{L3} and C_E^{II} for the Section 8 and Paper II sieve bounds. Define

$$C_E := \max\{C_E^{\mathrm{L3}}, C_E^{\mathrm{II}}\}. \tag{9.13}$$

By (A5), these constants are independent of σ and N. Here $C_{\mathrm{AA}} = 4C_{\mathrm{AA}37}$ by (7.39). The constant C_E dominates the absolute constant of (3.22) used in (8.25). The underlying sieve bounds come from Paper I, Theorem 2.6, (2.31)

[26, Theorem 2.6, (2.31)]. Choose one rational $\sigma_* > 0$ satisfying

$$\begin{aligned} 0 < \sigma_* &\le 10^{-10}, \\ \sigma_* &\le \frac{713332}{625 \cdot 10^{21} C_E}, \\ \sigma_* &\le \frac{1}{400000 A_K(b) C_{\mathrm{AA}}}, \\ \sigma_* &\le \left(\frac{9999}{3884000000000000000} \right)^{1/9}, \\ \sigma_* &\le \frac{1111}{65000000000000 C_E}, \\ \sigma_* &\le \left(\frac{27775}{4854488806315800000} \right)^{1/9}, \\ \tau_* &= (\sigma_*)^9. \end{aligned} \tag{9.14}$$

By (9.10) and (A5), every right side in (9.14) is positive and independent of σ. Hence such a rational σ_* exists. The second restriction preserves Paper II's fixed error bound. It is exactly

$$\sigma_* \le \frac{4 g_{\mathrm{AA}}}{C_E W_*}, \qquad g_{\mathrm{AA}} = \frac{178333}{6250000000}, \qquad W_* = 10^{14} \quad [27, (1.26), (1.27), (6.6)]. \tag{9.15}$$

The other four restrictions control new fixed errors. Freeze every finite partition and logarithmic precision next. Only then choose the common threshold.

The ten allocations are printed in Table 9.3. The two zero rows are unused capacity.

Table 9.3: The error bound $\delta_{\mathrm{err}} = 10^{-4}$

	Error term	Allocation
1	$4 A_K(b) C_{\mathrm{AA}} \sigma_*$	$\delta_{\mathrm{err}}/10$
2	$\frac{3884000000000}{9999} \tau_*$	$\delta_{\mathrm{err}}/10$
3	$\frac{650000000}{1111} C_E \sigma_*$	$\delta_{\mathrm{err}}/10$
4	$\frac{48544888063158}{27775} \tau_*$	$\delta_{\mathrm{err}}/10$
5	$E'(N;\sigma_*) + V_{\mathrm{sw}}(N;\sigma_*)$	$\frac{\delta_{\mathrm{err}}}{20} + \frac{\delta_{\mathrm{err}}}{20}$
6	0	$\delta_{\mathrm{err}}/10$
7	$E_{\mathrm{RA}}(N;\sigma_*)$	$\delta_{\mathrm{err}}/10$
8	$r_K(N;b,\sigma_*)$	$\delta_{\mathrm{err}}/10$
9	$E_{\mathrm{exc}}(N)$	$\delta_{\mathrm{err}}/10$
10	0	$\delta_{\mathrm{err}}/10$

Its capacities sum exactly:

$$4\frac{\delta_{\mathrm{err}}}{10} + \left(\frac{\delta_{\mathrm{err}}}{20} + \frac{\delta_{\mathrm{err}}}{20} \right) + 5\frac{\delta_{\mathrm{err}}}{10} = \delta_{\mathrm{err}}. \tag{9.16}$$

The first four bounds follow from (9.14). The five remaining nonzero bounds follow from the stated limits. Each limit is taken at fixed σ_*.

Proposition 9.1 (Error budget)**.** *Assume* (A1)–(A5)*, and let σ_* be rational with* (9.14)*. Then every even $N \ge N_0^*(\sigma_*)$ satisfies*

$$\mathcal{E}_B(N;b,\sigma_*) \le \frac{8}{10}\delta_{\mathrm{err}} < \delta_{\mathrm{err}}. \tag{9.17}$$

Proof. Rows 1 and 3. The third and fifth lines of (9.14) give

$$4A_K(b)C_{\mathrm{AA}}\sigma_* \le \frac{4}{400000} = \frac{\delta_{\mathrm{err}}}{10}, \qquad \frac{650000000}{1111}C_E\sigma_* \le \frac{650000000}{65000000000000} = \frac{\delta_{\mathrm{err}}}{10}.$$

Rows 2 and 4. The first line gives $\tau_* \le 10^{-90}$. This is below $9999/388400000000000000$ and $27775/4854488806315800000$, so the fourth and sixth lines hold, and

$$\frac{3884000000000}{9999}\tau_* \le \frac{\delta_{\mathrm{err}}}{10}, \qquad \frac{48544888063158}{27775}\tau_* \le \frac{\delta_{\mathrm{err}}}{10}.$$

Rows 5, 7 and 8. At the fixed σ_*, $E' \to 0$ by (A1), $V_{\mathrm{sw}} \to 0$ by (8.73) under (A4), $E_{\mathrm{RA}} \to 0$ by (A2), and $r_K \to 0$ by (A3). By (2.6), the thresholds $T_{E'}, T_V, T_{E_{\mathrm{RA}}}, T_r, T_{E_{\mathrm{exc}}}$ of (2.5) exist, and above them $E' \le \delta_{\mathrm{err}}/20$, $V_{\mathrm{sw}} \le \delta_{\mathrm{err}}/20$, $E_{\mathrm{RA}} \le \delta_{\mathrm{err}}/10$ and $r_K \le \delta_{\mathrm{err}}/10$. Row 9. Lemma 4.1 gives $E_{\mathrm{exc}}(N) = 0$ for $N \ge N_{\mathrm{pt}}$, so $T_{E_{\mathrm{exc}}} \le N_{\mathrm{pt}}$. Rows 6 and 10 are zero. Every threshold used is bounded by the common maximum in (2.5). Also $N_{\mathrm{pt}} < N_{\mathrm{tr}}$ by Lemma 9.2. Hence all of them hold for $N \ge N_0^*(\sigma_*)$. By (1.11), (9.8) and (9.9),

$$\begin{aligned}\mathcal{E}_B \le \big(E' + V_{\mathrm{sw}}\big) + E_{\mathrm{RA}} + \Big(\frac{650000000}{1111}C_E\sigma_* + \frac{48544888063158}{27775}\tau_*\Big)\\ + \Big(4A_KC_{\mathrm{AA}}\sigma_* + \frac{3884000000000}{9999}\tau_* + r_K\Big) + E_{\mathrm{exc}}.\end{aligned}$$

The bounds above give at most $\frac{\delta_{\mathrm{err}}}{10} + \frac{\delta_{\mathrm{err}}}{10} + \frac{2\delta_{\mathrm{err}}}{10} + \frac{3\delta_{\mathrm{err}}}{10} + 0$, which is below the nonzero allocations $\frac{8\delta_{\mathrm{err}}}{10}$ of Table 9.3. □

Take the named source, application and error thresholds listed in Section 2. The Section 7 definitions of T_K and T_r already include the main-mass threshold $N_{\mathrm{mass}}(b, \sigma_*)$ and the AA threshold $N_{\mathrm{AA}}(b, \sigma_*)$. Thus the common maximum in $\mathcal{T}_*$ below includes both requirements at the fixed b, σ_*. The common threshold is the single definition

$$N_0^*(\sigma_*) \quad \text{as defined in (2.5).} \tag{9.18}$$

Its existence depends on all those estimates. The floor alone supplies none of them. By its definition and Lemma 9.2, $N_0^*(\sigma_*) \ge \max\{N_{\mathrm{tr}}, N_6\} > N_{\mathrm{pt}}$. The Section 2 threshold definitions include N_{pt} through T_{src} and N_6 through T_{RA}. The bound N_{tr} enters directly in the common maximum. Lemma 4.1 holds for $N \ge N_{\mathrm{pt}}$ and Lemma 4.2 for $N \ge N_6$. Thus the order is

$$\text{constants} \longrightarrow \sigma_* \longrightarrow \text{finite choices} \longrightarrow N_0^*(\sigma_*) \longrightarrow N. \tag{9.19}$$

No parameter $\sigma_*(N)$ is used.

Proposition 9.1 gives

$$\mathcal{E}_B(N; b, \sigma_*) \le \delta_{\mathrm{err}} \qquad (N \ge N_0^*(\sigma_*),\ N \in 2\mathbb{Z}). \tag{9.20}$$

Combining (9.7) and (9.20) gives

$$\boxed{\frac{4G_b(N)}{W(N)} \ge D_B - c_{K2,G0}I(b).} \tag{9.21}$$

It holds for every even $N \ge N_0^*(\sigma_*)$ under (A1)–(A5).

9.3. **The numerical margin.** All constants in (9.1) have exact definitions. The exterior price uses the G0 coefficient in (7.36). The same coefficient prices L3. No G_*, U_0 sharper domination is used in this arithmetic or in the main-count hypotheses. The constants have the following outward enclosures:

$$\begin{aligned}1.967088903489691335582703173494 < D_B,\\ D_B < 1.967088903489691335582703173495,\\ 1.966859949178424046530804965343 < c_{K2,G0}I(b),\\ c_{K2,G0}I(b) < 1.966859949178424046530804965344,\\ 0.000228954311267289051898208151 < D_B - c_{K2,G0}I(b),\\ D_B - c_{K2,G0}I(b) < 0.000228954311267289051898208152.\end{aligned} \tag{9.22}$$

Here

$$c_{K2,G0} = \frac{68368251638755177562868463216507543729310946 89}{10^{45}}, \qquad I(b) = \log\frac{4000007}{2999993}. \tag{9.23}$$

The coefficient is defined in (7.36). The final comparison is (7.67). Hence

$$D_B - c_{K2,G0}I(b) > 0.0002289 > \frac{1}{5000} > 0. \tag{9.24}$$

The table below identifies each exact constant by its defining display and lists the comparisons that use it unchanged.

Table 9.4: Constants and the comparisons that use them

Constant	Definition	Comparison
D_{II}	(1.8), [27, (6.3a)]	Proposition 5.10, by (5.13)
$5719/10^6$	(5.25)	Proposition 5.10, by Proposition B.1
$M_{5,\mathrm{internal}}$, K_{hi}, L_{G1}	(9.3), Table 9.1	Proposition 5.10, by Proposition B.2
$g_{\mathrm{RA,lo}}$	(6.70)	Proposition 6.13, (6.72)
$\overline{H}$, $c_{K2,G0}$	(9.1), (7.36)	(8.72), with (8.61)
$c_{K2,G0}$, $I(b)$	(9.23)	Proposition 7.5, (7.67)
$A_K(b)$, C_{AA}, P_K	(9.10), (7.39), (7.67)	Table 9.3, rows 1 and 2
C_{sw}, P_{sw}	(8.61)	Table 9.3, rows 3 and 4
δ_{err}	(9.1)	Proposition 9.1

No other constant enters D_B or the error budget.

The logarithm enclosure uses rational arithmetic. Put $t_{\log} = 500007/3500000$ and

$$S_m = 2\sum_{k=0}^{m} \frac{t_{\log}^{2k+1}}{2k+1}, \qquad S_m < I(b) < S_m + \frac{2t_{\log}^{2m+3}}{(2m+3)(1-t_{\log}^2)}. \tag{9.25}$$

Here $I(b) = 2\operatorname{arctanh} t_{\log}$ and $0 < t_{\log} < 1$. Bounding each remaining denominator by $2m+3$ proves the upper bound. Taking $m = 24$ gives (9.22). Thus no rounded logarithm enters the margin test.

For reference, put $\zeta = 10^{-6}$ and $\mathfrak{E}_* = \exp(D_B/c_{K2,G0})$. The buffered critical exponent is

$$a^*_{\mathrm{buf}} = \frac{\mathfrak{E}_* + 1}{(1-\zeta)\mathfrak{E}_* - \zeta}. \tag{9.26}$$

To derive it, write $b_* = 1/(1+\mathfrak{E}_*) + \zeta$. The equation $a(1-b_*) = 1$ gives (9.26). For the present constants,

$$1.749974884137892433433207484622 < a^*_{\mathrm{buf}} < 1.749974884137892433433207484623 < \frac{7}{4}. \tag{9.27}$$

This enclosure also uses rational bounds. For $x = D_B/c_{K2,G0}$ and $m = 36$, put

$$T_m = \sum_{k=0}^{m} \frac{x^k}{k!}, \qquad T_m < e^x < T_m + \frac{x^{m+1}}{(m+1)!}\frac{1}{1-x/(m+2)}. \tag{9.28}$$

Here $0 < x < 1$. The tail ratio is at most $x/(m+2)$. The function in (9.26) decreases with $\mathfrak{E}_*$. Substitution of the two rational bounds proves (9.27).

The endpoint in (9.27) is equivalent to the positive scalar margin at this fixed b. It supplies no theorem at a smaller a without new comparisons at the corresponding cutoff. If the optional Lemma 5.3 is used, D_B increases by $g_6 = 0.0020549454896034927680\ldots$ of (5.18). The margin then becomes $0.0022838998008707818199\ldots$, and a^*_{buf} becomes $1.7497494980881623276859\ldots$. Theorem 1 does not use this.

For an optional arithmetic comparison, define $D_{B,c} = D_B + (c_{K2,G0} - c)\overline{H}/4$. Using c in both K2 and L3 would require two further inputs. This paragraph concerns the optional scalar comparison at the same fixed cutoff; it supplies no new cutoff-dependent estimate. The first is the separate sharper functional domination $G_* \le U_0$ from equation B.28. The second is a certified $2e^{-\gamma}U_0 \le c$. It does not follow from the coarse interval in equation B.27. The main proof uses neither. Its scalar endpoint is

$$E_c = \exp(D_{B,c}/c), \qquad a^*_{\mathrm{buf},c} = \frac{E_c + 1}{(1-\zeta)E_c - \zeta}. \tag{9.29}$$

The same rational Taylor enclosure gives

$$1.749889173139093190128834678 < a^*_{\mathrm{buf},c} < 1.749889173139093190128834679 < \frac{7}{4}. \tag{9.30}$$

Thus, on the optional sharper branch, $a^*_{\mathrm{buf},c}$ is $1.7498891731390\ldots$. The main theorem uses the explicitly certified G0 coefficient defined and bounded in (7.36); the display above is only the optional exponent endpoint.

9.4. **The transfer.**

Lemma 9.2 (Transfer)**.** *The parameters* (2.3) *satisfy*

$$\kappa_{\mathrm{tr}} = \frac{3}{4}(1-b) - b = \frac{7}{4000000}, \qquad N_{\mathrm{tr}} = 10^{12000000/7} > 10^{12} + 1 = N_{\mathrm{pt}}. \tag{9.31}$$

For even $N \ge N_{\mathrm{tr}}$, every $n = rq \in \mathcal{A}_N$ with primes $r \le q$ and $r < N^b$ satisfies

$$q > \varepsilon N^{1-b}, \qquad q^{3/4} > \varepsilon^{3/4} N^{b+\kappa_{\mathrm{tr}}} \ge N^b > r. \tag{9.32}$$

Proof. Since $a = 7/4$, $a(1-b)-1 = \frac{3}{4}(1-b)-b$, and $b = 2999993/7000000$ gives $7/4000000$. Also $\varepsilon^{-(a-1)/\kappa_{\mathrm{tr}}} = 10^{4\cdot 3000000/7}$, and $12000000/7 > 13$. For $n = rq \in \mathcal{A}_N$ we have $n > \varepsilon N$, so $q = n/r > \varepsilon N^{1-b}$ and $q^{3/4} > \varepsilon^{3/4} N^{3(1-b)/4} = \varepsilon^{3/4} N^{b+\kappa_{\mathrm{tr}}}$. Finally $N^{\kappa_{\mathrm{tr}}} \ge \varepsilon^{-3/4}$ exactly when $N \ge N_{\mathrm{tr}}$. □

Proof of Theorem 1

Use the established inputs (A1)–(A5) and fix a rational σ_* satisfying (9.14). Let $N \ge N_0^*(\sigma_*)$ be even. Then $N \ge \max\{N_{\mathrm{tr}}, N_6\} > N_{\mathrm{pt}}$. Lemma 4.1 gives the first line of (1.10). At $\sigma = \sigma_*$, the established comparisons (A1), (A2) and (A4) give its other three lines, and (A3) gives (1.13). The zero-gain combination Lemma 4.3 adds these four comparisons to (9.7). Proposition 9.1 gives (9.20), hence (9.21). Equation (9.24) then gives $G_b(N) > 0$, since $W(N) > 0$ by (2.2). Hence the set in (1.3) is nonempty.

Choose one counted integer $n = N - p$. If n is prime, take $q = n$ and $r = 1$. Otherwise $n = rq$ with primes $r \le q$ and $r < N^b$, and Lemma 9.2 gives $r < q^{3/4}$. Therefore $N = p + rq$ and $r \le q^{3/4}$. □

Appendices

Appendix A proves three auxiliary estimates. Appendix B records finite calculations, numerical certificates, and exact arithmetic. All cross-references to these appendices are internal to this document.

Appendix A. Proofs of three estimates of Section 6

This section proves Lemmas 6.5, 6.8 and 6.9. Numbers (6.k) and 6.k refer to Section 6. Numbers (A.1) and later refer to this section. The only inputs are Papers I and II and the classical results cited at the point of use. Throughout, τ_j is the j-fold divisor function and $\tau = \tau_2$. Let $\omega(n)$ be the number of distinct prime divisors of n. Let $P^+(n)$ and $P^-(n)$ be its largest and least prime factors. The sum $\sum^*_{b \bmod k}$ runs over reduced residues.

A.1. **Elementary inequalities.** For integers $j \ge 1$, $k \ge 1$ and $Y \ge 1$,

$$\tau(n)^j \le \tau_{2^j}(n), \qquad \sum_{n \le Y} \frac{\tau_k(n)}{n} \le (1+\log Y)^k, \qquad \sum_{n \le Y} \tau_k(n) \le Y(1+\log Y)^{k-1}. \tag{A.1}$$

The first is multiplicative. At a prime power it is $(a+1)^j \le \binom{a+2^j-1}{2^j-1}$, which follows by induction on j from

$$(a+1)\binom{a+k-1}{k-1} \le \binom{a+2k-1}{2k-1}.$$

The right side is the coefficient of p^a in the convolution of two k-fold divisor functions. Each of its $a+1$ terms is at least the left factor $\binom{a+k-1}{k-1}$, since $(j'+i)(a-j'+i) \ge i(a+i)$ for $0 \le j' \le a$. The two sums in (A.1) follow by writing $n = n_1 \cdots n_k$, enlarging the constraint to $n_i \le Y$, and summing one factor last. Also

$$\frac{n}{\varphi(n)} \le \tau(n), \qquad \varphi(mn) \ge \varphi(m)\varphi(n), \qquad \tau(mn) \le \tau(m)\tau(n), \tag{A.2}$$

prime power by prime power, and $\tau(n) \ll_\epsilon n^\epsilon$.

For a finitely supported sequence ξ, a modulus k and an integer $h \ge 1$, put

$$\Delta_\xi(k,b;h) = \sum_{\substack{n \equiv b \ (k) \\ (n,h)=1}} \xi_n - \frac{1}{\varphi(k)} \sum_{(n,kh)=1} \xi_n, \qquad \Delta_\xi(k,b) = \Delta_\xi(k,b;1), \tag{A.3}$$

$$V(k;\xi) = \sum_{b \bmod k}^{*} |\Delta_\xi(k,b)|^2. \tag{A.4}$$

If ξ is supported in an interval of length at most Y, then

$$V(k;\xi) \le \sum_{b \bmod k} \Big| \sum_{n \equiv b \ (k)} \xi_n \Big|^2 \le \Big(\frac{Y}{k} + 1\Big) \|\xi\|_2^2. \tag{A.5}$$

Here $\Delta_\xi(k,b)$ is the deviation of the reduced class sums from their mean. The variance is at most the second moment. Cauchy's inequality in each class gives the second bound.

The sequence $e_n = \mathbf{1}_{H < n \le 2H}$ satisfies, for all $k, h \ge 1$ and $(b,k) = 1$,

$$|\Delta_e(k,b;h)| \le 3\tau(h). \tag{A.6}$$

By inclusion–exclusion over $d \mid h$, $(d,k) = 1$, the class count equals $(H/k)\prod_{p \mid h,\, p \nmid k}(1-1/p) + O(\tau(h))$, with error at most $\tau(h)$ in absolute value. The comparator has the same main term and error at most $\tau(h)\tau(k)/\varphi(k) \le 2\tau(h)$.

Let $\ell = \log\log N$, $z = N^{1/\ell^3}$ and $y = N^{1/\ell}$. Here N is the large parameter of the lemma at hand. For each fixed $C \geq 0$, Rankin's method gives

$$\sum_{\substack{d>y\\ P^+(d)\leq z}} \frac{\tau(d)^C}{d} \ll_C (\log N)^{K_C} e^{-\ell^2} \tag{A.7}$$

with a fixed K_C. With $\lambda = 1/\log z$, the left side is at most

$$y^{-\lambda} \prod_{p\leq z} \Big(1 + \sum_{e\geq 1} (e+1)^C p^{-e(1-\lambda)}\Big), \qquad y^{-\lambda} = e^{-\ell^2}.$$

For $p \leq z$ one has $p^\lambda \leq e$ and $1-\lambda \geq 1/2$, so the factor at p is $1+O_C(1/p)$. Mertens' theorem gives $\sum_{p\leq z} 1/p \leq \ell + O(1)$. Hence the product is $\ll (\log N)^{K_C}$. Since $e^{-\ell^2} = (\log N)^{-\ell}$, the right side of (A.7) is $O_{C,A}((\log N)^{-A})$ for every fixed A.

A.2. **Lemma 6.5.** Recall the setting of Section 6. Let ξ be complex, supported in $[1, Z]$, $Z \geq 3$, and fix $B_{\mathrm{div}} \geq 0$. The hypothesis (6.35) is that, for every $P > 0$, uniformly in $h, k \geq 1$ and reduced $b \bmod k$,

$$\left| \sum_{\substack{n\equiv b\ (k)\\ (n,h)=1}} \xi_n - \frac{1}{\varphi(k)} \sum_{(n,kh)=1} \xi_n \right| \ll_P \|\xi\|_2 Z^{1/2} \tau(h)^{B_{\mathrm{div}}} (\log(2Z))^{-P}, \tag{6.35}$$

with constants common to the family of pairs (Z, ξ). The conclusion (6.36) is

$$\sum_{k\leq Z(\log Z)^{-B_1(A)}} {\sum_{b \bmod k}}^{*} |\Delta_\xi(k,b)|^2 \ll_A \|\xi\|_2^2 Z (\log Z)^{-A}. \tag{6.36}$$

In the selected-factor form of Paper I, (3.48), $\xi_n = \alpha_n 1_{(n,r)=1}$, $Z = \lceil 2T \rceil$, and (6.37) reads

$$\sum_{q\leq 3Q} {\sum_{b \bmod q}}^{*} |\Delta_\xi(q,b)|^2 \ll N_{\mathcal{I}} T L^{-A_3}, \qquad 3Q \leq Z(\log Z)^{-B_F}, \quad B_F = A_3 + 15, \tag{6.37}$$

where $N_{\mathcal{I}}$ is the interval mass in Paper I, (3.8).

Lemma 6.5 (Transferred mean-square estimate)**.**

(i) *If* (6.35) *holds, then* (6.36) *holds for every fixed $A > 0$. Here $B_1(A)$ and the implied constant depend only on A, B_{div} and the common constants.*

(ii) *Assume the hypotheses (3.1), (3.2), (3.6), (3.8) with its length condition, (3.11) and (3.25)–(3.27) of Paper I. Then the estimate* (6.37) *holds uniformly over the parameter boxes and selected factors of Paper I, Theorem 3.2. Hence that theorem holds with this input in place of* [3, Theorem 0(a)] *at (3.48).*

Proof of (i). Put $E = \|\xi\|_2^2$, $\lambda = \log(2Z)$ and, for $1 \leq Q \leq Z$,

$$S(Q) = \sum_{k\leq Q} V(k;\xi).$$

Since $2Z \leq Z^2$, $\log Z \leq \lambda \leq 2\log Z$.

Characters. Orthogonality on the reduced classes gives $\Delta_\xi(k,b) = \varphi(k)^{-1} \sum_{\chi\neq\chi_0} \overline{\chi}(b) \sum_n \xi_n \chi(n)$. The centring removes exactly the principal character. Hence

$$S(Q) = \sum_{k\leq Q} \frac{1}{\varphi(k)} \sum_{\substack{\chi \bmod k\\ \chi\neq\chi_0}} \Big|\sum_n \xi_n \chi(n)\Big|^2. \tag{A.8}$$

Each $\chi \neq \chi_0$ modulo k is induced by a unique primitive $\psi \bmod f$ with $f \mid k$, $f > 1$. Write $k = fe$. Then $\chi(n) = \psi(n)\mathbf{1}_{(n,e)=1}$, since ψ already vanishes on primes of f. The pair (e,ψ) determines (k,χ), and $\varphi(fe) \geq \varphi(f)\varphi(e)$ by (A.2). Therefore

$$S(Q) \leq \sum_{e\leq Q} \frac{1}{\varphi(e)} \sum_{1<f\leq Q/e} \frac{1}{\varphi(f)} {\sum_{\psi \bmod f}}^{\mathrm{prim}} |C_e(\psi)|^2, \qquad C_e(\psi) = \sum_{(n,e)=1} \xi_n \psi(n). \tag{A.9}$$

Small conductors. Let $F \ge 1$ and $1 < f \le F$. Since $\sum_b^* \psi(b) = 0$,

$$C_e(\psi) = \sum_{b \bmod f}^{*} \psi(b)\Big\{ \sum_{\substack{n \equiv b \ (f) \\ (n,e)=1}} \xi_n - \frac{1}{\varphi(f)} \sum_{(n,ef)=1} \xi_n \Big\}.$$

The bracket is the left side of (6.35) with $k = f$ and $h = e$. For a fixed $a > 0$ it is at most $C_a E^{1/2} Z^{1/2} \tau(e)^{B_{\mathrm{div}}} \lambda^{-a}$. There are at most $\varphi(f)$ primitive characters modulo f, so

$$\sum_{1<f\le F} \frac{1}{\varphi(f)} \sum_{\psi}^{\mathrm{prim}} |C_e(\psi)|^2 \le C_a^2 E Z \tau(e)^{2B_{\mathrm{div}}} \lambda^{-2a} \sum_{f \le F} \varphi(f)^2 \le C_a^2 E Z \tau(e)^{2B_{\mathrm{div}}} F^3 \lambda^{-2a}. \tag{A.10}$$

Large conductors. The multiplicative large sieve, Paper I, (2.19), applied to $c_n = \xi_n \mathbf{1}_{(n,e)=1}$ with support in $[1, Z]$ and $\|c\|_2^2 \le E$, gives

$$\sum_{f \le 2V} \sum_{\psi}^{\mathrm{prim}} |C_e(\psi)|^2 \ll (V^2 + Z)E.$$

If $p_1 < \cdots < p_s$ are the primes of f, then $p_i \ge i + 1$ and $f/\varphi(f) \le \prod_i (i+1)/i = s + 1 \le 1 + \log f / \log 2$. Thus $1/\varphi(f) \le 3\lambda/V$ for $V < f \le 2V$, and

$$\sum_{V<f\le 2V} \frac{1}{\varphi(f)} \sum_{\psi}^{\mathrm{prim}} |C_e(\psi)|^2 \ll \lambda E\Big(V + \frac{Z}{V}\Big).$$

Summing over $V = 2^j F < Q$ gives, uniformly in e,

$$\sum_{F<f\le Q/e} \frac{1}{\varphi(f)} \sum_{\psi}^{\mathrm{prim}} |C_e(\psi)|^2 \ll \lambda E\Big(Q + \frac{Z}{F}\Big). \tag{A.11}$$

If $F \ge Q$, this range is empty.

Summation over e. Let $r = \lceil 2B_{\mathrm{div}} + 1 \rceil$ and $m = 2^r$. By (A.2) and (A.1),

$$\sum_{e \le Q} \frac{\tau(e)^{2B_{\mathrm{div}}}}{\varphi(e)} \le \sum_{e \le Q} \frac{\tau(e)^r}{e} \le (1 + \log Q)^m, \qquad \sum_{e \le Q} \frac{1}{\varphi(e)} \le (1 + \log Q)^2.$$

Inserting (A.10) and (A.11) in (A.9) gives

$$S(Q) \ll_a E\lambda^{m+3}\Big(ZF^3\lambda^{-2a} + \frac{Z}{F} + Q\Big). \tag{A.12}$$

Put

$$B_1(A) = A + 2^{\lceil 2B_{\mathrm{div}}+1 \rceil} + 7, \qquad F = \lambda^{B_1}, \qquad a = 2B_1. \tag{A.13}$$

For $Q \le Z(\log Z)^{-B_1} \le 2^{B_1} Z\lambda^{-B_1}$ each term in (A.12) is $\ll EZ\lambda^{m+3-B_1} \le EZ\lambda^{-A}$. This is (6.36). The constants depend only on A, B_{div} and the constant C_{2B_1} of (6.35). For $B_{\mathrm{div}} = 1$ one has $r = 3$, $m = 8$ and $B_1(A) = A + 15$. □

Proof of (ii). Fix a box and a selected factor r of Paper I, (3.26). Put $\xi_n = \alpha_n \mathbf{1}_{(n,r)=1}$, with α as in Paper I, (3.8). Then ξ is supported in $I_T \subset (T, 2T] \subset [1, Z]$, $Z = \lceil 2T \rceil$, $|\xi_n| \le 1$, and $E = \|\xi\|_2^2 \le N_{\mathcal{I}} + 1$. Its discrepancy (A.3) at $h = 1$ is $\Delta(\alpha^{(r)}; q, b)$ of Paper I, (3.46). The support of ψ_0 in Paper I, (3.27), gives $q \le 5Q/2 < 3Q$.

Low norm. If $E \le N_{\mathcal{I}} L^{-A_3 - 1}$, then (A.5), $Q < T$ and $\log(3Q) \le L$ give

$$\sum_{q \le 3Q} V(q; \xi) \le (N_{\mathcal{I}}(1 + \log 3Q) + 3Q)E \ll TLE \le N_{\mathcal{I}} T L^{-A_3}.$$

This is the first case of Paper I, (3.48).

High norm. Let $E > N_{\mathcal{I}} L^{-A_3 - 1}$. For $d, k \ge 1$ and $(l, k) = 1$,

$$\Delta_\xi(k, l; d) = \sum_{\substack{n \in I_T,\ n \equiv l \ (k) \\ (n, dr) = 1}} \alpha_n - \frac{1}{\varphi(k)} \sum_{\substack{n \in I_T \\ (n, dkr) = 1}} \alpha_n,$$

which is the discrepancy of Paper I, (3.25), with $I = I_T$ and auxiliary divisor dr. Hence, for each $A_{\mathrm{SW}} > 0$,

$$|\Delta_\xi(k, l; d)| \ll_{A_{\mathrm{SW}}} T\tau(d)\tau(r) L^{-A_{\mathrm{SW}}}. \tag{A.14}$$

By Paper I, (3.26), r is u_0-rough and $r \le X^{0.6} \le N^{0.6}$. With $\log u_0 = \ell_0 \sigma^2 \log N$ from Paper I, (3.2), the number $\Omega(r)$ of prime factors with multiplicity satisfies $\Omega(r) \le 0.6/(\ell_0 \sigma^2)$, so

$$\tau(r) \le 2^{\Omega(r)} = O_\sigma(1).$$

By Paper I, (3.8), $N_{\mathcal{I}} \asymp TL^{-C_{\mathrm{box}}}$, hence

$$E^{1/2}Z^{1/2} \gg TL^{-(C_{\mathrm{box}}+A_3+1)/2}.$$

By Paper I, (3.11), $T \geq X^{60\sigma}$, so $\log(2Z) \asymp_\sigma L$. Take $A_{\mathrm{SW}} = P + (C_{\mathrm{box}} + A_3 + 1)/2$ in (A.14). This gives (6.35) for ξ with $B_{\mathrm{div}} = 1$, at every precision P. The constants depend only on σ, C_{box}, A_3, P and the constants of Paper I, (3.25). These are common to all boxes and selected factors.

By part (i) with $B_{\mathrm{div}} = 1$ and $A = A_3$, the range parameter is $B_1(A_3) = A_3 + 15 = B_F$. Paper I, (3.6), gives $Q < TX^{-29\sigma}$, hence

$$3Q < 3TX^{-29\sigma} \leq Z(\log Z)^{-B_F}$$

for all sufficiently large X. Part (i) yields

$$\sum_{q \leq 3Q} V(q;\xi) \ll EZ(\log Z)^{-A_3} \ll N_{\mathcal{I}}TL^{-A_3}.$$

This is (6.37), uniformly in the box and the selected factor.

In the proof of Paper I, Theorem 3.2, [3, Theorem 0(a)] is used only to obtain the second case of (3.48). See the list of inputs after the statement of that theorem in Paper I. The two cases above give (3.48) for every box and selected factor, with the same constants and quantifiers. The rest of that proof is unchanged. □

A.3. **Lemma 6.8.** Recall the notation of Section 6. Put $L = \log N$, $\ell = \log\log N$,

$$z = N^{1/\ell^3}, \qquad y = N^{1/\ell}, \qquad v = v_{\mathrm{RA}} = \frac{5227}{10000}, \qquad w = \frac{9}{1000}, \qquad \sigma_{\mathrm{p}} = 10^{-6}, \qquad \theta = \frac{7}{32},$$

and, for an integer q and a residue e,

$$\mathcal{K}_q(e) = \mathbf{1}_{e \equiv N \ (q)} - \frac{\mathbf{1}_{(e,q)=1}}{\varphi(q)} \tag{6.21}$$

(the last line of (6.21)). For a parent $(H, 2H]$ with $N^{1/3}/8 \leq H \leq 2N^{.46+O(1/L)}$, (6.38) defines

$$D_1 = HN^{-\sigma_{\mathrm{p}}}, \qquad D_2 = QN^{w+\sigma_{\mathrm{p}}}/H, \qquad Q = N^vL^{-B}. \tag{6.38}$$

For a sequence f on $(H, 2H]$, (6.32) defines $\Delta_f(I;k,b;h)$ as (A.3) with n restricted to the interval I. Then (6.34) is the bound

$$|\Delta_f(I;k,b;h)| \ll_P H\tau(h)L^{-P}, \qquad (b,k) = 1, \tag{6.34}$$

uniformly for $I \subset (H, 2H]$ and $k, h \geq 1$.

Lemma 6.8 (Two-factor convolution estimate)**.** *Let $N^{1/3}/8 \leq H \leq 2N^{.46+O(1/L)}$ and $HM \asymp N$. Let a_n, $n \sim H$, be one-bounded, supported on $P^-(n) > z$, and satisfy* (6.34) *at the original parent H, where $z = N^{1/(\log\log N)^3}$. Let b_m, $m \sim M$, be one-bounded. Let U_1, V_1 be dyadic modulus scales, allowing the singleton scale $1/2$ for the integer 1, such that*

$$U_1 \leq D_1, \qquad V_1 \leq D_2, \qquad U_1V_1 \leq N^{v_{\mathrm{RA}}}.$$

Let $f(r)$, $r \sim U_1$, and $g(s)$, $s \sim V_1$, be arbitrary one-bounded arrays, with the unary restrictions $r < D_1$, $s < D_2$, $(r,N) = (s,N) = 1$ in force. Neither a pointwise condition $rs < Q$ nor a joint condition $(r,s) = 1$ is imposed on the pair. Then, for every fixed $A > 0$, uniformly in these occupied-parent scales, f, g, unary interval cuts of a_n and b_m, and intervals $I \subset [N/8, 2N]$,

$$\sum_{r,s} f(r)g(s) \sum_{\substack{n,m \\ nm \in I}} a_nb_m\mathcal{K}_{rs}(nm) \ll_A NL^{-A}. \tag{6.45}$$

Write $H = N^h$. Fixed-factor enlargements of the parents change h by $O(1/L)$, so

$$\tfrac{1}{3} - o(1) \leq h \leq \tfrac{23}{50} + o(1), \qquad M \gg N^{.54-o(1)}. \tag{A.15}$$

All implied constants below may depend on A, on the fixed comparison constants and on the constants in (6.34). They do not depend on N, H, f, g, a or b. All fixed precisions are chosen before the final threshold for N.

Step 0: reductions. A unary cut $a_n\mathbf{1}_J(n)$ satisfies (6.34) with the same constants, since $\Delta_{a\mathbf{1}_J}(I;k,b;h) = \Delta_a(I \cap J;k,b;h)$. The error is still measured at H. A unary cut of b_m keeps it one-bounded. The product cut $nm \in I$ is removed in Step 9. Until then we prove, for every fixed $A > 0$,

$$\mathcal{S} := \sum_{r \sim U_1, \ s \sim V_1} f(r)g(s) \sum_{n \sim H, \ m \sim M} a_nb_m\mathcal{K}_{rs}(nm) \ll_A NL^{-A}. \tag{A.16}$$

Step 1: a shifted divisor moment and the mean square. For each fixed j and each interval J of length $O(M)$ in $(M/2, 4M]$,

$$\sum_{\substack{n\sim H,\ m\in J\\ mn\neq N}} \tau(|mn-N|)^j \ll_j MHL^{C_j}. \tag{A.17}$$

Let $1 \le u \le N^3$. Its prime factors above $N^{1/8}$ have total multiplicity at most 24. Group the others greedily into at most 25 factors, each at most $N^{1/4}$. Submultiplicativity of τ gives

$$\tau(u)^j \le 2^{24j} \sum_{\substack{d\mid u\\ d\le N^{1/4}}} \tau(d)^{25j}.$$

For fixed n and d, put $h_0 = (d,n)$. The congruence $d \mid mn - N$ has no solution unless $h_0 \mid N$, and otherwise $O(Mh_0/d+1)$ solutions $m \in J$. Using $(d,n) \le \sum_{e\mid(d,n)} e$ and (A.1),

$$\sum_{d\le N^{1/4}} \tau(d)^{25j}\Big(\frac{M(d,n)}{d}+1\Big) \ll L^C\big(M\tau(n)^{25j+1} + N^{1/4}\big).$$

Summing over n with (A.1), and using $N^{1/4} \le M$, proves (A.17).

Put $Z_H = \lceil 2H \rceil$. For each fixed $P > 0$ and every $K' \le HN^{-\sigma_{\mathrm{p}}/2}$,

$$\sum_{k\le 5K'} \tau(k)V(k;a) \ll_P H^2L^{-P}. \tag{A.18}$$

First let $R > 0$ be fixed. If $\|a\|_2^2 \le HL^{-R}$, then (A.5) gives $\sum_{k\le 5K'} V(k;a) \ll (H(1+\log 5K')+K')\|a\|_2^2 \ll H^2L^{1-R}$. If $\|a\|_2^2 > HL^{-R}$, then $\|a\|_2 Z_H^{1/2} \ge HL^{-R/2}$. Taking $I = (H, 2H]$ in (6.34) at precision $P' + R/2$ gives (6.35) for $\xi = a$ with $Z = Z_H$, $B_{\mathrm{div}} = 1$ and common constants, since $\log(2Z_H) \asymp L$. This argument is uniform at every requested precision P (after choosing P' accordingly). Lemma 6.5(i) with $A = P'$ and $B_1(P') = P' + 15$ gives

$$\sum_{k\le Z_H(\log Z_H)^{-P'-15}} V(k;a) \ll \|a\|_2^2 Z_H(\log Z_H)^{-P'} \ll H^2L^{-P'}.$$

The range contains $k \le 5K'$, because $5K'/H \le 5N^{-\sigma_{\mathrm{p}}/2}$ beats every fixed power of L. Thus in both cases $\sum_{k\le 5K'} V(k;a) \ll H^2L^{-\min(R-1,P')}$. To insert $\tau(k)$, split at $\tau(k) = L^{R'}$. Below the split the cost is $L^{R'}$. Above it, $\tau(k) \le L^{-R'}\tau(k)^2$, and $V(k;a) \le (2H/k+1)(H+1)$ by (A.5); hence by (A.1)

$$\sum_{\substack{k\le 5K'\\ \tau(k)>L^{R'}}} \tau(k)V(k;a) \ll L^{-R'}H^2 \sum_{k\le 5K'} \frac{\tau(k)^2}{k} \ll H^2L^{4-R'}.$$

Choosing $R' = P + 4$ and then $R, P' \ge P + R' + 1$ proves (A.18).

For a z-rough integer $t \le N^C$ put $a_n^{(t)} = a_n \mathbf{1}_{(n,t)=1}$. Since $\omega(t) \le \log t/\log z \ll \ell^3$ and each prime $p \mid t$ exceeds z,

$$\|a - a^{(t)}\|_2^2 \le \sum_{p\mid t}\Big(\frac{2H}{p}+1\Big) \ll \frac{HL}{z},$$

and by (A.5) and (A.1), uniformly in t,

$$\sum_{k\le 5K'} \tau(k)V(k;a-a^{(t)}) \ll \frac{H^2L^3}{z}. \tag{A.19}$$

Step 2: large smooth parts. Write $s = dt$ with $P^+(d) \le z < P^-(t)$. Consider the pairs (r,s) with $d > y$. Then the z-smooth part of $q = rs$ exceeds y. A modulus q has at most $\tau(q)$ factorizations $q = rs$, and $q \le 4U_1V_1 \le 4N^v$. For $(q,N) = 1$, counting m for each n coprime to q gives

$$\Big|\sum_{n,m} a_nb_m\mathcal{K}_q(nm)\Big| \ll \frac{N}{q} + H + \frac{N}{\varphi(q)}.$$

By (A.2), the contribution of these pairs to $\mathcal{S}$ is

$$\ll N \sum_{\substack{q\le 4N^v\\ q=d'h',\ d'>y,\ P^+(d')\le z}} \frac{\tau(q)^2}{q} + H\sum_{q\le 4N^v}\tau(q) \ll NL^C \sum_{\substack{d'>y\\ P^+(d')\le z}} \frac{\tau(d')^2}{d'} + HN^vL.$$

Here $\tau(q)^2 \le \tau(d')^2\tau(h')^2$ and the sum over h' is bounded by (A.1). By (A.7) the first term is $O_A(NL^{-A})$. The second is at most $N^{.9827+o(1)}$, since $h + v \le .46 + .5227 + o(1)$. It remains to treat $d \le y$.

Step 3: the first Cauchy square. Split d and t into dyadic shells $d \sim D$, $t \sim S$, with $D \le y$, $DS \asymp V_1$. There are $O(L)$ choices of D and $O(1)$ of S. Put

$$\lambda_d(t) = g(dt)\mathbf{1}_{dt\sim V_1}\mathbf{1}_{P^-(t)>z}\mathbf{1}_{P^+(d)\le z}, \qquad K = U_1 D, \qquad k = rd.$$

Then $|\lambda_d(t)| \le 1$, $rs = kt$, and $(kt, N) = 1$ on the supports. One block of $\mathcal{S}$ is

$$\mathcal{S}_0 = \sum_{r,d,m} f(r) b_m \sum_t \lambda_d(t) \sum_n a_n \mathcal{K}_{kt}(mn).$$

Fix a smooth $\psi \ge 0$ supported in $[1/2, 5/2]$, equal to 1 on $[1, 2]$. Cauchy's inequality in (r, d, m) gives

$$|\mathcal{S}_0|^2 \le CKM\,\mathcal{S}', \qquad \mathcal{S}' = \sum_{r,d} \psi\Big(\frac{r}{U_1}\Big)\mathbf{1}_{(rd,N)=1}\mathbf{1}_{d\sim D} \sum_m \psi\Big(\frac{m}{M}\Big)\Big|\sum_t \lambda_d(t) \sum_n a_n \mathcal{K}_{kt}(mn)\Big|^2. \tag{A.20}$$

The indicator $\mathbf{1}_{(rd,N)=1}$ is part of the original support and is kept. The scale conditions and (A.15) give, for large N,

$$\frac{K}{H} \le N^{-\sigma_{\mathrm{p}}/2}, \qquad \frac{KS}{M} \le N^{-.0172}, \qquad M \gg N^{.54-o(1)}, \tag{A.21}$$

because $K \le D_1 y$, $KS \asymp U_1 V_1 \le N^v$ and $v - .54 = -.0173$. In particular $kt = o(M)$ and $K = o(H)$.

Expand $\mathcal{S}' = \mathcal{S}_3 - 2\,\mathrm{Re}\,\mathcal{S}_2 + \mathcal{S}_1$ over tuples $\mathbf{t} = (r, d, t_1, t_2, n_1, n_2)$ with coefficient

$$c(\mathbf{t}) = \psi(r/U_1)\mathbf{1}_{(rd,N)=1}\mathbf{1}_{d\sim D}\lambda_d(t_1)\overline{\lambda_d(t_2)}a_{n_1}\overline{a_{n_2}}, \qquad \omega(\mathbf{t}) = \mathbf{1}_{(n_1,kt_1)=(n_2,kt_2)=1}:$$

$$\mathcal{S}_3 = \sum_{\mathbf{t}} c \sum_m \psi(m/M)\mathbf{1}_{kt_1 \mid mn_1 - N}\mathbf{1}_{kt_2 \mid mn_2 - N},$$

$$\mathcal{S}_2 = \sum_{\mathbf{t}} c \sum_m \psi(m/M)\mathbf{1}_{kt_1 \mid mn_1 - N}\frac{\mathbf{1}_{(mn_2,kt_2)=1}}{\varphi(kt_2)},$$

$$\mathcal{S}_1 = \sum_{\mathbf{t}} c \sum_m \psi(m/M)\frac{\mathbf{1}_{(mn_1,kt_1)=1}\mathbf{1}_{(mn_2,kt_2)=1}}{\varphi(kt_1)\varphi(kt_2)}.$$

In $\mathcal{S}_3$ the factor ω is automatic, since $(kt_i, N) = 1$.

Step 4: restoration before the Chinese remainder theorem. Call a tuple *gcd-bad* if one of

$$(n_1, n_2) > 1, \quad (t_1, t_2) > 1, \quad (n_1, t_2) > 1, \quad (n_2, t_1) > 1, \quad (k, t_1 t_2) > 1 \tag{A.22}$$

holds. Call a tuple *near-diagonal* if $|n_1 - n_2| < HL^{-C_M}$; this condition is charged separately below. On the other tuples k, t_1, t_2 are pairwise coprime. For a set $\mathcal{B}$ of tuples define the formal terms

$$MT(\mathcal{B}) = \sum_{\mathcal{B}} \frac{c\,\omega\,M\hat{\psi}(0)}{kt_1 t_2 \varphi(k)}, \qquad \mathcal{Z}(\mathcal{B}) = \sum_{\mathcal{B}} \frac{c\,\omega\,\mathbf{1}_{n_1\equiv n_2\ (k)}M\hat{\psi}(0)}{kt_1 t_2}. \tag{A.23}$$

We show

$$\begin{aligned} &\mathcal{S}' = M\mathcal{E} + \mathcal{Z}(\mathrm{all}) - MT(\mathrm{all}) + O(R_0), \\ &R_0 \ll_P \frac{MH^2}{K}L^{-P} + MHL^C\Big[\frac{H}{zK} + 1\Big] + N^{o(1)}\Big[\frac{H^2}{K} + H^2 S\Big] + N^{o(1)}\frac{(H + KS)^2}{K}, \end{aligned} \tag{A.24}$$

where $\mathcal{E}$ is the nonzero-frequency sum on good tuples defined in Step 6. The exponent C is fixed after P and the truncation parameters.

Resonant m. If $mn_i = N$, then $m \mid N$. There are $\tau(N) = N^{o(1)}$ such m. For fixed (r, d) and such m, the congruence terms over (t, n) number at most $\sum_t (2H/(kt) + 1) \ll H/K + S$. The comparator terms are $\ll L^C H/K$. Hence these m contribute $\ll N^{o(1)}K(H/K + S)^2$ to $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and to every subset of tuples below. This is the last term of R_0.

Divisor truncation. For the other m put $u_i = |mn_i - N| \ge 1$. Let Σ^* be the event $\tau(u_1), \tau(u_2) \le L^{C_\Sigma}$. On the part of the complement with $\tau(u_1) \ge \tau(u_2)$, the triples (r, d, t_1) with $kt_1 \mid u_1$ number at most $\tau(u_1)^3$. The $t_2 \mid u_2$ number at most $\tau(u_1)$, and n_2 lies in one class modulo k, with $O(H/K)$ choices. With $\mathbf{1}_{\tau(u_1)>L^{C_\Sigma}} \le \tau(u_1)L^{-C_\Sigma}$, (A.17) gives a contribution to $\mathcal{S}_3$ of

$$\ll L^{-C_\Sigma}\frac{H}{K}\sum_{m,n_1} \psi(m/M)\tau(u_1)^5 \ll \frac{MH^2}{K}L^{C_5 - C_\Sigma}. \tag{A.25}$$

The other part is symmetric. Choose $C_\Sigma \ge C_5 + P$.

Bad tuples in $\mathcal{S}_3$. In this paragraph, "bad" means Σ^* intersected with the five gcd failures in (A.22); the near-diagonal set is not included in this term and is charged separately below. On Σ^* fix (m, n_1). There are at most L^{3C_Σ} triples (r, d, t_1) and at most $O(L)$ candidate primes p for a failure in (A.22). Once n_2 is fixed, there are at most L^{C_Σ} choices of t_2. Every witnessing prime exceeds z: it divides a z-rough n_i or t_i. Since $(k, N) = 1$ and $k \mid mn_1 - N$, we have $(m, k) = (n_1, k) = 1$ and $n_2 \equiv n_1 \pmod k$. Each failure places n_2 in one class modulo kp:

- $p \mid (n_1, n_2)$: $p \nmid k$; combine $n_2 \equiv 0\ (p)$ with $n_2 \equiv n_1\ (k)$.
- $p \mid (t_1, t_2)$, $p \nmid k$: $p \nmid m$, since $p \mid mn_1 - N$ and $p \nmid N$; combine $mn_2 \equiv N\ (p)$ with $n_2 \equiv n_1\ (k)$. If $p \mid k$, then $kt_2 \mid mn_2 - N$ and $p \mid t_2$ give $kp \mid mn_2 - N$.
- $p \mid (n_1, t_2)$: $p \nmid k$ and $p \nmid m$; combine $mn_2 \equiv N\ (p)$ with $n_2 \equiv n_1\ (k)$.
- $p \mid (n_2, t_1)$: $p \nmid k$; combine $n_2 \equiv 0\ (p)$ with $n_2 \equiv n_1\ (k)$.
- $p \mid (k, t_1t_2)$: since $p > z \geq P^+(d)$, $p \mid r$. If $p \mid t_2$, then $kt_2 \mid mn_2 - N$ gives $kp \mid mn_2 - N$. If $p \mid t_1$ only, exchange the two copies first.

The number of n_2 in one class modulo kp is $O(H/(zK) + 1)$. Hence

$$\mathcal{S}_3(\text{bad}) \ll MHL^{4C_\Sigma+1}\Big[\frac{H}{zK} + 1\Big]. \tag{A.26}$$

The near-diagonal $|n_1 - n_2| < HL^{-C_M}$ is counted in the same way, with H/z replaced by HL^{-C_M}. Choose $C_M \geq P + 4C_\Sigma + 1$. This gives the first two terms of R_0. No joint condition $(r, s) = 1$ has been used. A prime $p \leq z$ dividing (r, s) divides d and only enlarges k. A prime $p > z$ dividing (r, s) divides k and some t_i. This is the last case of (A.22).

Bad tuples in the density terms. For $\mathcal{S}_1$, $\mathcal{S}_2$ and MT, dropping coprimality leaves the positive density

$$\ll L^{C_\varphi}\frac{M}{k^2t_1t_2},$$

since $kt_1 = o(M)$ and $q/\varphi(q) \leq L^{C_\varphi}$ for $q \leq N^{O(1)}$. (In $\mathcal{S}_2$ the m-sum is nonzero only if n_1 is invertible modulo kt_1, by $(k, N) = (t_1, N) = 1$ and $kt_1 \mid mn_1 - N$. Then m lies in one class modulo kt_1.) For $\mathcal{Z}$ retain ω and $n_2 \equiv n_1\ (k)$, with weight $M/(kt_1t_2)$. In particular $(n_1, k) = (n_2, k) = 1$ there. Without the bad condition these weights sum to $\ll L^{C_\varphi}MH^2/K$ and $\ll MH^2/K$, using $\sum_{t\sim S} 1/t \leq 1$, $k = rd \asymp K$ and, in $\mathcal{Z}$, $O(H/K)$ values of n_2 in one class modulo k. As in $\mathcal{S}_3$ every witnessing prime p exceeds z. In each case below the prime p divides n_1 or t_1, which is fixed first together with p. Since $\omega(n_1), \omega(t_1) \ll L$, there are $O(L)$ choices of p. We use

$$\#\{n_2 \sim H :\ p \mid n_2\} \leq \frac{H}{p} + 1, \qquad \sum_{\substack{t_2\sim S\\ p\mid t_2}}\frac{1}{t_2} \leq \frac{1}{S}\cdot\frac{2S}{p} = \frac{2}{p},$$

$$\#\{r \leq \tfrac{5}{2}U_1 :\ p \mid r\} \leq \frac{5U_1}{2p};$$

the last two have no $+1$, because they count multiples of p in $[1, 2S]$ and $[1, \frac{5}{2}U_1]$.

- $p \mid (n_1, n_2)$: fix n_1 and $p \mid n_1$. In $\mathcal{S}_1, \mathcal{S}_2, MT$ there are $\leq H/z + 1$ values of n_2. In $\mathcal{Z}$, $p \nmid k$ since $(n_1, k) = 1$; combining $n_2 \equiv 0\ (p)$ with $n_2 \equiv n_1\ (k)$ leaves $O(H/(zK) + 1)$ values of n_2.
- $p \mid (t_1, t_2)$: fix t_1 and $p \mid t_1$; the t_2-sum is at most $2/p \leq 2/z$. The n_2-count is unchanged; $p \mid k$ is allowed.
- $p \mid (n_1, t_2)$: fix n_1 and $p \mid n_1$; the t_2-sum is at most $2/z$, and the n_2-count is unchanged.
- $p \mid (n_2, t_1)$: fix t_1 and $p \mid t_1$. In $\mathcal{S}_1, \mathcal{S}_2, MT$ there are $\leq H/z + 1$ values of n_2. In $\mathcal{Z}$, $p \mid k$ would contradict $(n_2, k) = 1$, so $p \nmid k$; combining $n_2 \equiv 0\ (p)$ with $n_2 \equiv n_1\ (k)$ leaves $O(H/(zK) + 1)$ values of n_2.
- $p \mid (k, t_1t_2)$: the weights are symmetric in the two copies, so let $p \mid t_1$; fix t_1 and $p \mid t_1$. Since $p > z \geq P^+(d)$, $p \mid r$, and there are at most $5U_1/(2p) \leq 5U_1/(2z)$ values of r. The d-, t_2- and n-counts are unchanged.

In every case the bad part of each density term is $\ll MHL^{C_\varphi+1}[H/(zK) + 1]$. This is part of R_0. The near-diagonal is counted in the same way, with H/z replaced by HL^{-C_M}. There are $O(HL^{-C_M} + 1)$ values of n_2, and $O(HL^{-C_M}/K + 1)$ in one class modulo k. Hence the near-diagonal is $\ll MH^2L^{C_\varphi-C_M}/K + MHL^{C_\varphi}$. This is part of R_0 once also $C_M \geq P + C_\varphi$. Thus all bad tuples may be removed from $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{Z}, MT$ at the cost R_0.

Good tuples. On good tuples the two congruences in $\mathcal{S}_3$ are compatible exactly on the support $\omega(\mathbf{t}) = 1$ and $n_1 \equiv n_2 \pmod k$. Then m lies in one class μ modulo kt_1t_2, with

$$\frac{\mu}{kt_1t_2} \equiv \frac{N\overline{n_1t_2}}{kt_1} + \frac{N\overline{n_2kt_1}}{t_2} \pmod 1. \tag{A.27}$$

Poisson summation gives

$$\sum_{m\equiv\mu\ (kt_1t_2)}\psi\Big(\frac{m}{M}\Big) = \frac{M}{kt_1t_2}\sum_{h\in\mathbb{Z}}\hat{\psi}\Big(\frac{hM}{kt_1t_2}\Big)e\Big(\frac{h\mu}{kt_1t_2}\Big). \tag{A.28}$$

The term $h = 0$ gives $\mathcal{Z}(\text{good})$. All three expanded summands vanish unless $\omega(\mathbf{t}) = 1$. The following main-term computations are on that support. For $\mathcal{S}_1$ the smooth coprime m-sum is $M\hat{\psi}(0)\varphi(kt_1t_2)/(kt_1t_2) + O(\tau(kt_1t_2))$. Division by $\varphi(kt_1)\varphi(kt_2) = \varphi(k)^2\varphi(t_1)\varphi(t_2)$ gives $MT(\text{good})$ with total error $N^{o(1)}H^2/K$. For $\mathcal{S}_2$ expand $\mathbf{1}_{(m,t_2)=1}$ by Möbius inversion. The main term is $M\hat{\psi}(0)\varphi(t_2)/(kt_1t_2)$, with error $O(\tau(t_2))$. Division by $\varphi(kt_2) = \varphi(k)\varphi(t_2)$ gives $MT(\text{good})$ with total error $N^{o(1)}H^2S$. Completing $\mathcal{Z}$ and MT back to all tuples costs R_0. This proves (A.24).

Step 5: the zero mode. Expanding the centred class counts gives the exact identity, valid for all tuples including those with gcd failures,

$$\mathcal{Z}(\text{all}) - MT(\text{all}) = M\hat{\psi}(0) \sum_{r,d,t_1,t_2} \frac{\psi(r/U_1)\mathbf{1}_{(rd,N)=1}\lambda_d(t_1)\overline{\lambda_d(t_2)}}{kt_1t_2} \sideset{}{^*}\sum_{b \bmod k} \Delta_{a^{(t_1)}}(k,b)\overline{\Delta_{a^{(t_2)}}(k,b)}. \tag{A.29}$$

For fixed (k,t_1,t_2) both sides equal

$$\sideset{}{^*}\sum_{b \bmod k} A_1(b)\overline{A_2(b)} - \frac{1}{\varphi(k)} \sideset{}{^*}\sum_{b \bmod k} A_1(b) \sideset{}{^*}\sum_{b \bmod k} \overline{A_2(b)}, \qquad A_i(b) = \sum_{\substack{n \equiv b \ (k) \\ (n,t_i)=1}} a_n.$$

Use $|u\bar{v}| \le (|u|^2+|v|^2)/2$ and

$$V(k;a^{(t)}) \le 2V(k;a) + 2V(k;a-a^{(t)}).$$

The sums $\sum_{t\sim S} 1/t$ are $O(1)$, a given k has at most $\tau(k)$ factorizations $k=rd$, and $k \le 5K$. By (A.18) with $K'=K$ and (A.19),

$$|\mathcal{Z}(\text{all}) - MT(\text{all})| \ll_P \frac{MH^2}{K}\Big[L^{-P} + \frac{L^3}{z}\Big]. \tag{A.30}$$

Only $|\lambda_d| \le 1$ is used, so λ_d may vary with d. The factor $1/K$ is cancelled by KM in (A.20).

By (A.20), (A.24), (A.30) and (A.21), the contributions to $|\mathcal{S}_0|$ of all terms except $M\mathcal{E}$ are

$$\ll NL^{-P/2} + NL^C\Big[z^{-1/2} + \Big(\frac{K}{H}\Big)^{1/2}\Big] + N^{1+o(1)}\Big[M^{-1/2} + \Big(\frac{KS}{M}\Big)^{1/2}\Big] + N^{.857}. \tag{A.31}$$

For the last term, $M^{1/2}(H+KS) \le N^{(1+h)/2+o(1)} + N^{(1-h)/2+v+o(1)}$, and $(1+h)/2 \le .73+o(1)$, $(1-h)/2+v \le .856034$.

Step 6: nonzero frequencies. Fix $d \sim D$ and let $\mathcal{E}_d$ be the part of $\mathcal{E}$ with this d:

$$\mathcal{E}_d = \sum_{\substack{\text{good } \mathbf{t},\ \omega(\mathbf{t})=1 \\ n_1 \equiv n_2 \ (k)}} \frac{c(\mathbf{t})}{kt_1t_2} \sum_{h\neq 0} \hat{\psi}\Big(\frac{hM}{kt_1t_2}\Big) e\Big(\frac{h\mu}{kt_1t_2}\Big).$$

Put $\eta = 10^{-9}$ and

$$H_* = C\frac{KS^2N^\eta}{M}, \tag{A.32}$$

with C larger than every $kt_1t_2/(KS^2)$. Since $|\hat{\psi}(\xi)| \ll_j |\xi|^{-j}$, the terms $|h| > H_*$ contribute $O(N^{-90})$ in total, taking $j > 200/\eta$. If $H_* < 1$, $\mathcal{E}_d$ is negligible. This covers $t=1$, $S=1/2$, where $H_* \ll KN^\eta/M \le N^{-.07}$. Hence we may assume $S \ge 1$ and $H_* \ge 1$.

Reciprocity. For coprime u,v, $\bar{u}/v + \bar{v}/u \equiv 1/(uv) \pmod 1$. Applied to the first term of (A.27), with $u = n_1t_2$ and $v = kt_1$, it gives

$$e\Big(\frac{h\mu}{kt_1t_2}\Big) = e\Big(\frac{Nh(n_1-n_2)\overline{n_2kt_1}}{n_1t_2}\Big) e\Big(\frac{Nh}{kt_1t_2n_1}\Big).$$

Put $f_0 = (n_1-n_2)/k$, a nonzero integer on good tuples. Cancelling k,

$$e\Big(\frac{Nhf_0\overline{n_2t_1}}{n_1t_2}\Big). \tag{A.33}$$

The conditions on t_1 are

$$(t_1, Nc) = 1, \qquad (t_1, J) = 1, \qquad b = n_2, \quad c = n_1t_2, \quad J = kn_2. \tag{A.34}$$

The first is kept. To separate the second, colour every prime in $[z, N^3]$ blue or red independently with probability $1/2$. Since t_1 is z-rough,

$$\mathbf{1}_{(t_1,J)=1} = \mathbb{E}_{\text{col}}\Big[2^{\omega(t_1)+\omega_z(J)}\mathbf{1}_{\text{all primes of } t_1 \text{ blue}}\mathbf{1}_{\text{all primes } \ge z \text{ of } J \text{ red}}\Big], \tag{A.35}$$

where $\omega_z(J)$ counts the primes $\ge z$ of J. A shared prime makes the event empty. Otherwise its probability is the reciprocal of the power of 2. Both counts are $O(\ell^3)$, so the multipliers are $\exp(O(\ell^3)) = N^{o(1)}$. Fix one colouring.

Split f_0 and h into signed dyadic blocks $|f_0| \sim F$ and $h = \varepsilon h_+$, $h_+ \sim H_0 > 0$, with $F \ll H/K$ and $H_0 \le H_*$. Below, h_+ is renamed $h > 0$. Fix d, this block, a sign ε, and one prime colouring from (A.35). The *outer* tuple is $\mathbf{o} = (r, n_1, n_2, t_2)$, with $k = rd$, $b = n_2$, $c = n_1t_2$ and $f_0 = (n_1-n_2)/k$. Its support contains $k \mid n_1 - n_2$, the near-diagonal exclusion, and the conditions on (r,d,n_1,n_2,t_2) inherited from the good tuples and $\omega = 1$. It also requires every prime $\ge z$ of $J = kn_2$ to be red. The remaining t_1-conditions are $(t_1, Nc) = 1$, $t_1 \sim S$ and all primes of t_1 blue. The colouring enforces $(t_1, J) = 1$. Write

$$u = \frac{hS}{H_0t_1}, \quad Y_2(\mathbf{o}) = \frac{\varepsilon H_0M}{kt_2S}, \quad Y_1(\mathbf{o}) = \frac{\varepsilon NH_0}{kt_2n_1S}, \quad W_{\mathbf{o}}(u) = w_0(u)\hat{\psi}(Y_2(\mathbf{o})u)e(Y_1(\mathbf{o})u),$$

where w_0 is a fixed smooth cutoff supported in $(1/4,4)$ and equal to 1 on $[1/2,2]$. Since $H_0 \le H_*$, $k \asymp K$, $t_2 \asymp S$, and $N/(n_1 M) \asymp 1$, uniformly in *all* outer tuples, $|Y_1(\mathbf{o})| + |Y_2(\mathbf{o})| \ll N^{\eta}$. On the original support, the remaining Fourier weight is exactly $W_{\mathbf{o}}(hS/(H_0 t_1))$. Define the Mellin transform and its uniform majorant by

$$\widetilde{W}_{\mathbf{o}}(it) = \int_0^\infty W_{\mathbf{o}}(u) u^{it} \frac{du}{u}, \qquad m(t) = C_{\psi,w_0} \min\{1, N^{2\eta}(1+|t|)^{-2}\}.$$

The compact support, two integrations by parts, and the bounds on Y_1, Y_2 give

$$\sup_{\mathbf{o},t} \frac{|\widetilde{W}_{\mathbf{o}}(it)|}{m(t)} \le 1, \qquad \int_{\mathbb{R}} m(t)\, dt \ll_{\psi,w_0} N^{\eta},$$

with the fixed constant C_{ψ,w_0} chosen large enough. Mellin inversion is the *outer-dependent* identity

$$W_{\mathbf{o}}\Big(\frac{hS}{H_0 t_1}\Big) = \frac{1}{2\pi} \int_{\mathbb{R}} m(t) \underbrace{\Big(\frac{H_0}{S}\Big)^{it} \frac{\widetilde{W}_{\mathbf{o}}(it)}{m(t)}}_{\text{outer coefficient}} \underbrace{h^{-it} t_1^{it}}_{\text{inner factors}}\, dt.$$

In particular $\widetilde{W}_{\mathbf{o}}(it)/m(t)$ is not an inner t_1-weight: it depends on k, t_2, n_1 and the fixed sign and block.

The t-integral precedes Cauchy. Let $B_{\mathrm{in}} \ge \sup_{t_1} 2^{\omega(t_1)}$ and $B_{\mathrm{out}} \ge \sup_{\mathbf{o}} 2^{\omega_z(kn_2)}$ on this block, with $B_{\mathrm{in}} B_{\mathrm{out}} = \exp(O(\ell^3)) = N^{o(1)}$. For fixed real Mellin parameter t set

$$\begin{aligned}
\lambda'_v(t) &= \frac{S}{v} \lambda_d(v) \frac{2^{\omega(v)}}{B_{\mathrm{in}}} \mathbf{1}_{\text{all primes of } v \text{ blue}}\, v^{it},\\
\eta_h(t) &= h^{-it} \mathbf{1}_{h \sim H_0} \mathbf{1}_{h \le H_*},\\
A_{\mathbf{o}}(t) &= \frac{KS}{kt_2}\, \psi(r/U_1) \overline{\lambda_d(t_2)} a_{n_1} \overline{a_{n_2}} \frac{2^{\omega_z(kn_2)}}{B_{\mathrm{out}}} \mathbf{1}_{\text{all primes} \ge z \text{ of } kn_2 \text{ red}} \Big(\frac{H_0}{S}\Big)^{it} \frac{\widetilde{W}_{\mathbf{o}}(it)}{m(t)}.
\end{aligned}$$

The factor $\mathbf{1}_{h \le H_*}$ is the one-bounded mask on a terminal frequency block; it is independent of the outer tuple. The sum over $\mathbf{o}$ is restricted to the outer support just specified. Choose the fixed cutoff ψ with $0 \le \psi \le 1$. On the dyadic support $KS/(kt_2) \le 2$. By the definitions of m, B_{in} and B_{out}, every other factor above has modulus at most one. Hence $|A_{\mathbf{o}}(t)| \le 2$ and $|\lambda'_v(t)|, |\eta_h(t)| \le 1$, uniformly in the real t and all original outer parameters. Using $1/(kt_1t_2) = (KS^2)^{-1}(KS/(kt_2))(S/t_1)$, the contribution of this block and colouring to $\mathcal{E}_d$ is

$$\frac{B_{\mathrm{in}} B_{\mathrm{out}}}{2\pi KS^2} \int_{\mathbb{R}} m(t) \sum_{\mathbf{o}} A_{\mathbf{o}}(t) \sum_{\substack{v \sim S \\ (v,Nc)=1}} \sum_{h \sim H_0} \lambda'_v(t) \eta_h(t) e\Big(\frac{\varepsilon N h f_0 \overline{bv}}{c}\Big)\, dt.$$

Here $(b,c) = 1$ on the good support. The colour average and the $O(L^2)$ signed dyadic blocks are taken *after* the estimate. Their cost is $N^{o(1)}$.

Second square, pointwise in t. Group the outer tuples by (b,c,f_0). For fixed (b,c,f_0), the factorization $n_1 t_2 = c$ has at most $\tau(c) = N^{o(1)}$ choices. Then $k = (n_1 - b)/f_0$ and $r = k/d$ are determined. The number of outer tuples is $\ll N^{o(1)} H^2 S$. Let $\rho \ge 0$ be a smooth majorant, at least 1 on their ranges $b \ll H$, $c \ll HS$. Since $b = n_2$ is z-rough, $(b, p_0) = 1$ for the least prime p_0 not dividing N, as $p_0 \ll \log N < z$. Define, for each real t,

$$\begin{aligned}
W_4(t) &= \sum_{\substack{(b,c)=1 \\ (b,p_0)=1}} \rho(b,c) \sum_{f_0 \sim F} \Big| \sum_{\substack{v \sim S \\ (v,Nc)=1}} \sum_{h \sim H_0} \lambda'_v(t) \eta_h(t) e\Big(\frac{\varepsilon N h f_0 \overline{bv}}{c}\Big)\Big|^2,\\
|\mathcal{E}_d| &\ll N^{o(1)} \frac{H}{KS^{3/2}} \int_{\mathbb{R}} m(t) W_4(t)^{1/2}\, dt \le N^{\eta + o(1)} \frac{H}{KS^{3/2}} (W_4^*)^{1/2},\\
W_4^* &= \sup_{\substack{\text{signed blocks, colourings} \\ t \in \mathbb{R}}} W_4(t).
\end{aligned} \tag{A.36}$$

Here $W_4(t)$ is formed with a fixed block and colouring; the supremum is taken afterwards. The displayed bound for the full $\mathcal{E}_d$ follows from the colour expectation and the sum over the $O(L^2)$ blocks. Their cost is a uniform $N^{o(1)}$. Thus no outer coefficient is moved through Cauchy or the t-integral. Cauchy is applied *inside* the integral for each t. No outer-dependent Mellin factor appears in $\lambda'_v(t)$.

For every fixed t, expand the square with $z_1 = vv'$, $k_0 = h_1 v' - h_2 v$ and $y_1 = f_0 k_0$. Since $|\lambda'_v(t)|, |\eta_h(t)| \le 1$, the diagonal $k_0 = 0$ obeys, uniformly in t,

$$W_{\mathrm{diag}}(t) \le N^{o(1)} H_* S \cdot H \cdot F \cdot HS \le N^{\eta + o(1)} \frac{H^4 S^4}{N}. \tag{A.37}$$

For $k_0 \neq 0$, write $m = |y_1| \ll FH_*S$ and separate the two signs of y_1. The off-diagonal is a sum of expressions

$$\sum_{z_1,m} b^{\pm}_{z_1,m}(t) \sum_{(b,c)=(b,p_0)=(z_1,c)=1} \rho(b,c) e\Big(\pm \frac{Nm\overline{z_1 b}}{c}\Big), \tag{A.38}$$

where $b^{\pm}_{z_1,m}(t)$ depends only on $(d, S, F, H_0, \varepsilon, \text{colouring}, t; z_1, m)$, and not on (b, c, f_0) or the original outer tuple. Its summands are products of the one-bounded weights $\lambda'_v(t), \eta_h(t)$, and $|b^{\pm}_{z_1,m}(t)| \le B_{z_1,\pm m}$ for the counting majorant defined below. The support has $(z_1, Np_0) = 1$.

Coefficient norm. Fix $C > 0$. For $1 \le S \le N^C$, $1 \le H' \le S$ and $1/2 \le F \le N^C$ let B_{z_1,y_1} count $t, t' \sim S$, $(tt', N) = 1$, $1 \le |h_1|, |h_2| \le H'$, $f_0 \sim F$, with $tt' = z_1$ and $f_0(h_1 t' - h_2 t) = y_1 \neq 0$. Write y_N for the largest divisor of y composed of primes dividing N. Then

$$\sum_{z_1, y_1 \neq 0} B^2_{z_1,y_1} (y_{1,N})^{\theta} \le N^{o(1)} F H'^2 S^2. \tag{A.39}$$

At fixed (z_1, y_1) there are at most $\tau(z_1)\tau(|y_1|) = N^{o(1)}$ choices of (t, t', f_0). By Cauchy's inequality the left side is at most

$$N^{o(1)} \Big[\sum_{f \le 2F} f_N^{\theta}\Big] \mathcal{Q}, \qquad \mathcal{Q} = \sum_{t,t'} \sum_{k_0 \neq 0} (k_{0,N})^{\theta} \mathcal{N}_{t,t'}(k_0)^2,$$

where $\mathcal{N}_{t,t'}(k_0)$ counts (h_1, h_2) with $h_1 t' - h_2 t = k_0$. Since $f_N^{\theta} \le \sum_{d|f,\ d|N^{\infty}} d^{\theta}$, the f-sum is at most $2FP_0$, with $P_0 = \prod_{p|N} (1 - p^{\theta-1})^{-1}$.

Write $g_0 = (t, t')$, $t = g_0 u$, $t' = g_0 v$ and $S' = S/g_0$. A fibre of $(h_1, h_2) \mapsto k_0$ is an arithmetic progression with step (u, v). It has at most $1 + 2H'/S'$ points, and at most one if $S' \ge 2H'$. Also

$$(k_{0,N})^{\theta} = \sum_{d|k_0,\ d|N^{\infty}} \phi_{\theta}(d), \qquad 0 \le \phi_{\theta}(d) \le d^{\theta},$$

by telescoping each prime power. Let $A_d(t, t')$ count (h_1, h_2) with $k_0 \neq 0$ and $d \mid k_0$. Then

$$\mathcal{Q} \le \sum_{\substack{d|N^{\infty} \\ d \le 4H'S}} d^{\theta} \Big[\sum_{t,t'} A_d(t,t') + \sum_{t,t':\, S' < 2H'} \frac{2H'}{S'} A_d(t,t')\Big]. \tag{A.40}$$

For the first sum write $(h_1, h_2) = e\,\mathbf{w}$ with $\mathbf{w}$ primitive, $\|\mathbf{w}\|_{\infty} = j \le H'$. There are at most $8j$ such $\mathbf{w}$ and $e \le H'/j$. Put $d_e = d/(d, e)$. Since tt' is prime to d, a contributing $\mathbf{w}$ has $(w_1, d_e) = 1$ (and $w_1 = 0$ forces $d_e = 1$). Counting t and then t' in one class modulo d_e, and using $\sum_{e \le E} (d, e) \le \tau(d) E$,

$$\sum_{t,t'} A_d(t,t') \le C\Big[\tau(d) \frac{H'^2 S^2}{d} + H'^2 S\Big]. \tag{A.41}$$

For the second sum $(g_0, d) = 1$. There are $O(H'S'/d)$ nonzero levels of $h_1 v - h_2 u$ divisible by d. Each has $O(H'/S')$ points, so $A_d \le CH'^2/d$. Summing over g_0, with $\sum_{g_0 \le 2S} (S/g_0 + 1)^2 (g_0/S) \le S(8 + \log 2S)$,

$$\sum_{t,t':\, S' < 2H'} \frac{2H'}{S'} A_d \le C(1 + \log 2S) \frac{H'^3 S}{d} \le C(1 + \log 2S) \frac{H'^2 S^2}{d}. \tag{A.42}$$

Finally

$$\sum_{d|N^{\infty}} \tau(d) d^{\theta-1} = P_0^2, \qquad \sum_{\substack{d|N^{\infty} \\ d \le 4H'S}} d^{\theta} \le (4H'S)^{1/2} P_2, \qquad P_2 = \prod_{p|N} (1 - p^{\theta - 1/2})^{-1}.$$

Since $H' \le S$, $H'^2 S (H'S)^{1/2} \le H'^2 S^2$. Both P_0 and P_2 are $N^{o(1)}$: with the primes of N in increasing order, $\log P_2 \ll \sum_{i \le \omega(N)} p_i^{\theta - 1/2} \ll (\log N)^{\theta + 1/2}$. This proves (A.39).

With $H' = H_*$ we have $H_*/S \ll KSN^{\eta}/M \le N^{-.0171}$ by (A.21). For every real Mellin parameter t, fixed $d, S, F, H_0, \varepsilon$ and colouring, the coefficient sequences $b^{\pm}_{z_1,m}(t)$ in (A.38) satisfy the *uniform* weighted estimate

$$\sum_{z_1, m \ge 1} (m_N)^{\theta} |b^{\pm}_{z_1,m}(t)|^2 \le \sum_{z_1, y_1 \neq 0} B^2_{z_1,y_1} (y_{1,N})^{\theta} \ll N^{o(1)} F H_*^2 S^2.$$

This follows from (A.39), since the inner complex weights have modulus at most one. In particular the implied constant is independent of t and of all original outer data $(r, k, n_1, n_2, t_2, b, c, f_0)$. The dependence on these data was placed in $A_{\mathbf{o}}(t)$ *before* Cauchy. Writing $\widetilde{b}^{\pm}_t(j)$ for the slices of Paper I, (2.43), with $n_T = p_0 m$ and $a_T = \pm N$, gives

$$\sup_{t \in \mathbb{R}} \sum_{j | N^{\infty}} j^{\theta} \|\widetilde{b}^{\pm}_t(j)\|_2^2 \ll N^{o(1)} F (H_* S)^2, \qquad \sup_{t \in \mathbb{R}} \|b^{\pm}_t\|_2^2 \ll N^{o(1)} F (H_* S)^2. \tag{A.43}$$

Since $(p_0, N) = 1$, the N-part of $n_T = p_0 m$ is precisely m_N. This is the norm required by Paper I, (2.42)–(2.43), for every t, without any Mellin factor in the inner coefficients.

Application of Paper I, Theorem 2.8. In (A.38) put

$$\begin{aligned} r_T = z_1 \sim Z_1, \quad s_T = p_0, \quad R_T = Z_1, \quad S_T = p_0/2, \\ d_T = b, \quad c_T = c, \quad n_T = p_0 m, \quad Y := FH_*S, \quad N_T = Cp_0 Y \le N^{o(1)} Y, \quad p_0 \ll \log N, \quad a_T = \pm N, \end{aligned} \tag{A.44}$$

with $A = N$. Since $t, t' \sim S$, $z_1 = tt' \in (S^2, 4S^2]$, which meets two dyadic windows. Split the z_1-sum into $z_1 \in (S^2, 2S^2]$ and $z_1 \in (2S^2, 4S^2]$. Apply the theorem to each, with $Z_1 = S^2$ and $Z_1 = 2S^2$ respectively. This costs a factor 2 in the final bound, and $Z_1 \asymp S^2$ in both windows. Then $s_T \in (S_T, 2S_T]$ and $S_T \ge 3/2$, since N is even and $p_0 \ge 3$. The support has $(a_T, r_T s_T) = 1$. The domain of the theorem is exactly

$$(z_1 b, p_0 c) = 1 \iff (b, c) = (z_1, c) = (z_1, p_0) = (b, p_0) = 1,$$

and $(z_1, p_0) = 1$ holds by roughness. Exactly as in Paper I, (3.82)–(3.84), reducing the inverse modulo c gives

$$e\Big(\frac{a_T n_T \overline{z_1 b}}{p_0 c}\Big) = e\Big(\pm \frac{Nm\overline{z_1 b}}{c}\Big),$$

also when $p_0 \mid c$. Split ρ into finitely many smooth pieces on strict dyadic windows in (b, c). Multiply by fixed smooth cutoffs equal to 1 on the supported n_T, z_1, p_0. At $n_T \ge 1$ use a cutoff vanishing below $1/2$. All derivatives satisfy Paper I, (2.39), with uniform seminorms. Take $\epsilon = \epsilon_0 = 10^{-12}$. All parameters lie in $[1, N^{10}]$, so $\mathcal{X} = 2 + ACDNRS \le 3N^{60}$ in Paper I, (2.41). The prefactor there satisfies $\mathcal{X}^{\epsilon + 18\epsilon_0} = \mathcal{X}^{19 \cdot 10^{-12}} \le N^{1.2 \cdot 10^{-9}}$ for large N. Write $\delta_X = 1.2 \cdot 10^{-9}$ for this fixed loss. This loss and the factors $N^\eta = N^{10^{-9}}$ are far below three power savings obtained below. These are $N^{-.00829}$, $N^{-.1317}$ and $N^{-.2914}$, one for each term of the bound. It leaves the final saving $N^{-.001}$ intact. Powers of p_0 are $N^{o(1)}$.

Put $B = H$ and $C_{\mathrm{aux}} = HS$. Then $Y \ll Z_1$, since $Y/Z_1 \ll N^\eta H^2 S/N \le N^{-.008}$ (see (A.47)). On a nonempty f_0-block the near-diagonal removal gives $F \ge cHL^{-C_M}/K$, hence $NY \ge cH^2 S^3 N^\eta L^{-C_M} \ge C_{\mathrm{aux}} B Z_1$. All exponents of B, C_{aux} in Paper I, (2.42), are nonnegative. By (2.42) and (A.43),

$$J_*^2 \le N^{o(1)} \big[N^\theta (C_{\mathrm{aux}} B Z_1 + NY) + C_{\mathrm{aux}} B Z_1 (C_{\mathrm{aux}}/Z_1)^{2\theta} + B^2 Y\big] Z_1 F (H_* S)^2;$$

the regular spectral product is $C_{\mathrm{aux}}(C_{\mathrm{aux}} B\sqrt{Z_1})^{2\theta} Z_1^{1-\theta} (BZ_1)^{1-2\theta} = C_{\mathrm{aux}}^{1+2\theta} B Z_1^{2-2\theta}$. With $B^2 < N$, $F \ll H/K$ and (A.32),

$$\frac{J_*^2}{H^4 S^6} \le N^{3\eta + o(1)} [L_A + L_R], \qquad L_A = \frac{N^\theta HKS^5}{N^2}, \qquad L_R = \Big(\frac{H}{S}\Big)^{2\theta} \frac{HKS^5}{N^2}. \tag{A.45}$$

Fix a real t. Apply the diagonal bound (A.37), Paper I, Theorem 2.8 with the uniform coefficient norm (A.43), and (A.45). The resulting estimate is the same for every real t. Thus their supremum W_4^* satisfies

$$|\mathcal{E}_d| \ll N^{\eta + o(1)} \frac{H}{KS^{3/2}} (W_4^*)^{1/2}, \qquad \frac{W_4^*}{H^2 S^3} \ll N^{\eta + o(1)} \frac{H^2 S}{N} + N^{3\eta/2 + \delta_X + o(1)} (L_A + L_R)^{1/2}. \tag{A.46}$$

All Mellin and colouring costs remain $N^{\eta + o(1)}$ as recorded in (A.36). No outer-dependent coefficient enters the norm.

Step 7: exponent reserves. With $S \ll V_1/D$, $K = U_1 D$ and the three scale conditions, $H^2 S \le H^2 V_1$, $KS^5 \le U_1 V_1^5$ and $H^{2\theta} K S^{5-2\theta} \le H^{2\theta} U_1 V_1^{5-2\theta}$. The smooth part D only helps. Using $U_1 \le HN^{-\sigma_{\mathrm{p}}}$, $V_1 \le N^{v+w+\sigma_{\mathrm{p}}}/H$ and $U_1 V_1 \le N^v$,

$$\begin{aligned} 1 - v - h - \sigma_{\mathrm{p}} - w &\ge \tfrac{8299}{1000000}, \\ 2 - \theta - 5v + 3h - 4(\sigma_{\mathrm{p}} + w) &\ge \tfrac{65873}{500000}, \\ 2 - (5 - 2\theta)v + (3 - 4\theta)h - (4 - 2\theta)(\sigma_{\mathrm{p}} + w) &\ge \tfrac{13989529}{48000000}, \end{aligned} \tag{A.47}$$

on $1/3 \le h \le 23/50$. The first minimum is at $h = .46$, the other two at $h = 1/3$. These are the savings of $H^2 S/N$, L_A and L_R. Hence $H^2 S/N \le N^{-.00829}$, $L_A \le N^{-.1317}$, $L_R \le N^{-.2914}$, and (A.46) gives

$$|\mathcal{E}_d| \le C \frac{H^2}{K} N^{-.001}. \tag{A.48}$$

Summing over the $O(D)$ values of d, $|\mathcal{E}| \le CDH^2 N^{-.001}/K$. Its contribution to $|\mathcal{S}_0|$ through (A.20) is

$$\big[KM \cdot M \cdot DH^2 N^{-.001}/K\big]^{1/2} \le CND^{1/2} N^{-.0005} \le N^{1-.0004},$$

since $D \le y = N^{o(1)}$.

Step 8: parameter order and conclusion of (A.16). Fix A. Choose $P \ge 2A+20$; then C_Σ; then $C_M \ge \max(P+4C_\Sigma+1,\, P+C_\varphi)$. Here C_φ is the absolute exponent with $q/\varphi(q) \le L^{C_\varphi}$. The smooth weights, η, ϵ and ϵ_0 are fixed numbers. Choose the threshold for N last. By (A.31) and Step 7,

$$|\mathcal{S}_0| \ll NL^{-A-3} + N^{1-\sigma_{\mathrm{p}}/8}L^{C},$$

since $(K/H)^{1/2} \le N^{-\sigma_{\mathrm{p}}/4}$ and $z^{-1/2}$ beats every power of L. Sum over the $O(L)$ shells of D and the $O(1)$ shells of S, and add Step 2. This proves (A.16), uniformly in f, g, a, b and the scales. Of f and g we used only one-boundedness and the unary support $(rs, N) = 1$.

Step 9: the product interval. Let $I \subset [N/8, 2N]$ and $C_{\mathrm{grid}} \ge 1$. Divide $(H, 2H]$ and $(M, 2M]$ into $O(L^{C_{\mathrm{grid}}})$ consecutive intervals of relative length at most $L^{-C_{\mathrm{grid}}}$. Call a rectangle $J_1 \times J_2$ interior if $J_1J_2 \subset I$, exterior if $J_1J_2 \cap I = \emptyset$, and boundary otherwise. On each interior rectangle apply (A.16) to $a_n\mathbf{1}_{J_1}(n)$ and $b_m\mathbf{1}_{J_2}(m)$, with saving $A + 2C_{\mathrm{grid}}$. By Step 0 the first array satisfies (6.34) at the original parent H. For fixed n, the m in boundary rectangles lie in $O(1)$ intervals of total length $O(ML^{-C_{\mathrm{grid}}})$. A modulus $q \le 4N^v$ has at most $\tau(q)$ factorizations $q = rs$. Counting m for each n,

$$\sum_{q\le 4N^v} \tau(q) \sum_{n\sim H} \Big[\frac{ML^{-C_{\mathrm{grid}}}}{q} + 1 + \frac{ML^{-C_{\mathrm{grid}}}}{\varphi(q)}\Big] \ll NL^{C_1-C_{\mathrm{grid}}} + HN^vL,$$

with a fixed C_1. The last term saves a fixed power, since $h + v \le .9827 + o(1)$. Taking $C_{\mathrm{grid}} = A + C_1$ proves (6.45). □

A.4. **Lemma 6.9.** Recall (6.47):

$$H_{\mathrm{c}}M_{\mathrm{c}} \asymp N, \qquad N^{.50} \ll H_{\mathrm{c}} \ll N^{.54}, \qquad Q_1 \ll N^{.4545}, \qquad Q_2 \ll N^{.0682}. \tag{6.47}$$

Lemma 6.9 (Central rectangular distribution)**.** *Let x be even and $H_{\mathrm{c}}M_{\mathrm{c}} \asymp x$. Let the bounds in* (6.47) *hold with N replaced by x, with fixed comparison constants. Let a_n, b_m be one-bounded on the complete dyadic rectangle $n \sim H_{\mathrm{c}}, m \sim M_{\mathrm{c}}$. Let the tested a_n satisfy* (6.34) *at its original parent, with one fixed divisor exponent and uniform constants at every requested precision. Write*

$$\mathcal{D}(q,a) = \sum_{n\sim H_{\mathrm{c}}, m\sim M_{\mathrm{c}}} a_nb_m \left(1_{nm\equiv a\ (q)} - \frac{1_{(nm,q)=1}}{\varphi(q)}\right).$$

Then, for every fixed $A > 0$,

$$\sum_{r\sim Q_1} \max_{(b,r)=1} \sum_{\substack{s\sim Q_2 \\ \mu(rs)^2=1,\ (rs,x)=1}} \max_{\substack{a \bmod rs,\ (a,rs)=1 \\ a\equiv b\ (r)}} |\mathcal{D}(rs,a)| \ll_A x(\log x)^{-A}. \tag{6.48}$$

The constant is uniform over these families, rectangles, squarefree modulus supports and unary interval restrictions of a_n and b_m. All errors are measured at the original parents. One outer residue b is chosen before s varies. In the application it is $N \bmod r$.

Write $H = H_{\mathrm{c}}$, $M = M_{\mathrm{c}}$, $L = \log x$, $\ell = \log\log x$, $z = x^{1/\ell^3}$, $y = x^{1/\ell}$. Let $c_0 \ge 0$ be the fixed divisor exponent in the hypothesis, so that for every $D > 0$, uniformly in intervals $I \subset (H, 2H]$, $k, h \ge 1$ and $(u, k) = 1$,

$$|\Delta_a(I; k, u; h)| \le C_D H\tau(h)^{c_0}L^{-D}. \tag{A.49}$$

Since $\Delta_{a\mathbf{1}_J}(I; k, u; h) = \Delta_a(I \cap J; k, u; h)$, a unary restriction of a satisfies (A.49) with the same constants. A unary restriction of b keeps it one-bounded. We may therefore assume that a and b are the restricted arrays. We keep $(rs, x) = 1$ in (6.48) and throughout the signed residue-difference argument. The individual maxima in (6.48) are nonnegative, but the signed differences in Step 2 are not. Enlarging their support does not preserve order. Every selected coefficient below is extended by zero outside $\mu(rs)^2 = 1$ and $(rs, x) = 1$. On this support $(r, s) = 1$ and $\varphi(rs) = \varphi(r)\varphi(s)$. Throughout, $rs \le 4Q_1Q_2 \le Cx^v$ with $v = .5227$, and $M \gg x^{.46}$.

Step 1: the projected comparator. For $t = rs$ and reduced $a \equiv b \pmod r$ put $S_t(a) = \sum_{n,m} a_nb_m\mathbf{1}_{nm\equiv a\ (t)}$ and

$$\Delta(a, b; r, s) = S_t(a) - \frac{\varphi(r)}{\varphi(t)} \sum_{\substack{(nm,t)=1 \\ nm\equiv b\ (r)}} a_nb_m,$$

$$\mathcal{F}(b; r, s) = \frac{\varphi(r)}{\varphi(t)} \sum_{(nm,t)=1} a_nb_m\Big[\mathbf{1}_{nm\equiv b\ (r)} - \frac{1}{\varphi(r)}\Big].$$

Then exactly

$$\mathcal{D}(t, a) = \Delta(a, b; r, s) + \mathcal{F}(b; r, s). \tag{A.50}$$

With $a_n^{(s)} = a_n \mathbf{1}_{(n,s)=1}$,

$$\mathcal{F}(b;r,s) = \frac{1}{\varphi(s)} \sum_{\substack{m\sim M\\(m,rs)=1}} b_m \Delta_{a^{(s)}}(r, b\overline{m}). \tag{A.51}$$

Mean square for $a^{(s)}$. The identity $\Delta_{a^{(s)}}(I;k,u;h) = \Delta_a(I;k,u;sh)$ and (A.2) give

$$|\Delta_{a^{(s)}}(I;k,u;h)| \le C_D H \tau(s)^{c_0}\tau(h)^{c_0} L^{-D}.$$

Put $h_s = a^{(s)}/\tau(s)^{c_0}$, which is one-bounded, and $e_n = \mathbf{1}_{H<n\le 2H}$. By the parallelogram identity $\|h_s+e\|^2+\|h_s-e\|^2 \ge 2\|e\|^2$, there is $\varepsilon_s \in \{\pm 1\}$ with

$$g_s = h_s + \varepsilon_s e, \qquad \|g_s\|_2^2 \ge \|e\|_2^2 \ge H/2, \qquad \|g_s\|_2^2 \le 4\|e\|_2^2 \le 8H.$$

Let $Z = \lceil 2H\rceil$ and $c_1 = \max(c_0, 1)$. By (A.49) for h_s, by (A.6) for e, and since $H \le 2^{1/2}\|g_s\|_2 Z^{1/2}$ and $\|e\|_2 Z^{1/2} \ge H/2$, both g_s and e satisfy (6.35) with $B_{\mathrm{div}} = c_1$. The constants are independent of s, the sign and the restriction. Here $\log(2Z) \asymp L$, as $H \gg x^{.50}$. Lemma 6.5(i) applies to both. Since $\Delta_{a^{(s)}} = \tau(s)^{c_0}(\Delta_{g_s} - \varepsilon_s\Delta_e)$ and $|u-w|^2 \le 2|u|^2+2|w|^2$, for every fixed $S_0 > 0$,

$$\sum_{k\le 2Q_1} V(k; a^{(s)}) \ll_{S_0} \tau(s)^{2c_0} H^2 L^{-S_0}. \tag{A.52}$$

The range condition $2Q_1 \le Z(\log Z)^{-B_1(S_0)}$ holds for large x, because $Q_1/H \ll x^{.4545-.50}$.

Bound for $\mathcal{F}$. Each reduced class modulo r contains at most $M/r+1$ values $m \sim M$. The map $m \mapsto b\overline{m}$ permutes the reduced classes. Cauchy's inequality in (A.51) gives, uniformly in b,

$$\Big|\sum_m b_m \Delta_{a^{(s)}}(r, b\overline{m})\Big|^2 \le 2M\Big(\frac{M}{r}+1\Big) V(r; a^{(s)}).$$

Cauchy's inequality over $r \sim Q_1$ and (A.52) then give

$$\sum_{r\sim Q_1} \max_b \Big|\sum_m b_m \Delta_{a^{(s)}}(r, b\overline{m})\Big| \ll \big[Q_1 M(M/Q_1+1)\tau(s)^{2c_0} H^2 L^{-S_0}\big]^{1/2} \ll \tau(s)^{c_0} HML^{-S_0/2},$$

since $Q_1 \le M$ by a fixed power. Summing over s with (A.2) and (A.1),

$$\sum_{r\sim Q_1} \max_{(b,r)=1} \sum_{s\sim Q_2} |\mathcal{F}(b;r,s)| \ll xL^{-S_0/2} \sum_{s\le 2Q_2} \frac{\tau(s)^{c_0}}{\varphi(s)} \ll_A xL^{-A}, \tag{A.53}$$

choosing S_0 after A (the full order is fixed in Step 8).

Step 2: residue differences. For coprime r, s there are exactly $\varphi(s)$ reduced lifts $a' \bmod t$ of $b \bmod r$. Every nm coprime to t with $nm \equiv b\ (r)$ lies in exactly one of them. Hence

$$\Delta(a,b;r,s) = \frac{1}{\varphi(s)} \sum_{\substack{a' \bmod t,\ (a',t)=1\\ a'\equiv b\ (r)}} \big[S_t(a) - S_t(a')\big]. \tag{A.54}$$

For each r, choose one reduced class $b_r \pmod r$ maximizing the masked sum of the projected differences, before s varies. For each s with $\mu(rs)^2 = 1$ and $(rs,x) = 1$, choose a reduced lift $a_{r,s}$ maximizing $|\Delta(a, b_r; r, s)|$. By (A.54) and the triangle inequality, some reduced lift $a'_{r,s} \equiv b_r \pmod r$ satisfies

$$|S_{rs}(a_{r,s}) - S_{rs}(a'_{r,s})| \ge |\Delta(a_{r,s}, b_r; r, s)|.$$

Choose $|\gamma_{r,s}| = 1$ to make this difference nonnegative real, and put $\gamma_{r,s} = 0$ outside the masked squarefree support. By (A.50), (A.53) and (A.54), it suffices to prove, uniformly for all these residues and phases,

$$\mathcal{Z}_0^{(x)} = \sum_{r\sim Q_1} \sum_{\substack{s\sim Q_2\\ \mu(rs)^2=1\\ (rs,x)=1}} \gamma_{r,s}\big[S_{rs}(a_{r,s}) - S_{rs}(a'_{r,s})\big] \ll_A xL^{-A}. \tag{A.55}$$

This signed sum keeps the original mask.

Step 3: large smooth parts and large divisor functions. We quote the completed-hyperbola lemma of Paper I:

Let $c \bmod q$ be a unit. For integer intervals J_1, J_2, put

$$\mathcal{D} = \#\{(u,v) \in J_1 \times J_2 : uv \equiv c(q)\} - \frac{\#J_1^* \#J_2^*}{\varphi(q)}. \tag{3.107}$$

The Weil–Estermann bound [11, 12]

$$|S(h,k;q)| \le \tau(q) q^{1/2} (h,k,q)^{1/2}. \tag{3.108}$$

implies, uniformly in interval lengths and positions,

$$|\mathcal{D}| \ll \tau(q) q^{1/2} \log^2(2q) + \tau(q)^2 \log^2(2q)/\varphi(q). \tag{3.109}$$

For (A.56), set the lemma's $q = t = rs$, $J_1 = (H, 2H] \cap \mathbb{Z}$, $J_2 = (M, 2M] \cap \mathbb{Z}$, and take $c = a$ or $c = a'$. Both are units modulo t by the reduced-lift condition. Here $J_i^* = \{u \in J_i : (u,t) = 1\}$. The source's $\mathcal{D}$ is an unweighted counting discrepancy. It differs from our weighted $\mathcal{D}(t,a)$. Define

$$C_t(c) = \#\{(n,m) \in J_1 \times J_2 : nm \equiv c \pmod t\}.$$

The quoted lemma gives $C_t(c) \le \#J_1^* \#J_2^*/\varphi(t) + O(E_t)$, where $E_t = \tau(t) t^{1/2} \log^2(2t) + \tau(t)^2 \log^2(2t)/\varphi(t)$. Since $|a_n|, |b_m| \le 1$, one has $|S_t(c)| \le C_t(c)$. This is the only passage from unweighted counts to the arbitrary complex arrays. Every pair counted has $(nm, t) = 1$ automatically because c is a unit. No coprimality hypothesis on the arrays is inferred. Consequently,

$$|S_t(a) - S_t(a')| \le C_t(a) + C_t(a') \le \frac{2\#J_1^* \#J_2^*}{\varphi(t)} + O(E_t) \le \frac{8Cx}{\varphi(t)} + O(E_t), \tag{A.56}$$

using $HM \le Cx$. The mask $(t,x) = 1$ remains on the summation support but is not a hypothesis of Lemma 3.4. In the smooth-part and divisor-function eliminations below, we first take absolute values termwise. Only then are the nonnegative majorants enlarged. Write $s = k\varrho$ with $P^+(k) \le z < P^-(\varrho)$. If $k \ge y$, then the z-smooth part of t is at least y. A given t has at most $\tau(t)$ factorizations $t = rs$. By (A.56), (A.2), (A.1) and (A.7), the sum of the absolute values of these masked terms in $\mathcal{Z}_0^{(x)}$ is at most

$$\ll x \sum_{h' \le Cx^v} \frac{\tau(h')^2}{h'} \sum_{\substack{d' \ge y \\ P^+(d') \le z}} \frac{\tau(d')^2}{d'} + \sum_{t \le Cx^v} \tau(t) E_t \ll_A xL^{-A} + x^{3v/2} L^K,$$

and $3v/2 = .78405 < 1$. In the same way the pairs with $\tau(rs) > L^{D_\tau}$ contribute

$$\ll xL^{-D_\tau} \sum_{t \le Cx^v} \frac{\tau(t)^3}{t} + x^{3v/2} L^K \ll xL^{K' - D_\tau} + x^{3v/2} L^K.$$

Choose $D_\tau = D_\tau(A) \ge K' + A$ after the requested precision A. From now on $k < y$, and every remaining modulus and every divisor of it has $\tau \le L^{D_\tau}$.

Step 4: the first Cauchy square. Rename the outer modulus r as $q \sim Q := Q_1$ and write the inner one as $s = kr$, with $k \sim K \le y$ smooth and $r \sim R$ squarefree and z-rough. Then $KR \asymp Q_2$. On the support of (A.55), q, k, r are pairwise coprime and squarefree, $(qkr, x) = 1$, and $QKR \ll x^v$. There are $O(L^2)$ choices of (K, R). Extend the selected coefficients by zero and put

$$\Gamma_{qkr} = \gamma_{q,kr} \mathbf{1}_{\mu(qkr)^2 = 1} \mathbf{1}_{(qkr,x)=1}, \qquad |\Gamma_{qkr}| \le 1.$$

The Γ_{qkr} are identical in both choices a, a' of residue. For one shell the *masked signed sum* is

$$\mathcal{Z}^{(x)} = \sum_{\substack{q \sim Q,\ k \sim K \\ (qk,x)=1}} \sum_m b_m \sum_{\substack{r \sim R,\ n \\ (r,x)=1}} \Gamma_{qkr} a_n \big[\mathbf{1}_{mn \equiv a_{qkr} \ (qkr)} - \mathbf{1}_{mn \equiv a'_{qkr} \ (qkr)} \big],$$

where $a_{qkr} \equiv a'_{qkr} \equiv b_q \pmod q$ and b_q does not depend on k, r. Put

$$\mathcal{F}_{qk}^{(x)}(m) = \sum_{\substack{r \sim R,\ n \\ (r,x)=1}} \Gamma_{qkr} a_n \big[\mathbf{1}_{mn \equiv a_{qkr} \ (qkr)} - \mathbf{1}_{mn \equiv a'_{qkr} \ (qkr)} \big].$$

With ψ as in Step 3 of the proof of Lemma 6.8,

$$|\mathcal{Z}^{(x)}|^2 \ll QKM \sum_{\substack{q \sim Q,\ k \sim K \\ (qk,x)=1}} \sum_m \psi(m/M) |\mathcal{F}_{qk}^{(x)}(m)|^2. \tag{A.57}$$

For $A_i = A_i(q,k,r_i)$ chosen from a_{qkr_i}, a'_{qkr_i}, the four correlations, with signs $+,-,-,+$, are precisely

$$\mathcal{C}_x(A_1,A_2) = \sum_{\substack{q\sim Q,\ k\sim K\\ (qk,x)=1}} \sum_m \psi(m/M) \sum_{\substack{r_1,r_2\sim R\\ (r_1r_2,x)=1}} \Gamma_{qkr_1}\overline{\Gamma_{qkr_2}} \times \sum_{n_1,n_2} a_{n_1}\overline{a_{n_2}}\mathbf{1}_{mn_1\equiv A_1\ (qkr_1)}\mathbf{1}_{mn_2\equiv A_2\ (qkr_2)}.$$

We prove at arbitrary fixed precision $P>0$ that

$$\mathcal{C}_x(A_1,A_2) = MT_x + O_P\Big(\frac{MH^2}{QKL^P}\Big), \tag{A.58}$$

where MT_x contains the same mask and selected coefficients. It is independent of the choices of A_1, A_2. Since the coefficients and support are the same in all four correlations, their main terms cancel. Equation (A.57) then gives $|\mathcal{Z}^{(x)}| \ll xL^{-P/2}$ before the $O(L^2)$ shell sum.

Write $g=(r_1,r_2)$, $r_1=gu$, $r_2=gv$, $(u,v)=1$, with $g\sim G$ dyadic. Put $d_0=qkg$ and $U=R/G$. Then q,k,g,u,v are pairwise coprime. The product of the two Γ-supports imposes $(qkguv,x)=1$. This modulus-only restriction remains in the correlation and its zero mode. It may be omitted only from a termwise absolute-value majorant. It may not be omitted from a signed expression or from an inner sum before squaring. The two congruences $mn_1\equiv A_1\ (d_0u)$, $mn_2\equiv A_2\ (d_0v)$ are soluble exactly when

$$n_2 \equiv A_2\overline{A_1}\,n_1 \pmod{d_0}, \qquad (n_1,d_0u)=(n_2,d_0v)=1, \tag{A.59}$$

and then m lies in one reduced class modulo $l=d_0uv$. Modulo q both residues equal b_q, so $n_2\equiv n_1 \pmod q$.

Step 5: large common factor. Let $G\ge H/(QKL^{C_g})$, with C_g fixed later. The target is $\mathcal{B}=MH^2/(QK)\asymp xH/(QK)$. In the large-$G$ estimate we first take absolute values of each weighted correlation term. Hence its nonnegative counting majorant may be enlarged to all (q,k,g,u,v), while $\mathcal{C}_x$ itself stays masked. For fixed (q,k,g,u,v) the pairs (n_1,n_2) in (A.59) lie in a congruence lattice of determinant d_0. Let $\mathbf{w}$ be a shortest nonzero vector. If the line $\mathbb{R}\mathbf{w}$ meets the positive rectangle, choose its sign so its coordinates are positive; otherwise the $\beta=0$ contribution is empty. Complete it to a reduced basis $(\mathbf{w},\mathbf{w}')$. The component of $\mathbf{w}'$ orthogonal to $\mathbf{w}$ has length $h_\perp = d_0/|\mathbf{w}| \ge \sqrt{3}|\mathbf{w}|/2$. In a disc of radius CH the points $\alpha\mathbf{w}+\beta\mathbf{w}'$ with $\beta\neq 0$ have $|\beta|\ll H/h_\perp$ and, for fixed β, $O(H/|\mathbf{w}|+1)$ values of α. Such points exist only if $h_\perp \ll H$. Then $|\mathbf{w}|\ll h_\perp \ll H$ and $(H/h_\perp)^2 \ll H^2/d_0$, so they number $O(H^2/d_0)$. Each such pair has $O(M/l+1)$ values of m. Hence these points contribute

$$\ll QKGU^2\cdot\frac{H^2}{QKG}\Big(1+\frac{M}{QKGU^2}\Big) = H^2U^2 + \frac{MH^2}{QKG}.$$

For $\beta=0$, $(n_1,n_2)=\alpha\mathbf{w}$ with $\alpha\ge 1$, and the two congruences fix $\alpha m\le Cx$ in one unit class $c \pmod l$. Here $l=d_0uv=qkguv\ge 1$, and c is a unit. Each original congruence has a unit right-hand side A_i. Hence m and α are units modulo every factor of l on this line. For $X\le Cx$, cover the positive integer hyperbola $\alpha m\le X$ by the $O(L)$ integer rectangles $J_{1,0}=\{1\}$, $J_{2,0}=[1,\lfloor X\rfloor]$, and, for $j\ge 1$ with $2^{j-1}<X$,

$$J_{1,j}=(2^{j-1},2^j]\cap\mathbb{Z}, \qquad J_{2,j}=[1,\lfloor X/2^{j-1}\rfloor]\cap\mathbb{Z}.$$

These rectangles may overcount. This is harmless for the positive quantity $\sum_{e\le X, e\equiv c\ (l)}\tau(e)$. Each has $\#J_{1,j}\#J_{2,j}\ll X$, and their number is $O(L)$. Apply the quoted Paper I, Lemma 3.4 with its modulus $q=l$, unit c, and these two integer intervals. Empty intervals contribute zero. The positive counts give, uniformly in c and X,

$$\sum_{\substack{e\le X\\ e\equiv c\ (l)}}\tau(e) \ll \frac{XL}{\varphi(l)} + \Big[\tau(l)l^{1/2}+\frac{\tau(l)^2}{\varphi(l)}\Big]L^3 \tag{A.60}$$

uniformly in the unit c and $X\le Cx$. A modulus l has at most $\tau_5(l)\le\tau(l)^4$ representations $l=qkguv$. Summing (A.60) with (A.1),

$$\ll xL^C + (QKGU^2)^{3/2}L^C.$$

With $W=QK\le x^{.4545+o(1)}$, $R\ll x^{.0682}$ and $WR\ll x^{.5227}$, on this range of G,

$$\frac{MH^2/(QKG)}{\mathcal{B}} = \frac{1}{G} \le x^{-.0455+o(1)}, \quad \frac{H^2U^2}{\mathcal{B}} \ll \frac{W^3R^2L^{2C_g}}{xH} \le x^{-.0001+o(1)}, \quad \frac{x}{\mathcal{B}} \le x^{-.0455+o(1)},$$

and $(QKGU^2)^{3/2}\le (W^2R^2L^{C_g}/H)^{3/2}\le x^{.8181+o(1)}$, while $\mathcal{B}\ge x^{1.0455-o(1)}$. So the large-$G$ part of $\mathcal{C}_x$ is $O(\mathcal{B}L^{-P})$ at any requested fixed precision P. The mask was removed only after taking absolute values.

Step 6: small common factor, Poisson and the zero mode. Let $G < H/(QKL^{C_g})$. For one class $c \bmod l$,

$$\sum_{m\equiv c\ (l)} \psi\Big(\frac{m}{M}\Big) = \frac{M}{l}\sum_{h\in\mathbb{Z}} \hat\psi\Big(\frac{hM}{l}\Big) e\Big(\frac{hc}{l}\Big), \tag{A.61}$$

and the terms $|h| > x^\eta l/M$ contribute $O(x^{-100})$ per tuple. We take $\eta = 10^{-8}$.

The zero mode is $M\hat\psi(0)/l$ times the weighted number of compatible (n_1, n_2). It still carries the product of the two Γ-supports, and thus $(qkguv, x) = 1$. After taking absolute values of the *error*, its positive majorant can be enlarged. This is the only use of the unmasked range in (A.62). Put $f_v = a\mathbf{1}_{(n,v)=1}$. For fixed n_1, the n_2-sum in (A.59) is a sum of $\overline{f_v}$ over one reduced class $c(n_1)$ modulo d_0. Replace it by $\varphi(d_0)^{-1}\sum_{(n_2,d_0v)=1}\overline{a_{n_2}}$. The error is $\overline{\Delta_{f_v}(d_0, c(n_1))}$. The map $n_1 \mapsto c(n_1)$ hits each reduced class $O(H/d_0)$ times, so Cauchy's inequality gives

$$\sum_{n_1} |\Delta_{f_v}(d_0, c(n_1))| \ll Hd_0^{-1/2} V(d_0; f_v)^{1/2}.$$

A given d_0 has at most $\tau_3(d_0) \le L^{2D_\tau}$ representations, and $d_0 \asymp D_0 = QKG$. Multiplying by $M/(d_0uv)$ and summing over u, v, the zero-mode error is

$$\ll \frac{MH}{D_0} L^{2D_\tau} \sup_v \Big[\sum_{d_0\le C'D_0} V(d_0; f_v)\Big]^{1/2}. \tag{A.62}$$

Since v divides a remaining modulus, $\tau(v) \le L^{D_\tau}$. The identity $\Delta_{f_v}(I; k, u; h) = \Delta_a(I; k, u; vh)$ and (A.49) show that f_v satisfies (A.49) at precision $D - c_0D_\tau$. As in Step 1, add $\varepsilon_v e$ to make the norm at least $(H/2)^{1/2}$. Both sequences satisfy (6.35) with $B_{\mathrm{div}} = c_1$ and uniform constants. Lemma 6.5(i) and linearity give, uniformly in v,

$$\sum_{d_0\le C'D_0} V(d_0; f_v) \ll_{S_0} H^2L^{-S_0},$$

because $C'D_0 \le C'H/L^{C_g} \le Z(\log Z)^{-B_1(S_0)}$ once $C_g > B_1(S_0) + 1$. With $D_\tau = D_\tau(A)$ fixed in Step 3, choose first $S_0 > 4D_\tau + 2A$, then $C_g > B_1(S_0) + 1$, then the threshold for x. Since $D_0 \ge cQK$, (A.62) is $O(\mathcal{B}L^{2D_\tau - S_0/2}) = O(\mathcal{B}L^{-A})$.

What remains of the zero mode is the *masked* main term

$$MT_{x,G} = M\hat\psi(0) \sum_{\substack{q,k,g,u,v\\(qkguv,x)=1}} \Gamma_{qkgu}\overline{\Gamma_{qkgv}} \frac{\big[\sum_{(n_1,d_0u)=1} a_{n_1}\big]\big[\sum_{(n_2,d_0v)=1}\overline{a_{n_2}}\big]}{d_0uv\varphi(d_0)}, \tag{A.63}$$

with the previously imposed coprime/squarefree and dyadic conditions understood. This expression depends on the modulus support and Γ, but not on A_1, A_2. Put $MT_x = \sum_{G\ \mathrm{small}} MT_{x,G}$.

Step 7: the second square. It remains to bound the frequencies $h \ne 0$ in the masked correlation. Write

$$\frac{M}{l} = \frac{M}{QKGU^2}\cdot\frac{QKGU^2}{qkguv};$$

the second factor is a product of bounded weights in (q, k, g), in u and in v. The mask factors exactly as

$$\mathbf{1}_{(qkguv,x)=1} = \mathbf{1}_{(qkg,x)=1}\mathbf{1}_{(u,x)=1}\mathbf{1}_{(v,x)=1}.$$

Keep the first factor with the outer (q, k, g, n_1, n_2) data and the last two with the (u, v, h)-sum, together with $\Gamma_{qkgu}\overline{\Gamma_{qkgv}}$. No masked term is reintroduced into a signed sum.

Every z-rough $u, v \le x^2$ has $O(\ell^3)$ prime factors. Colour each prime below x^2 red or blue independently. A coprime pair (u, v) is covered if all primes of u are red and all of v blue. This has probability at least $2^{-O(\ell^3)}$. With $J_0 = \lceil 2^{O(\ell^3)}(10\log x + 1)\rceil$ independent colourings, the probability that one of the at most x^4 pairs is never covered is < 1. Fix such a list and assign each pair to its first covering colouring. This partitions the coprime pairs into $J \le J_0 = x^{o(1)}$ classes, and $(uu', vv') = 1$ for any two pairs in one class.

On one class let $\mathcal{Y}$ be the sum over compatible (n_1, n_2), (u, v) and $0 < |h| \le \mathsf{F}$, after removing $M/(QKGU^2)$, where

$$\mathsf{F} = Cx^\eta QKGU^2/M. \tag{A.64}$$

If $\mathsf{F} < 1$ there is nothing to bound. By CRT the class of m in the phase is

$$\frac{c}{l} \equiv \frac{A_1\overline{n_1}\,\overline{v}}{d_0u} + \frac{A_2\overline{n_2}\,\overline{d_0u}}{v} \pmod 1.$$

Cauchy's inequality over the *masked* outer data (q, k, g, n_1, n_2), $(qkg, x) = 1$, with $n_2 \equiv n_1\ (q)$ removes a and gives

$$|\mathcal{Y}|^2 \ll H^2KG\,\mathcal{D}_3, \tag{A.65}$$

since there are at most $QKG\cdot O(H(H/Q+1)) \ll H^2KG$ outer tuples ($Q<H$). Here $\mathcal{Y}$ and $\mathcal{D}_3$ keep the x-mask. The sum $\mathcal{D}_3$ runs over $(qkg,x)=1$ and $n_2\equiv n_1\ (q)$. Its terms are a smooth majorant times the square of the (u,v,h)-sum. This inner sum contains $\mathbf{1}_{(u,x)=1}\mathbf{1}_{(v,x)=1}$ and the selected Γ coefficients. The congruence modulo kg stays inside the square. In the expansion of this square each pair of copies keeps its zero–one masks. The later diagonal, Weil and Ramanujan bounds are termwise bounds after taking absolute values of the expanded contributions. Hence they may discard zero–one factors. We do not use the false inequality $|\sum_{(uv,x)=1} z_{uv}|^2 \le |\sum_{u,v} z_{uv}|^2$.

Expanding gives (u,v,h) and (u',v',h') with $0<|h|,|h'|\le \mathsf{F}$. Put $A=A_1(u)$, $A'=A_1(u')$, $B=A_2(v)$, $B'=A_2(v')$ and

$$\Delta = hu'v'-h'uv.$$

The two congruences modulo kg must agree, or the term vanishes. Let $t\equiv B\overline{A}\pmod{d_0}$. The (n_1,n_2)-sum is

$$\mathcal{K}=\sum_{\substack{n_2\equiv tn_1\ (d_0)\\ (n_1,d_0uu')=1,\ (n_2,vv')=1}} \psi\Big(\frac{n_1}{H}\Big)\psi\Big(\frac{n_2}{H}\Big)e\Big(\frac{c_1\overline{n_1}}{d_0uu'}\Big)e\Big(\frac{c_2\overline{n_2}}{vv'}\Big),$$

$$c_1\equiv (Ahu'v'-A'h'uv)\overline{vv'}\pmod{d_0uu'},\qquad c_2\equiv (Bhu'v'-B'h'uv)\overline{d_0uu'}\pmod{vv'}.$$

All inverses exist, since $(uu',vv')=1$ and both are prime to d_0. Completing first in n_2 and then in n_1,

$$\mathcal{K}=\frac{H^2}{d_0^2uu'vv'}\sum_{\ell_1,\ell_2\in\mathbb{Z}}\hat{\psi}\Big(\frac{\ell_2H}{d_0vv'}\Big)\hat{\psi}\Big(\frac{\ell_1H}{d_0uu'}\Big)S(c_2,\ell_2\overline{d_0};vv')\,S(c_1,\ell_1+\ell_2c_3;d_0uu'),\tag{A.66}$$

with $c_3\equiv uu't\,\overline{vv'}\pmod{d_0uu'}$ and the Kloosterman sum $S(a,b;m)$ of Paper I, (2.35). The first completion produces the phase $e(\ell_2tn_1\overline{vv'}/d_0)$, which combines with the n_1-phase to give c_3. Let

$$\Lambda = C_1x^{\eta}QKGU^2/H.\tag{A.67}$$

Frequencies with $|\ell_i|>\Lambda$ contribute $O(x^{-100})$ per tuple. If $\Lambda<1$, only $(\ell_1,\ell_2)=(0,0)$ remains.

The target for $\mathcal{D}_3$ is $T_3=H^2U^4G/K$. By (A.65), a bound $\mathcal{D}_3\ll T_3x^{-\rho}$ gives $|\mathcal{Y}|\ll H^2U^2Gx^{-\rho/2}$.

Diagonal $\Delta=0$. Then $uv\mid hu'v'$, so for fixed (h,u',v') there are $x^{o(1)}$ choices of (h',u,v). Bounding $\mathcal{K}$ trivially by $O(H^2/d_0)$, as $d_0<H$,

$$\mathcal{D}_3^{\mathrm{diag}}\ll x^{o(1)}H^2\mathsf{F}U^2,\qquad \frac{\mathcal{D}_3^{\mathrm{diag}}}{T_3}\ll x^{\eta+o(1)}\frac{QK^2H}{x}\le x^{-.0055+\eta+o(1)}.$$

The divisor-counting bound $x^{o(1)}$ for the tuple count immediately above is loose. Only this $x^{o(1)}$ factor is used in the estimate for $\mathcal{D}_3^{\mathrm{diag}}$.

Gcd bound. Put $e=(u,u')$, $f=(v,v')$, $u_0=u/e$, $u_1=u'/e$, $v_0=v/f$, $v_1=v'/f$. Then

$$(c_1,uu')(c_2,vv')\le (h,u_0v_0)(h',u_1v_1)(c_1,e^2)(c_2,f^2).\tag{A.68}$$

The four factors are supported on disjoint sets of primes. If $p\mid u$, $p\nmid u'$, then $c_1\equiv Ahu'v'\overline{vv'}\pmod p$ with a unit multiplier, so $(c_1,p)=(h,p)$. The cases $p\mid u'$ only, $p\mid v$ only and $p\mid v'$ only are identical, with (h',p), (h,p) and (h',p). If $p\mid e$, the numerator of c_1 is $p(Aha'v'-A'h'a''v)$ with $u=pa''$, $u'=pa'$. Thus the p-part of (c_1,e^2) is at most p^2. The case $p\mid f$ is similar. Modulo q all four residues equal b_q, so $c_1\equiv b_q\overline{vv'}\Delta\pmod q$ and $(c_1,q)=(\Delta,q)$. For $p\mid kg$ we use only $(c_1,kg)\le kg$. Hence

$$(c_1,d_0uu')(c_2,vv')\le kg\,(\Delta,q)\,(h,uv)(h',u'v')(u,u')^2(v,v')^2.\tag{A.69}$$

Long dual range, $\Lambda\ge 1$, $\Delta\ne 0$. There are $O(\Lambda^2)$ pairs (ℓ_1,ℓ_2). Paper I, (2.35), gives

$$|S(a,b;m)|\le \tau(m)m^{1/2}(a,m)^{1/2}.$$

Use the square root of (A.69) and $(kg)^{1/2}(QKG)^{1/2}\ll Q^{1/2}KG$. Then

$$\begin{aligned}|\mathcal{K}|&\ll x^{2\eta+o(1)}Q^{1/2}KGU^2\\ &\quad\times (h,uv)^{1/2}(h',u'v')^{1/2}(u,u')(v,v')(\Delta,q)^{1/2}.\end{aligned}$$

The averages

$$\sum_{1\le|h|\le\mathsf{F}}(h,a)\le 2\mathsf{F}\tau(a),\qquad \sum_{q\sim Q}(\Delta,q)\le CQ\tau(|\Delta|),\qquad \sum_{u,u'\sim U}(u,u')\ll U^2\log(2U)$$

follow from $(h,a)=\sum_{d\mid(h,a)}\varphi(d)$ and counting multiples. Summing over the $O(KG)$ choices of (k,g),

$$\mathcal{D}_3^{\mathrm{long}}\ll x^{2\eta+o(1)}Q^{3/2}K^2G^2U^6\mathsf{F}^2,\qquad \frac{\mathcal{D}_3^{\mathrm{long}}}{T_3}\ll x^{4\eta+o(1)}\frac{Q^{7/2}K^5R^6}{G^3x^2}\le x^{-.00005+4\eta+o(1)},$$

using $U = R/G$, $G \geq \frac{1}{2}$, $K \leq y$; the possible absolute factor $G^{-3} \leq 8$ is absorbed in the implied constant and in $x^{o(1)}$, and then

$$2 - \tfrac{7}{2}(.4545) - 6(.0682) = .00005.$$

Short dual range, $\Lambda < 1$, $\Delta \neq 0$. Now

$$\mathcal{K} = \frac{H^2\hat{\psi}(0)^2}{d_0^2 uu'vv'}\, c_{vv'}(c_2)\, c_{d_0uu'}(c_1) + O(x^{-100}),$$

with Ramanujan sums $c_m(a) = S(a, 0; m)$, and $|c_m(a)| \leq (a, m)$. By (A.69) each term is

$$\ll \frac{H^2}{Q^2KGU^4}(\Delta, q)(h, uv)(h', u'v')(u, u')^2(v, v')^2.$$

With $\sum_{u,u' \sim U}(u, u')^2 \ll U^3$ and the averages above,

$$\mathcal{D}_3^{\text{short}} \ll x^{o(1)}\frac{H^2\mathsf{F}^2U^2}{Q}, \qquad \frac{\mathcal{D}_3^{\text{short}}}{T_3} \ll x^{2\eta+o(1)}\frac{H^2QK^3R^2}{Gx^2} \leq x^{-.3291+2\eta+o(1)},$$

since $2(.54) + .4545 + 2(.0682) - 2 = -.3291$.

Step 8: conclusion. After the $x^{o(1)}$ factors the three ratios are at most $x^{-.00005+5\eta}$. These factors include J, the divisor bounds, $K \leq y$ and fixed powers of L. Take $\rho = 10^{-6}$. Then $|\mathcal{Y}| \ll H^2U^2Gx^{-\rho/2}$ on each masked colouring class. Restoring $M/(QKGU^2)$ and the $J = x^{o(1)}$ classes, the nonzero frequencies contribute $\ll \mathcal{B}x^{-\rho/3}$ to each small-G block of $\mathcal{C}_x$. The modulus mask was preserved through both Cauchy squares. Only termwise absolute-value majorants were enlarged. There are $O(L)$ common-factor blocks. Together with Steps 5 and 6 this proves (A.58) with the *same* masked MT_x in each of its four choices of residues. Their signed combination is $MT_x - MT_x - MT_x + MT_x = 0$. Fix a requested final exponent A. First choose the correlation precision $P > 2A+10$. Then choose $D_\tau = D_\tau(P) \geq K' + P$ (Step 3). Then choose $S_0 > 4D_\tau + 2P$, also large enough to make (A.53) smaller than xL^{-A-10}. Next take $C_g > B_1(S_0) + 1$, and finally the threshold for x. Equation (A.57) gives $\mathcal{Z}^{(x)} \ll xL^{-P/2}$ on each shell. Sum over its $O(L^2)$ choices of (K, R) and add the masked Step 1 and Step 3 errors. This proves exactly (6.48), with $(rs, x) = 1$ as stated. Every constant depends only on A, the comparison constants in (6.47) and the constants in (A.49). □

Appendix B. Finite calculations

This appendix gives the finite quantities used above. Each terminating decimal denotes an exact rational number. All decimal interval endpoints are rounded outwards; decimal approximations in explanatory prose are not additional certified bounds. The comparisons with prime sums occur in the main text.

B.1. **Series bounds.** For $|z| < 1$,

$$\left|\operatorname{arctanh} z - \sum_{k=0}^{m-1}\frac{z^{2k+1}}{2k+1}\right| \leq \frac{|z|^{2m+1}}{(2m+1)(1-z^2)}. \tag{B.1}$$

Write any positive rational as $t = 2^a v$, where $2/3 \leq v < 4/3$. Then use

$$\log t = a\log 2 + 2\operatorname{arctanh}\frac{v-1}{v+1}, \qquad \log 2 = 2\operatorname{arctanh}(1/3). \tag{B.2}$$

Also, for $0 \leq z < 1$,

$$0 \leq \operatorname{Li}_2(z) - \sum_{n=1}^{m}\frac{z^n}{n^2} \leq \frac{z^{m+1}}{(m+1)^2(1-z)}. \tag{B.3}$$

These inequalities enclose the elementary functions below. Integer comparisons then give the printed rational bounds.

B.2. **Four profile differences.** Put

$$\alpha = \frac{100}{1149}, \quad \beta = \frac{50}{309}, \quad \mu_* = \frac{25}{128}, \quad \lambda = \frac{93}{400}, \quad h_G = \frac{9}{5000}, \quad K = \frac{57}{64} - \frac{1}{500},$$
$$\delta_h = \frac{h_G}{\alpha}, \qquad C = K - \frac{11}{5}\alpha, \qquad B(x) = \min\{\beta, \max\{\alpha, C - 5x/2\}\}. \tag{B.4}$$

Define the elementary functions

$$\widehat{f}(s) = \frac{\log(s-1)}{s} \qquad (2 \le s \le 4),$$

$$\widehat{F}(s) = \frac{1}{s} \qquad (1 < s \le 3), \tag{B.5}$$

$$s\widehat{F}(s) = 1 + \log(s-2)\log(s-1) - \mathrm{Li}_2\left(\frac{s-2}{s-1}\right) - \frac{\log^2(s-1)}{2} + \mathrm{Li}_2(1/2) + \frac{\log^2 2}{2} \qquad (3 < s < 5), \tag{B.6}$$

The two sieve normalizations satisfy

$$\widehat{F} = F/(2e^\gamma), \qquad \widehat{f} = f/(2e^\gamma).$$

The raw multiplier $2/(e^\gamma\alpha)$, a signed row coefficient $\pm 1/4$, and $F = 2e^\gamma\widehat{F}$ or $f = 2e^\gamma\widehat{f}$ combine to the displayed $1/\alpha$ factor. The directed integrals below use these hatted functions. No further $1/4$ or $2e^\gamma$ factor is applied. Also $\mathrm{Li}_2(t) = \sum_{n\ge1} t^n/n^2$ for $0 \le t < 1$. Write

$$D_F(s) = \widehat{F}(s) - \widehat{F}(s+\delta_h), \qquad D_f(s) = \widehat{f}(s+\delta_h) - \widehat{f}(s).$$

For the three intervals in Table B.1, put

$$\Delta_j = \frac{1}{\alpha}\int_{I_j} D_F\left(\frac{K - 5x/2}{\alpha}\right)\frac{dx}{x} \quad (j = 4, 5, 7), \tag{B.7}$$

$$\Delta_{10} = \frac{1}{\alpha}\int_{\mu_*}^{\lambda}\int_{\alpha}^{B(x)} D_f\left(\frac{K - 5x/2 - y}{\alpha}\right)\frac{dy\,dx}{xy}. \tag{B.8}$$

These Δ_j are those of equation (5.22). Closed or half-open interval endpoints have measure zero. The curved domain is that of [27, (2.38)–(2.40)].

Proposition B.1. *A directed interval computation certifies the following outward enclosure of the four elementary integrals:*

$$\frac{572004}{10^8} < \Delta_4 + \Delta_5 + \Delta_7 + \Delta_{10} < \frac{606486}{10^8}. \tag{B.9}$$

In particular, their sum minus 10^{-20} *exceeds* $5719/10^6$. *Also,*

$$\frac{\log 3}{4\alpha^2}\left(\frac{5}{2}\lambda + \beta\right)\frac{\lambda - \mu_*}{\mu_*} < \frac{21137254516197687}{4120000000000000} < \frac{5137}{1000} < \frac{21}{4} < \frac{179}{25}. \tag{B.10}$$

TABLE B.1. Outward decimal bounds from the directed computation. Every endpoint is an exact terminating rational.

j	I_j	lower bound	upper bound
4	$[25/128, 1/4]$	0.00162068	0.00169539
5	$[1/4, 7109/28000]$	0.00040327	0.00041325
7	$[25/128, 93/400]$	0.00056463	0.00059051
10	curved domain	0.00313145	0.00336572
sum		0.00572003	0.00606487

Proof. The profiles are evaluated with directed interval arithmetic. An independent verifier rechecks the directed output. Put $h_G = 9/5000$, $p(s) = -\widehat{F}'(s)$, and let d be the right-branch derivative profile for $\widehat{f}$, so $d(s) = \widehat{f}'(s)$ for $2 < s < 4$ and $d(2) = 1/2$ by continuous extension. The three one-dimensional rows each use 2048 equal cells:

$$x_k = a_j + \frac{k(b_j - a_j)}{2048} \qquad (0 \le k \le 2048).$$

For a cell $[x_0, x_1] = [x_k, x_{k+1}]$, define

$$s_- = \frac{K - 5x_1/2}{\alpha}, \qquad s_+ = \frac{K - 5x_0/2 + h_G}{\alpha}.$$

Since $D_F(s) = \int_0^{h_G/\alpha} p(s+t)\,dt$, the code's lower and upper cell contributions are exactly

$$L = \frac{h_G}{\alpha^2}\frac{x_1 - x_0}{x_1}p(s_+), \qquad U = \frac{h_G}{\alpha^2}\frac{x_1 - x_0}{x_0}p(s_-). \tag{B.11}$$

These hold because $s_- \le (K - 5x/2)/\alpha + t \le s_+$ for every $x \in [x_0, x_1]$ and $0 \le t \le h_G/\alpha$; also $x_0 \le x \le x_1$.

For the fourth integral, split at

$$x_B = \frac{506848823}{2366940000}.$$

On each piece $[a,b]$ of $[\mu_*, x_B] \cup [x_B, \lambda]$, the S10 routine uses $x_i = a + i(b-a)/96$ and $z_j = j/96$ for $0 \le i, j \le 96$. For a cell $[x_i, x_{i+1}] \times [z_j, z_{j+1}]$, with $0 \le i, j < 96$, put

$$b_0 = B(x_i), \quad b_1 = B(x_{i+1}), \quad y_- = \alpha + (b_1 - \alpha) z_j, \quad y_+ = \alpha + (b_0 - \alpha) z_{j+1},$$

$$q_- = \frac{K - 5x_{i+1}/2 - y_+}{\alpha}, \qquad q_+ = \frac{K - 5x_i/2 - y_- + h_G}{\alpha}.$$

The substitution is $y = \alpha + (B(x) - \alpha) z$. Its executed lower and upper contributions are

$$L = \frac{h_G}{\alpha^2}(x_{i+1} - x_i)(z_{j+1} - z_j)\frac{b_1 - \alpha}{x_{i+1} y_+} d(q_+), \qquad U = \frac{h_G}{\alpha^2}(x_{i+1} - x_i)(z_{j+1} - z_j)\frac{b_0 - \alpha}{x_i y_-} d(q_-). \tag{B.12}$$

For every point in this cell and $0 \le t \le h_G/\alpha$, $q_- \le (K - 5x/2 - y)/\alpha + t \le q_+$, and

$$\frac{b_1 - \alpha}{x_{i+1} y_+} \le \frac{B(x) - \alpha}{xy} \le \frac{b_0 - \alpha}{x_i y_-}.$$

Hence the two positive factors are bounded separately, as in the code. The two rectangular grids contain $2 \cdot 96^2$ cells. The total executed cell count is

$$3 \cdot 2048 + 2 \cdot 96 \cdot 96 = 24576.$$

The endpoint bounds use the following branch formulas. For $2 < s < 4$,

$$d(s) = \frac{s/(s-1) - \log(s-1)}{s^2} > 0, \qquad d'(s) = -\frac{1}{s(s-1)^2} - \frac{2\{s/(s-1) - \log(s-1)\}}{s^3} < 0.$$

They extend continuously to $s = 2$ with $d(2) = 1/2$ and right derivative $d'_+(2) = -1$. Thus the branch profile is positive and decreasing on $[2, 4]$. For positivity: the numerator decreases, and at $s = 4$ it is $4/3 - \log 3 > 0$, since $e^{4/3} > 1 + 4/3 + (4/3)^2/2 + (4/3)^3/6 > 3$. For $s > 3$, put $A(s) = \widehat{F}(s) - \widehat{f}(s-1)$. Then

$$(sA(s))' = -s\,\widehat{f}'(s-1) < 0, \qquad p(s) = A(s)/s, \qquad p'(s) = -2A(s)/s^2 - \widehat{f}'(s-1)/s.$$

At $s = 3$ the two pieces give $p(3) = 1/9$, so p is continuous there; the displayed differential identities are used on $s > 3$. The exact directed evaluation at $s_{\max} = 73930107/16000000$ gives $A(s_{\max}) > 1597/100000 > 0$. Hence $A(s) > 0$ throughout the required range $3 < s \le s_{\max}$, and there $p > 0$ and $p' < 0$. For $s \le 3$, $p(s) = 1/s^2$ is also positive and decreasing. The S10 mixed-corner range is

$$\frac{17359415273}{7910400000} \le q_- \le q_+ \le \frac{57930107}{16000000} < 4, \qquad \frac{17359415273}{7910400000} > \frac{219}{100}.$$

These inequalities justify using the decreasing positive d in (B.12). The lower bound refers only to the executed mixed-corner interval.

The code evaluates the formulas by rational geometry and 256-bit Arb ball arithmetic. In `profile_direct.py`, logarithms use Arb's `log()` operation. Put $w = s - 2$. The dilogarithm uses Arb's `polylog(2)` operation at the negative argument $-w$, together with $\pi^2/12$. These are the code's evaluations of (B.5). By the Landen identity

$$\mathrm{Li}_2(-w) = -\mathrm{Li}_2\left(\frac{w}{1+w}\right) - \frac{1}{2}\log^2(1+w), \qquad \mathrm{Li}_2(1/2) + \frac{1}{2}\log^2 2 = \frac{\pi^2}{12}$$

the Arb expression equals the displayed formula. Arb supplies outward ball enclosures. The code does not use the atanh and dilogarithm tail inequalities above. The exact directed rational endpoints are written to the result record. The terminating decimals in the table are outward projections of each row's directed endpoint. Their displayed column sums are 0.00572003 and 0.00606487. The sharper bound (B.9) adds the unrounded rational endpoints in RESULT.json before decimal projection. Summing before rounding proves the stated integer comparisons. Finally, 64 terms of $\log 3 = 2\sum_{n \ge 0} 2^{-(2n+1)}/(2n+1)$ and its geometric remainder give $\log 3 < 10987/10000$. This proves (B.10). □

Proposition B.1 concerns elementary integrals and one strip coefficient. Its application to the prime sums requires the source comparisons, uniform errors and a single charge of the strip.

B.3. **The shifted three-variable integral.** Here F, f are the linear-sieve functions. Their normalization is

$$\begin{aligned} F(s) &= 2e^{\gamma}/s \quad (1 \le s \le 3), \qquad f(s) = 0 \quad (s \le 2), \\ (sF(s))' &= f(s-1) \quad (s > 3), \qquad (sf(s))' = F(s-1) \quad (s > 2), \\ 0 \le f \le 1 &\le F, \qquad f \text{ is nondecreasing.} \end{aligned} \tag{B.13}$$

These classical properties are the inputs to this calculation. Put $u = 1/500$, $\mu_* = 25/128$ and $h_G = 9/5000$. Define

$$\begin{aligned} v_1(s) &= \min\left(\frac{1+s}{2}, \frac{32-34s}{39}, \frac{57}{64} - \frac{3s}{2}\right), \qquad v(s) = v_1(s) - u, \\ \psi(x,y) &= \begin{cases} x, & x \ge y, \\ -\infty, & x < y, \end{cases} \\ \Theta_*(x,y,z) &= \max\{v(x), v(y), v(x+y), v(x+y+z), \\ &\qquad \psi(v(x+z), x+2y+z), \psi(v(y+z), 2x+y+z)\}. \end{aligned} \tag{B.14}$$

The unguarded branches include $v(x) > 0$ for $u \le x \le \mu_*$. Thus replacing the inactive value $-\infty$ by 0 leaves this maximum unchanged. For $d = 0, h_G$, put

$$\begin{aligned} J_d &= \int_{u \le z \le y \le x \le \mu_*} \frac{1 - f((\Theta_*(x,y,z) + d - x - y - z)/z)}{xyz^2}\, dx\, dy\, dz, \\ H_{\log} &= \log(\mu_*/u), \\ R_{\mathrm{up}} &= 500(F(30) - 1) + 500H_{\log}(1 - f(30)) + 250H_{\log}^2(F(30) - 1), \\ \mathcal{G}_d &= 1/\mu_* + R_{\mathrm{up}} + J_d. \end{aligned} \tag{B.15}$$

The unshifted $d = 0$ profile and functional come from [27, (4.70)–(4.75)]. The parameter d and the shifted functional $\mathcal{G}_d$ are defined in this paper. The retained-side argument uses the instance $d = h_G$, constructed in (5.52). Paper II's (5.4)–(5.5) are not the source of this shifted functional.

Proposition B.2. *The shifted scalar satisfies*

$$0 \le \mathcal{G}_{h_G} \le U_h < \frac{6073714261}{10^9}. \tag{B.16}$$

Let $F_h^+ \ge F(30)$, $f_h^- \le f(30)$, and $H_h^+ \ge H_{\log}$ be the directed table endpoints. Define

$$\begin{aligned} \mathcal{R}(X,Y,Z) &= 500(X-1) + 500Z(1-Y) + 250Z^2(X-1), \\ R_h &= \mathcal{R}(F_h^+, f_h^-, H_h^+), \\ U_h &= 1/\mu_* + R_h + J_h^+. \end{aligned} \tag{B.17}$$

The construction has $F_h^+ \ge 1$, $f_h^- \le 1$. Thus $R_{\mathrm{up}} \le R_h$. This directed endpoint comparison is the monotonicity step of [27, §4.6, (4.73)]. *Table B.2 encloses these finite upper bounds. The constants K_{hi} and L_{G1} are printed exactly in* (5.48) *and* (5.49)*, with their Paper II locators. The normalization identities are*

$$\begin{aligned} M_{5,\mathrm{internal}} &= -K_{\mathrm{hi}} \frac{6073714261}{10^9} - L_{G1}, \\ M_{5,\mathrm{internal}} + K_{\mathrm{hi}}\left(\frac{6073714261}{10^9} - U_h\right) &= -K_{\mathrm{hi}} U_h - L_{G1}. \end{aligned} \tag{B.18}$$

TABLE B.2. Directed bounds for the shifted profile with $n_{\mathrm{mesh}} = 512$.

Quantity	lower endpoint	upper endpoint
R_h	0.000004373236955	0.000004373236956
J_h^+	0.953709886776891	0.953709886776892
U_h	6.073714260013847	6.073714260013848
$6073714261/10^9 - U_h$	0.000000000986152	0.000000000986153

Proof. For the outer partition put $n = 512$, $x_0 = u$, $x_n = \mu_*$, and

$$x_i = 2^{-48}\left\lfloor 2^{48}\left(u + (\mu_* - u)\sqrt{i/n}\right)\right\rfloor \quad (1 \le i < n). \tag{B.19}$$

It is strictly increasing. For the inner integral write

$$g_d(x,y) = \int_u^y \frac{1 - f((\Theta_*(x,y,z) + d - x - y - z)/z)}{z^2}\, dz. \tag{B.20}$$

Each branch residual decreases in x, y. Each guard's active set is a down-set. Thus g_d is nondecreasing in x, y, guard jumps included.

Let w_i be an outward upper bound for $\log(x_i/x_{i-1})$, on the 2^{-128} lattice. Let v_{ij} enclose $g_{h_G}(x_i, x_j)$ upwards on that lattice. The diagonal cell has half the logarithmic square's area. Hence

$$J_{h_G} \le J_h^+ = \sum_{i=1}^{n} w_i \left(\sum_{j<i} w_j v_{ij} + \frac{1}{2} w_i v_{ii} \right). \tag{B.21}$$

Split the inner interval at every affine crossing. Include every guard crossing and each branch kink. On each resulting interval the argument is $A/z + B$. If $\Phi(t) = \int_1^t (1 - f(s))\, ds$, then

$$\int_p^q \frac{1 - f(A/z + B)}{z^2}\, dz = \frac{\Phi(A/p + B) - \Phi(A/q + B)}{A} \quad (A \neq 0). \tag{B.22}$$

The trivial bound is $1/p - 1/q$. The smaller of the two upper bounds may be used. For $A = 0$, the argument is constant.

The delay equations give the primitive explicitly:

$$P(t) = \frac{tF(t)}{2e^\gamma}, \qquad \Phi(t) = t - 1 - 2e^\gamma (P(t+1) - 1) \quad (t \ge 0). \tag{B.23}$$

For $t \ge 30$, monotonicity gives

$$\Phi(30) \le \Phi(t) \le \Phi(30) + (t - 30)(1 - f(30)). \tag{B.24}$$

We bound the trapezoid error. If g is either F or f and $[a, a + h_0] \subset [1, 30]$ is one mesh panel, then

$$\int_a^{a+h_0} g(s)\, ds - \frac{h_0}{2}\big(g(a) + g(a + h_0)\big) = -\frac{h_0^3}{12} g''(\xi), \qquad a < \xi < a + h_0, \tag{B.25}$$

with one-sided interpretations at the integer breakpoints. The mesh has $h_0 = 1/65536$ and every breakpoint is a mesh node. We prove the derivative bound independently of the quadrature table. Write $C = 2e^\gamma$. The elementary inequality $\gamma \le H_6 - \log 6$ and the first three positive terms of the atanh series give

$$\log 2 > \frac{842}{1215}, \qquad \log 3 > \frac{263}{240}, \qquad \gamma < \frac{12853}{19440} < \frac{2}{3} < \log 2.$$

Thus $0 < C < 4$ and $\log 2 < 1$. The defining functions satisfy

$$F(s) = C/s \ (1 \le s \le 3), \quad f(s) = 0 \ (1 \le s \le 2), \quad f(s) = C \log(s-1)/s \ (2 < s \le 3),$$
$$sF(s) = C + \int_1^{s-1} f(t)\, dt, \qquad sf(s) = \int_1^{s-1} F(t)\, dt \quad (s > 3).$$

Induction on unit intervals gives $0 \le F, f \le C$. If the delayed functions are in this range, the integrals have length $s - 2$, so $F(s) \le C(s-1)/s \le C$ and $f(s) \le C(s-2)/s \le C$. For either companion pair $(g, h) = (F, f)$ or (f, F), differentiating on open unit intervals gives

$$g'(s) = \frac{h(s-1) - g(s)}{s}, \qquad g''(s) = \frac{h'(s-1)}{s} + \frac{2(g(s) - h(s-1))}{s^2} \quad (s > 3).$$

For f, the first-derivative delay identity holds already for $t > 2$. For F on $2 < t < 3$, use $F'(t) = -C/t^2$. Hence $|F'(t)|, |f'(t)| \le C/t$ for $t > 2$. Since $g(s), h(s-1) \in [0, C]$, for $s > 3$

$$|g''(s)| \le \frac{C}{s(s-1)} + \frac{2C}{s^2} \le \frac{7C}{18} < \frac{14}{9} < 12.$$

On the initial intervals,

$$F''(s) = 2C/s^3 \ (1 < s < 3), \quad f''(s) = 0 \ (1 < s < 2),$$
$$f''(s) = C\left(\frac{2\log(s-1)}{s^3} - \frac{1}{s(s-1)^2} - \frac{2}{s^2(s-1)} \right) \ (2 < s < 3).$$

The formulas imply $0 \le F'' \le 8$ on $[1, 2]$, $0 \le F'' \le 1$ on $[2, 3]$, and $-4 \le f'' \le \log 2 < 1$ on $[2, 3]$. Inside the last parentheses each negative term has magnitude at most $1/2$. The positive term is at most $\log 2/4$. Therefore $|g''| \le 8 < 12$ on all smooth panel interiors of $[1, 30]$, with one-sided limits at the integer breakpoints.

The archived outward intervals and the independently computed exact rational intervals are recorded with the reproducibility materials. The derivative check starts from the proved invariant and then rebuilds the delay table. It

checks every computed unit-interval enclosure against the archived coarse intervals. It confirms that the computed global bound is ≤ 12, the recorded value:

range	F''	f''
$[1,2]$	$[0,8]$	$[0,0]$
$[2,3]$	$[-8,8]$	$[-4,4]$
$[3,30]$	$[-12,12]$	$[-12,12]$

The computed bound is 8. Hence the absolute local remainder is at most $8h_0^3/12 = (2/3)h_0^3 \le h_0^3$. The verifier checks this radius against the h_0^3 panel allowance before it accepts the regenerated table. The M5 computation builds the delay table with `DEN=65536`, `S=128`, 160-bit Arb interval arithmetic, and 1,900,545 generated nodes. The shifted computation then evaluates the exact $n_{\mathrm{mesh}} = 512$ triangular grid, with $512 \cdot 513/2 = 131,328$ inner-grid evaluations. It stores one aggregated row per outer index. Hence `SHIFT-N512-fresh.json` contains 512 rows for the 131,328 evaluations. The M5 computation uses no Taylor model and no width-1/32 step. The historical Taylor certificate provides background information and is not an input to this calculation. The table and shifted rows are regenerated before the directed inclusion check. The verifier compares recorded JSON fields and exact row values. It also checks SHA-256 hashes of the canonical compact-JSON integer arrays used by the real-inclusion verifier. These canonical-array hashes differ from hashes of regenerated JSON files. File formatting is not used as a certificate. Directed summation gives Table B.2. The positive last row proves the strict cap. The 512 aggregated rows give the exact 131328-term sum used for J_h^+. The directed real-inclusion result records the directed tail and the cap check. The reproducibility report documents the execution boundary. The two normalization identities follow by expansion. □

The proposition concerns the scalar functional only. The prime-sum comparison is required separately in Section 5.

B.4. **Unshifted comparison (historical and not used).**
Status: historical and non-certified. This historical unshifted comparison records an earlier $n = 8192$ calculation for context only. No current computation reproduces it. None of its values, table, constants or comparison is an input to the main proof. The comparison concerns the unshifted functional $G_* := \mathcal{G}_0$. It is distinct from the main-line coefficient bound $G_0 = 6088440363/10^9$. The main K2 and L3 comparisons use the latter via [27, (4.73), (4.75), (4.79), (A.10a)]. They do not use U_0. The cited bound $G_* \le G_0$ is the coefficient used in the main argument. It does not assert the separate, sharper $G_* \le U_0$ comparison. This appendix proves that comparison only conditionally, as an optional one. Its historical delay table was constructed separately, but no such table or producer is part of the current proof package. The panel width is $h_0 = 1/65536$. At 192-bit precision, each trapezoid receives radius h_0^3. On smooth panel interiors, $|F''|, |f''| < 12$. The panels meet the integer breakpoints. This bounds each local truncation remainder. The endpoint enclosures pass through the delay recursion in Arb ball arithmetic. Accumulated input radii are thus kept before outward rounding. All table endpoints are then rounded outwards to 2^{-128}. The primitive and true tail use (B.24). This table gives a historical directed bound for R_{up}. It is kept only to reproduce the earlier non-certified calculation. The separate earlier Taylor models are not part of the M5 computation. Let $F_0^+ \ge F(30)$, $f_0^- \le f(30)$, and $H_0^+ \ge H_{\log}$ be these directed endpoints. Define

$$R_0 = \mathcal{R}(F_0^+, f_0^-, H_0^+) \ge R_{\mathrm{up}}. \tag{B.26}$$

The earlier finite calculation used $n = 8192$ in (B.19). The outer sum has $n(n+1)/2 = 33558528$ nodes. Write J_0^+ for that upper-corner sum. Table B.3 groups its eight consecutive row ranges. Each row lists a lower and upper bound for the sum. The diagonal half-weights are already included.

TABLE B.3. Historical non-certified contributions to the unshifted sum.

Rows	nodes	lower endpoint	upper endpoint
1–1024	524800	0.000002727813799350	0.000002727813799351
1025–2048	1573376	0.001226034866061399	0.001226034866061400
2049–3072	2621952	0.019651743682986805	0.019651743682986806
3073–4096	3670528	0.080548405705743750	0.080548405705743751
4097–5120	4719104	0.146720281588383935	0.146720281588383936
5121–6144	5767680	0.200161682376550457	0.200161682376550458
6145–7168	6816256	0.242188572133550572	0.242188572133550573
7169–8192	7864832	0.275672089024516687	0.275672089024516688

The earlier, non-certified calculation reported

$$\begin{aligned} J_0^+ &< 0.966171537191592960294580903427, \\ R_0 &< 0.000004373236943357822206406687, \\ U_0 &= 1/\mu_* + R_0 + J_0^+ + \frac{1}{2\cdot 10^{85}} \\ &\in [6.086175910428536318116787310112545656706640625, \\ &\qquad 6.086175910428536318116787310112545656706640626]. \end{aligned} \tag{B.27}$$

The term $1/(2\cdot 10^{85})$ bounds directed outer-sum rounding. The shifted computation does not use this interval. The comparison with the continuous functional is one-sided:

$$\text{Historical comparison (not certified): } G_* = \mathcal{G}_0 \le U_0. \tag{B.28}$$

No claim $2e^{-\gamma}U_0 \le c$ follows from the printed interval. The finite row sums alone prove no prime-sum estimate. The source comparison and its error terms remain required.

B.5. **The sixth-sum constant.** The certified finite enclosure is

$$\mathfrak{g}_{\mathrm{RA}} \in [0.0989598353353,\ 0.0989598353355]. \tag{B.29}$$

Here $\mathfrak{g}_{\mathrm{RA}}$ denotes the integral quantity in Section 6. The constant used in the final bound is

$$g_{\mathrm{RA,lo}} = \frac{989598353}{10^{10}} = 0.0989598353. \tag{B.30}$$

Thus the certified enclosure gives

$$\frac{7499}{9999}\left(J_J(1/2) - J_J(5227/10000)\right) - \frac{5106819}{31557955000} = \mathfrak{g}_{\mathrm{RA}} > g_{\mathrm{RA,lo}}. \tag{B.31}$$

Only this conservative lower constant enters the final bound.

B.6. **Computer checks.** Table B.4 lists the supplementary scripts. The second column states what each one checks. The partitions and defining formulas are printed above. The exact-arithmetic checks use Python's integer and fraction types. Environment and timing details are recorded in the reproducibility materials. They are not theorem inputs. A full numerical regeneration is distinguished from a compact check. The certificate bundle is available at the location given in the Data availability statement; its SHA-256 manifest hash is `81fc00441c3a3f0924995cd41263feddece827d9b40376dae531191c86f90ed3`. The recorded M5 regeneration agrees with the exact 512-row certificate. The reproducibility certificate set is independent. It consists of the H6 `RESULT.json`, the 512-row M5 output, the Root–A enclosure, and the verifier scripts for those records. Each record is checked against the source hashes and the manifest. None replaces an analytic estimate.

TABLE B.4. Scripts and finite checks.

Script	Statements checked	Time
Reproducibility replay verifier	manifests, source hashes, TeX structure, finite consistency checks, and the producer verification run	replay
H6/M5 producer verification	executes the H6 computation and M5 $n_{\mathrm{mesh}} = 512$ table, shifted probe and real-inclusion verifier, computing the derivative enclosure and comparing regenerated outputs	replayed
H6 directed verifier	regenerates the directed H6 rows from the source inputs and Arb runtime	replayed
M5 derivative verifier	exact rational unit-interval bounds for F'' and f'' and the panel radius	replayed
M5 table, shift and inclusion verifiers	regenerates the $DEN = 65536$ delay table, $n_{\mathrm{mesh}} = 512$ rows, and real inclusion	replayed
Finite consistency verifier	exact finite arithmetic, orbit, and sign checks for the patches	replayed

The reproducibility verifier checks the recorded inputs and then runs the producer verification. That run regenerates the H6 output, the M5 delay table, the shifted rows and the real inclusion on the recorded platform. The earlier unshifted calculation used Python 3.12.3 and python-flint 0.9.0, with four workers. Its slowest worker took 7639 seconds, rounded upwards. The historical unshifted comparison is optional and not certified. The main proof does not use it, and it does not affect the shifted computation. Neither these scripts nor their agreement establishes a uniform estimate. The final proof requires the analytic comparisons in Sections 5–8.

B.7. **Rows of the retained side.** The index set $\mathcal{I}$ of Section 5 consists of Paper II's nineteen original row chains and one outer chain. The outer chain T5 covers the upper part of S_5, Lemma 5.1. Sixteen of the row chains are listed below, each with its source parent and raw coefficient. The remaining three are Paper II's single-prefix comparisons for S_3, S_8 and S_9. They keep Paper II's own sources, restorations and error entries unchanged. The four rows of $\mathcal{H}$ are restored to their parents S_4, S_5, S_7 and S_{10}. The lower part of S_5 has no separate restoration address. The four joint rows are restored once to the joint source of G_1 with weights $(2, 1, 1)$.

Row	Parent	Coefficient	Row	Parent	Coefficient
S1-0	S1	3	S4	S4	-1
S1-1	S1	-3	S5-low	S5	-1
S1-2	S1	3	S7	S7	-1
S1-3	S1	-3	S10-Ret	S10	1
S2-0	S2	1	G1-0	G1 $(2, 1, 1)$	-1
S2-1	S2	-1	G1-1	G1 $(2, 1, 1)$	1
S2-2	S2	1	G1-2	G1 $(2, 1, 1)$	-1
S2-3	S2	-1	G1-3	G1 $(2, 1, 1)$	1

These sixteen rows, their seven source-parent restorations and four outer terms carry 91 error addresses. Each address occurs exactly once. One outer term is the chain T5. The others are the strict BV piece of S_2, inside its joint group, and two G_1 comparator entries. The same table has seven further rows with 38 addresses. They treat S_3, S_8 and S_9 another way, and this paper does not use them.

B.8. **Statements for the sixth sum.** The sixth-sum argument uses the following statements.

Statement
Lemma 6.2 (rough-factor distribution)
Lemma 6.4 (four-family Siegel–Walfisz); range $N^{1/3}/8 \le H$
Lemmas 6.6 and 6.7 (selector and tensor, applied per shell pair)
Lemma 6.5 (transferred mean square)
Lemma 6.8 (two-factor convolution)
Lemma 6.9 (central rectangles)
Proper powers, (6.26)
Fact 3.2 (maximal prime estimates)
Paper I, (2.21), as used in Paper I, (3.25)

Use of AI tools

The author devised the method of proof. Codex (OpenAI, Astra 6) derived and checked the estimates, wrote the Python code for the certificates in Appendix B and ran blind reviews of the manuscript. Claude Code (Anthropic, Claude Fable 5) checked the proofs, the numerical certificates and the references. The author checked every step of the proof, every table and every reference, and takes full responsibility for the content of the paper.

Statements and declarations

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Data availability. The supplementary scripts, exact inputs and compact outputs are archived as `paper-iii-public-certificates-v1.tar.gz` with release v1 at `https://github.com/ryzfsu/goldbach-1plus1.75-theorem/releases/download/v1/paper-iii-public-certificates-v1.tar.gz`.
The SHA-256 hash of the archive's internal `SHA256SUMS` manifest is `81fc00441c3a3f0924995cd41263feddece827d9b40376dae531191c86f90ed3`.

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