A POWER SAVING BELOW TWO-THIRDS FOR EQUAL RADICALS OF CONSECUTIVE PRODUCTS

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ABSTRACT. Let $F(X)$ count positive integer pairs $m < n \le X$ with $\mathrm{rad}(m(m+1)) = \mathrm{rad}(n(n+1))$. We prove $F(X) \ll_\varepsilon X^{109/165+\varepsilon}$ for every $\varepsilon > 0$, improving the earlier exponent $2/3$. A four-corner parametrization divides these pairs into thirteen classes. Uniform estimates for those classes combine with a finite exponent calculation and one choice of error terms.

1. INTRODUCTION

For a positive integer r, let $\mathrm{rad}(r) = \prod_{p \mid r} p$. We count ordered pairs

$$F(X) = \#\{(m,n) \in \mathbb{Z}_{>0}^2 : m < n \le X,\ \mathrm{rad}(m(m+1)) = \mathrm{rad}(n(n+1))\}.$$

Theorem 1.1. *For every $\varepsilon > 0$ and $X \ge 2$,*

$$F(X) \ll_\varepsilon X^{109/165+\varepsilon}.$$

The exponent is $1/165$ below $2/3$. Section 2 introduces the four-corner gcd coordinates. It also records the source-specific identities used here. The new part has three pieces: a first-match partition into thirteen disjoint populations, uniform estimates for its new rows, and a finite optimization of the long-shift remainder.

The minimum S_0 below is the limiting minimum of the row savings. After the analytic loss β_0 and a conservative long-row charge $g_0/4$, the proof retains the smaller saving displayed below.

The earlier result [1, Theorem 1] gives exponent $2/3$ for this counting function. Its support-fibre lemma and low-radical estimate are used below; Appendix E states them and records the substitutions. The quotient-tail and Farey-spacing estimates in that paper are context only. They are not used. Two related papers by the author [2, 3] count primitive *abc* triples, not the pairs counted here, and do not enter this proof.

At the chosen rational point, the minimum row saving is

$$S_0 = \frac{30869}{5000000}, \qquad g_0 = S_0 - \frac{1}{165} = \frac{18677}{165000000}.$$

The source-uniform estimates spend a total analytic loss $\beta_0 = g_0/16$. The long row has three finite corrections: geometric, interval, and high-degree. Beyond the common threshold each is at most $g_0/4$. We charge their maximum, not their sum. The remaining margin is

$$S_0 - \beta_0 - \frac{g_0}{4} = \frac{1}{165} + \frac{11g_0}{16} > \frac{1}{165}.$$

Sections 2–3 give the canonical source and the exact row partition. Sections 4–6 treat the outer, joint-corner, short-shift and intermediate rows. Section 7 handles the long row. Sections 8–10 allocate losses and sum the dyadic shells. The appendices contain the self-contained square-fibre proof, the application of the cited estimates, and the branch arithmetic. The use of AI tools is described after the appendices.

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We use Heath–Brown's published Lemma 2 under its stated bihomogeneous hypotheses. The fixed cubic branch uses the unnumbered theorem of Bombieri–Schmidt with its 1989 correction [6]. Appendix E states both inputs and records the applications. Reuss's Theorem 1 [7] is the determinant-method input behind the low-radical estimate in [1]. That estimate is used here. Tao's Lemma 2.10 gives a comparison with the square-fibre estimate of Appendix B. The proof uses that appendix theorem.

2. Canonical source and parameters

For an integer $Y \geq 2$, let

$$\Omega_Y = \{(m,n) \in \mathbb{Z}^2_{>0} : m < n \leq Y,\ Y/2 < n,\ \operatorname{rad}(m(m+1)) = \operatorname{rad}(n(n+1))\}.$$

An element of Ω_Y is called a source pair. For each source pair define its four corner gcds by

$$A = \gcd(m,n), \quad B = \gcd(m,n+1), \quad C = \gcd(m+1,n), \quad D = \gcd(m+1,n+1).$$

These four integers are pairwise coprime. For example, a prime dividing A and B would divide both n and $n+1$. A prime dividing A and C would divide both m and $m+1$. The same two consecutive pairs settle the other four pairs. Each corner divides the two entries in its definition. Hence, by pairwise coprimality, $AB \mid m$, $CD \mid m+1$, $AC \mid n$, and $BD \mid n+1$. Define the positive integer residuals

$$a = \frac{m}{AB}, \qquad b = \frac{m+1}{CD}, \qquad c = \frac{n}{AC}, \qquad d = \frac{n+1}{BD}.$$

The residuals are pairwise coprime. First, $\gcd(a,b) = \gcd(c,d) = 1$, since each pair divides consecutive integers. For the four cross pairs, divide the defining corner gcds. This gives

$$\gcd(Ba, Cc) = \gcd(Aa, Dd) = \gcd(Db, Ac) = \gcd(Cb, Bd) = 1,$$

so $\gcd(a,c) = \gcd(a,d) = \gcd(b,c) = \gcd(b,d) = 1$ as well. The coordinates and mask are

$$\begin{gathered} m = ABa, \quad m+1 = CDb, \quad n = ACc, \quad n+1 = BDd, \\ Z = ABCD, \quad R = \operatorname{rad}(Z), \quad M = abcd, \\ K_{\mathrm{res}} = M/\operatorname{rad}(M), \quad L = cd - ab, \\ \alpha = ac, \quad \beta = bd, \quad \gamma = ad, \quad \delta = bc, \\ u = (m+n+1)/(BC), \qquad t = (n-m)/(AD), \\ CDb - ABa = 1, \qquad BDd - ACc = 1. \end{gathered} \tag{2.1}$$

The four cross-gcd identities above will be used below. The radical condition is exactly $\operatorname{rad}(M) \mid Z$. If $p \mid a$, then $p \mid m$. Equality of the original radicals forces $p \mid n$ or $p \mid n+1$. Hence $p \mid A$ or $p \mid B$. The other residuals give the corresponding corner. Conversely, every prime of Z divides both consecutive products. The mask ensures that no residual introduces a prime outside Z. Thus the common radical is

$$\operatorname{rad}(m(m+1)) = \operatorname{rad}(n(n+1)) = R.$$

The identities

$$Z^2 M = m(m+1)n(n+1), \quad ZL = (n-m)(m+n+1)$$

give $ut = L > 0$ once the shifts are integral. Their integrality follows from the corner identities

$$m+n+1 = C(Ac+Db) = B(Aa+Dd), \qquad n-m = A(Cc-Ba) = D(Bd-Cb).$$

Since $\gcd(B,C) = \gcd(A,D) = 1$, the first numerator is divisible by BC and the second by AD. Thus u, t are positive integers with product L. The shifts are units at the opposite corner pairs:

$$\gcd(u, AD) = \gcd(t, BC) = 1.$$

Modulo a prime dividing A, the integer $m+n+1$ is congruent to 1. Modulo a prime dividing D, it is congruent to -1. Modulo a prime dividing B, $n-m$ is congruent to -1. Modulo a prime dividing C, it is congruent to 1. The denominator factors BC and AD are coprime to these opposite corner pairs.

The two conics, scale identities, and inverse maps from the same parametrization are

$$\begin{aligned} \beta D^2-\alpha A^2&=u, & \delta C^2-\gamma B^2&=t,\\ \alpha A^2&=u\frac{mn}{m+n+1}, & \beta D^2&=u\frac{(m+1)(n+1)}{m+n+1},\\ \gamma B^2&=t\frac{m(n+1)}{n-m}, & \delta C^2&=t\frac{n(m+1)}{n-m},\\ B&=(cA+bD)/u, & C&=(aA+dD)/u,\\ A&=(dB-bC)/t, & D&=(cC-aB)/t. \end{aligned} \tag{2.2}$$

We use only source points. They satisfy positivity, integrality, the four corner gcds and the cross gcds. They also satisfy the shell condition and the radical mask. We assert no converse for arbitrary positive solutions of either conic.

The selected chart is deterministic:

$$(J,K;s)=\begin{cases}(A,D;u), & u\le t,\\ (B,C;t), & t<u.\end{cases} \tag{2.3}$$

Thus $u=t$ belongs to the first chart. Write $(f,g)=(\alpha,\beta)$ in the first chart and $(f,g)=(\gamma,\delta)$ in the second. For a positive integer x, let $v_p(x)$ be the exponent of p in its prime factorization. For a finite set of primes $\mathcal{P}$, put

$$x_{\mathcal{P}}=\prod_{p\in\mathcal{P}}p^{v_p(x)}, \qquad x_{\varnothing}=1.$$

Then

$$f_J=f_{\{p:p\mid J\}}, \quad g_K=g_{\{p:p\mid K\}}, \quad v=f/f_J, \quad w=g/g_K, \quad r_J=\operatorname{rad}(f_J), \quad r_K=\operatorname{rad}(g_K),$$

$$\Pi=\min(r_J,v)\min(r_K,w). \tag{2.4}$$

These supported parts are functions of the canonical source. They are not extra summation variables.

Set

$$\begin{gathered} S_0=\frac{30869}{5000000}, \quad \epsilon_0=k_0=S_0, \quad l_0=2S_0, \quad j_0=p_0=3S_0,\\ a_0=\frac{6}{125}, \quad b_0=\frac{4166537}{30000000}, \quad \Delta_{\exp}=3S_0,\\ q_{\min}=\frac{9907393}{15000000}=\frac{2}{3}-S_0, \quad q_{\max}=\frac{5092607}{7500000}=\frac{2}{3}+2S_0, \quad \tau_0=1-\frac{3}{2}S_0. \end{gathered} \tag{2.5}$$

We use the following notation.

Symbol	Meaning
Y	dyadic shell endpoint; $Y/2 < n \le Y$
A, B, C, D	four corner gcds
a, b, c, d	positive residual factors
Z, R, M, L	$ABCD,\ \mathrm{rad}(Z),\ abcd,\ cd - ab$
$\alpha, \beta, \gamma, \delta$	$ac,\ bd,\ ad,\ bc$
u, t	positive shifts, with $ut = L$
$J, K; s$	selected corner pair and its smaller shift
f_J, g_K, v, w, Π	supported coefficients, complements, and their mask
S_0, g_0, β_0	minimum saving, reserve, and allocated loss (values in the Introduction)

Define the high sector

$$\mathcal{C}_{\mathrm{hi}} = \{(m,n) \in \Omega_Y : R > Y^{5/3-\epsilon_0},\ M > Y^{q_{\min}},\ \mathrm{rad}(M) > Y^{2/3-k_0}\}. \tag{2.6}$$

The high sector is the complement of the first three predicates in (3.1). Those rows test the complementary inequalities. For sources with $R > Y^{5/3-\epsilon_0}$, the source identities give

$$M < \kappa_Y^2 Y^{q_{\max}}, \qquad L < \kappa_Y Y^{1/3+\epsilon_0}, \qquad \kappa_Y = 1 + Y^{-1}. \tag{2.7}$$

We have $m(m+1)n(n+1) \le \kappa_Y^2 Y^4$ and $ZL = (n-m)(m+n+1) < \kappa_Y Y^2$. Also $Z > Y^{5/3-\epsilon_0}$. On $\mathcal{C}_{\mathrm{hi}}$, the strict lower cuts on R and M also give

$$Y^{4-3\epsilon_0} < Z^2 M = m(m+1)n(n+1) \le 2\kappa_Y m^2 Y^2,$$

so $m^2 > Y^{2-3\epsilon_0}/(2\kappa_Y) \ge Y^{2-3\epsilon_0}/3$. Therefore

$$m > Y^{\tau_0}/\sqrt{3}. \tag{2.8}$$

Here $2\kappa_Y \le 3$ for $Y \ge 2$. This last inequality does not use the radical cut in $\mathcal{C}_{\mathrm{hi}}$.

After the $Q0$ row of (3.1) has failed, $K_{\mathrm{res}} = M/\mathrm{rad}(M) \le Y^{\Delta_{\exp}}$. After $Q1$ has failed, $L \ge Y^{1/3-l_0}$. After both $Q2$ chart tests have failed, the selected corners satisfy $J, K \ge Y^{1/3-j_0}$. These are the non-strict complements used in the following partition. Display (3.1) states the predicates and their order.

3. The first-match partition

Each predicate below is tested only after every earlier row has failed. Here $s = \min(u,t)$ in the chart selected by (2.3).

row	predicate
O_R	$R \le Y^{5/3-\epsilon_0}$
O_M	$M \le Y^{2/3-\epsilon_0}$
$O_{\mathrm{rad}\,M}$	$\mathrm{rad}(M) \le Y^{2/3-k_0}$
$Q0$	$K_{\mathrm{res}} > Y^{\Delta_{\exp}}$
$Q1$	$L < Y^{1/3-l_0}$
$Q2u$	$u \le t,\ \min(A,D) < Y^{1/3-j_0}$
$Q2t$	$t < u,\ \min(B,C) < Y^{1/3-j_0}$
$3u$	$u \le t,\ u \le Y^{a_0}$
$3t$	$t < u,\ t \le Y^{a_0}$
$4E$	$Y^{a_0} < s < Y^{b_0},\ \min(v,w) \le Y^{p_0}$
$4H$	$Y^{a_0} < s < Y^{b_0},\ v,w > Y^{p_0},\ \Pi \le Y^{p_0}$
$4P$	$Y^{a_0} < s < Y^{b_0},\ v,w > Y^{p_0},\ \Pi > Y^{p_0}$
$5W$	the remaining source.

(3.1)

Theorem 3.1 (Exact partition)**.** *The thirteen first-match sets in* (3.1) *are pairwise disjoint and have union* Ω_Y*. On* $5W$*,*

$$s \geq Y^{b_0}, \qquad K_{\mathrm{res}} \leq Y^{\Delta_{\mathrm{exp}}}, \qquad L \geq Y^{1/3-l_0}, \qquad J, K \geq Y^{1/3-j_0}. \tag{3.2}$$

Proof. The quantities in (3.1) are deterministic functions of the canonical source. Each of the first twelve rows is intersected with the complements of the preceding predicates. So these rows are disjoint. Their complement is exactly the last row, which proves exhaustiveness. Failure of $Q0$ gives $K_{\mathrm{res}} \leq Y^{\Delta_{\mathrm{exp}}}$. Failure of $Q1$ gives $L \geq Y^{1/3-l_0}$. On the complement of both chart-specific $Q2$ tests, the selected corners satisfy $J, K \geq Y^{1/3-j_0}$. Failure of the $3u$ and $3t$ tests gives $s > Y^{a_0}$. If $s < Y^{b_0}$, trichotomy gives exactly one of the $4E, 4H, 4P$ predicates. Therefore their complement has $s \geq Y^{b_0}$. □

The following boundary conventions apply after all earlier rows have failed. The first three rows use non-strict upper cuts. Equality at $K_{\mathrm{res}} = Y^{\Delta_{\mathrm{exp}}}$, $L = Y^{1/3-l_0}$, and $\min(J, K) = Y^{1/3-j_0}$ survives the corresponding strict test. The chart equality $u = t$ selects the u-chart. The full first-match order determines its row. The $3u, 3t$ tests include $s = Y^{a_0}$, while $5W$ contains $s = Y^{b_0}$. In the open middle interval, $4E$ contains the boundary $\min(v, w) = Y^{p_0}$. Also $4H$ contains $\Pi = Y^{p_0}$ when $v, w > Y^{p_0}$.

Remark 3.2 (Counting populations). The estimates below count the following populations:

(1) For O_R, fix n first. Then apply the support-fibre lemma in [1] to the partners m.
(2) $O_M, O_{\mathrm{rad}\, M}, Q0$ use the complete fixed-M quadratic fibre, including all ordered factorizations $M = (ab)(cd)$. $Q1$ instead fixes (P, L), with $P = ab$ and $Q_{\mathrm{prod}} = cd = P + L$.
(3) $Q2u, Q2t$ are disjoint chart populations. Their two analytic contributions apiece are additive bounds for each population.
(4) $3u, 3t, 4E, 4H, 4P$ retain the corresponding earlier-failure predicates and the canonical inverse filters. The $4P$ estimate is not applied to the $5W$ complement.
(5) $5W$ is exactly the remaining first-match set in (3.1).

Fix the ordered residual factors (a, b, c, d), both shifts (u, t), the selected chart and its conic coordinates. Then the inverse identities (2.2) reconstruct the remaining two corners. This is the full canonical label used for inverse reconstruction. The outer-row divisor bound includes ordered residual factorizations. The joint-row sums include their coefficient factorizations. The later estimates retain the stated divisor or factor multiplicities. A larger set used to estimate one row may intersect the set used for another row. The first-match partition makes the original populations disjoint. Each row is bounded separately.

4. Outer rows and inherited fibres

The small-radical input is Theorem 3 of [1]. Appendix E states it, and Appendix C records the shell substitution. With $H = Y/2$ and $V = Y^{5/3-\epsilon_0}$, it gives

$$\#\{n : Y/2 < n \leq Y,\ \mathrm{rad}(n(n+1)) \leq Y^{5/3-\epsilon_0}\} \ll_\eta Y^{2/3-S_0+\eta}. \tag{4.1}$$

The threshold in that application is large enough that $Y \geq 4$ and $Y^{1/3+\epsilon_0} > 4$. These are exactly the required conditions $H \geq 2$ and $V < H^2$.

Let Q_{sf} be a positive squarefree integer. For a positive integer Q, let $\omega(Q)$ denote the number of distinct prime divisors of Q. Apply the support-fibre estimate [1, Lemma 2.1] with $X = n$:

$$\#\{1 \leq m < n : \mathrm{rad}(m(m+1)) \mid Q_{\mathrm{sf}}\} \ll_\eta Y^{2\eta} \log(2Y). \tag{4.2}$$

Here $Q_{\mathrm{sf}} = \mathrm{rad}(n(n+1))$ is squarefree and at most $Y(Y+1)$. The source equality of radicals gives the divisibility condition for every partner. The bound follows from $2^{\omega(Q_{\mathrm{sf}})} \ll_\eta Q_{\mathrm{sf}}^{\eta}$. Choose

the internal exponents smaller than the requested final ε. Absorbing the logarithm then proves the O_R row bound.

Proposition 4.1 (Five outer rows)**.** *For every $\varepsilon > 0$,*

$$\begin{aligned} \#O_R,\ \#O_M,\ \#O_{\operatorname{rad} M},\ \#Q1 &\ll_\varepsilon Y^{2/3-S_0+\varepsilon}, \\ \#Q0 &\ll_\varepsilon Y^{\varepsilon}. \end{aligned} \tag{4.3}$$

Proof. The O_R estimate is (4.1) followed by the partner-fibre bound (4.2). We apply the latter with $X = n$ and with the squarefree common radical as Q. The defining equality of radicals implies the required divisibility $\operatorname{rad}(m(m+1)) \mid Q$. Each logarithm and divisor factor uses a separate portion of the arbitrary ε.

For O_M, fix $M \le Y^{2/3-\epsilon_0}$. Write $P = ab$, $Q_{\mathrm{prod}} = cd$, so $PQ_{\mathrm{prod}} = M$, $P < Q_{\mathrm{prod}}$, and $Q_{\mathrm{prod}} - P = L > 0$. There are at most $d_4(M) \ll_\eta M^\eta$ ordered choices of (a, b, c, d), where

$$d_4(M) = \#\{(a, b, c, d) \in \mathbb{Z}_{>0}^4 : abcd = M\}.$$

For each, the source identity

$$Q_{\mathrm{prod}} m(m+1) = Pn(n+1)$$

is equivalent, after setting $x = 2n + 1$ and $y = 2m + 1$, to

$$Px^2 + (Q_{\mathrm{prod}} - P) = Q_{\mathrm{prod}} y^2. \tag{4.4}$$

Appendix B bounds the number of such pairs by Y^η. The coefficients and nonzero shift have height bounded by a fixed power of Y. For the theorem's variable bound take $x_0 = 2Y + 1$, since $2n + 1 \le x_0$. The source identity gives $M \le Y^2(Y+1)^2 \le 4Y^4$, so $P, Q_{\mathrm{prod}}, |Q_{\mathrm{prod}} - P| = L \le 4Y^4 \le x_0^4$, and the fixed height exponent is $A_{\mathrm{ht}} = 4$. Thus the implied constant is uniform in the moving labels. Summing over M proves the O_M bound.

For $O_{\operatorname{rad} M}$, (2.7) gives $M \le 4Y^4$. If $R_0 = Y^{2/3-k_0}$, then

$$\#\{M \le 4Y^4 : \operatorname{rad}(M) \le R_0\} \le R_0 (4Y^4)^\sigma \sum_{M \ge 1} \frac{1}{\operatorname{rad}(M) M^\sigma}. \tag{4.5}$$

For fixed $\sigma > 0$, the series is the convergent product

$$\prod_q \left(1 + \frac{1}{q(q^\sigma - 1)}\right).$$

Fix the final target loss. Then choose a fixed $\sigma > 0$ so small that 4σ is below its allocated part. Choose the divisor and fibre exponents independently, each within its own remaining allocation. The resulting label count is $O_\sigma(Y^{2/3-k_0+4\sigma})$. For each M, sum the at most $d_4(M)$ residual factorizations and apply the same fixed-height fibre (4.4). Since $k_0 = S_0$, this gives the asserted exponent.

On $Q0$, failure of the first three rows and the $Q0$ predicate give

$$Y^{q_{\max}} < M < \kappa_Y^2 Y^{q_{\max}}.$$

The interval length is at most $(\kappa_Y^2 - 1)Y^{q_{\max}} \le 3Y^{q_{\max}-1} < 1$ for $Y \ge 256$. Thus at most one integer M occurs. Its complete ordered factorization sum and the fibre (4.4) cost Y^ε. We do not declare the row empty.

On $Q1$, enumerate $P = ab$ and $L = cd - ab$, and put $Q_{\mathrm{prod}} = cd = P + L$. Since $M = PQ_{\mathrm{prod}} \ge P^2$, (2.7) gives $P < \kappa_Y Y^{q_{\max}/2} \ll Y^{1/3+S_0}$. The $Q1$ predicate and the positivity of the integer L give $1 \le L < Y^{1/3-l_0} = Y^{1/3-2S_0}$. Hence the number of (P, L) labels is

$$\ll Y^{1/3+S_0} Y^{1/3-2S_0} = Y^{2/3-S_0}.$$

Each label fixes $Q_{\mathrm{prod}} = P + L$. Splitting P, Q_{prod} into their ordered factors costs at most $\tau_{\mathrm{div}}(P)\tau_{\mathrm{div}}(Q_{\mathrm{prod}}) \ll_\eta Y^\eta$. Then the fixed-height fibre (4.4) is used once. This row carries no additional divisor sum over M. □

Remark 4.2. The source and theorem hypotheses in this section hold above fixed thresholds. Their applications are recorded in Appendix C. The finitely many remaining Y are absorbed into the implied constants in Proposition 4.1.

5. Joint-corner and short-shift rows

The two $Q2$ populations are disjoint. This is because the chart rule (2.3) assigns each source to exactly one of $u \le t$ or $t < u$. In a dyadic sector for the selected corners, write

$$J \asymp J_0, \qquad K \asymp K_0, \qquad P_0 = J_0K_0, \qquad \mu_{\mathrm{sec}} = \min(J_0, K_0).$$

Here each dyadic sector means $J_0 \le J < 2J_0$ and $K_0 \le K < 2K_0$. These half-open sectors are disjoint. The $Q2$ predicate gives $\mu_{\mathrm{sec}} < Y^{1/3-j_0}$.

Lemma 5.1 (Sector count)**.** *Fix a positive selected shift s and a nonempty sector as above. Count the canonical-source coefficient-corner triples with $\gcd(f, K) = \gcd(g, J) = \gcd(s, JK) = 1$, with the second coefficient forced by the conic. Suppose that*

$$fJ^2 \le H, \qquad gK^2 \le H.$$

Then their number is

$$\ll_\eta Y^\eta H\left(\frac{1}{P_0} + \frac{\mu_{\mathrm{sec}}^3}{P_0^2}\right)$$

for every $\eta > 0$, with an implied constant depending only on η. In the $Q2u$ application take $H = uY$. In the $Q2t$ application we assert the natural cutoff $H \ll tY^{1+\epsilon_0+l_0}$ only on the first-match population. There the earlier O_R and $Q1$ rows have failed. That source-uniform bound is proved in (5.1)*.*

Proof. In the u chart the conic is $gK^2 - fJ^2 = u$. The imposed cutoff gives $fJ^2, gK^2 \le H$. For the $Q2u$ application, the source scale identities permit the choice $H = uY$. The full source filter gives $\gcd(f, K) = \gcd(g, J) = 1$ and $\gcd(u, JK) = 1$. These are the unit conditions for either orientation. Thus the quadratic congruence has at most $2^{\omega(K)+2}$ roots modulo K^2, or $2^{\omega(J)+2}$ roots modulo J^2. For odd prime powers each unit-square congruence has at most two roots. The prime 2 contributes at most four roots modulo a power of 2. So the factor $2^{\omega(\cdot)+2}$ is a uniform overcount, including the even modulus. Suppose first $J_0 \ge K_0$. For fixed f and K,

$$fJ^2 \equiv -u \pmod{K^2}.$$

This congruence has at most $2^{\omega(K)+2} \ll_\eta Y^\eta$ root classes modulo K^2. Each class contains at most $J_0/K^2 + 1$ integers J in the sector. There are $O(K_0)$ choices of K and $O(H/J_0^2)$ choices of f. The resulting count is

$$\ll_\eta Y^\eta H\left(\frac{1}{J_0K_0} + \frac{K_0}{J_0^2}\right) = Y^\eta H\left(\frac{1}{P_0} + \frac{\mu_{\mathrm{sec}}^3}{P_0^2}\right).$$

If $K_0 \ge J_0$, reduce the conic modulo J^2. This gives the same expression with J_0, K_0 interchanged.

For the t chart the full filter gives $\gcd(g, J) = \gcd(f, K) = 1$ and $\gcd(t, JK) = 1$. Use $gK^2 - fJ^2 = t$, now with $(J, K) = (B, C)$. Combine the scale identities in (2.2), the source bounds (2.7) and the failed $Q1$ inequality. They give

$$fJ^2,\ gK^2 \ll tY^{1+\epsilon_0+l_0}. \tag{5.1}$$

The source identity $Zcd = n(n+1)$ and $Z \geq R > Y^{5/3-\epsilon_0}$ give

$$cd = \frac{n(n+1)}{Z} < \kappa_Y Y^{1/3+\epsilon_0}.$$

The denominator satisfies

$$cd = ab + L \ll Y^{1/3+\epsilon_0}, \quad n - m = \frac{L\, n(n+1)}{cd(m+n+1)} \gg Y^{1-\epsilon_0-l_0}.$$

The second estimate uses $n > Y/2$, $L \geq Y^{1/3-l_0}$, $m + n + 1 \leq 2Y$, and the first estimate. Substitution in the last two scale identities of (2.2) proves (5.1). The same congruence argument applies, using $\delta C^2 \equiv t \pmod{B^2}$ or $\gamma B^2 \equiv -t \pmod{C^2}$. It gives the asserted sector bound. □

Proposition 5.2 (Joint-corner rows)**.** *For every $\varepsilon > 0$,*

$$\begin{aligned} \#Q2u &\ll_\varepsilon Y^{2/3-2222179/15000000+\varepsilon} + Y^{2/3-30869/1250000+\varepsilon}, \\ \#Q2t &\ll_\varepsilon Y^{2/3-972179/7500000+\varepsilon} + Y^{2/3-S_0+\varepsilon}. \end{aligned} \tag{5.2}$$

The terms in each line are additive contributions to one chart population. They are not separate first-match rows.

Proof. On $Q2u$ or $Q2t$, (2.7) and the failed earlier tests give

$$\begin{gathered} Y^{q_{\min}} < M < \kappa_Y^2 Y^{q_{\max}}, \qquad Y^{1/3-l_0} \leq L < \kappa_Y Y^{1/3+\epsilon_0}, \\ 1 \leq s < \sqrt{\kappa_Y}\, Y^{1/6+\epsilon_0/2}. \end{gathered} \tag{5.3}$$

The last inequality follows from $ut = L$ and $s = \min(u, t) \leq \sqrt{L}$.

We also need a lower bound for the product of the selected corners. Again $Zcd = n(n+1)$ and $Z \geq R > Y^{5/3-\epsilon_0}$, so $cd < \kappa_Y Y^{1/3+\epsilon_0}$. The identity

$$\frac{n-m}{L} = \frac{n(n+1)}{cd(m+n+1)}$$

and $n > Y/2$ imply

$$AD = u\frac{n-m}{L} \gg uY^{2/3-\epsilon_0}.$$

Reducing the exponent by $\frac{3}{2}\epsilon_0$ absorbs the fixed constant for all sufficiently large Y. A sector has $AD < 4P_0$. So its nonemptiness implies

$$P_0 > \frac{uY^{2/3-5\epsilon_0/2}}{32} \quad \text{in the } u \text{ chart.} \tag{5.4}$$

In the t chart, $BC = t(m+n+1)/L$, so (5.3) gives

$$BC \gg tY^{2/3-\epsilon_0}, \qquad P_0 > \frac{tY^{2/3-5\epsilon_0/2}}{32}. \tag{5.5}$$

We reduce the exponent by $3\epsilon_0/2$ from the scale bound $BC \gg tY^{2/3-\epsilon_0}$. This absorbs fixed dyadic constants uniformly. This gives the exponent used in (5.7).

Apply Lemma 5.1 in each occupied sector. In the u chart, its two terms are bounded by (5.4) and $\mu_{\mathrm{sec}} < Y^{1/3-j_0}$ as

$$Y^\eta \left(Y^{1/3+5\epsilon_0/2} + \frac{Y^{2/3-3j_0+5\epsilon_0}}{u} \right). \tag{5.6}$$

For the t chart, (5.1), (5.5), and the same corner bound give

$$Y^\eta \left(Y^{1/3+7\epsilon_0/2+l_0} + \frac{Y^{2/3-3j_0+6\epsilon_0+l_0}}{t} \right). \tag{5.7}$$

The source has $s \le \sqrt{\kappa_Y} Y^{1/6+\epsilon_0/2}$. Sum the first terms of (5.6)–(5.7) over this many shifts. This adds $1/6+\epsilon_0/2$ to their exponents. The second terms are harmonic in the selected shift and add only $O(\log Y)$. There are $O(\log^2 Y)$ corner sectors.

Fix a coefficient f, a sector corner pair and a shift. Then the conic forces the other coefficient g. The number of ordered residual splits of f, g is at most $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g) \ll_\eta Y^\eta$. The inverse maps (2.2) then determine the remaining corners. The full canonical filter is retained. Choose each internal exponent first. Then the divisor factors, root factors, dyadic logarithms and the harmonic sum use a total arbitrarily small loss.

Substitute $\epsilon_0 = k_0 = S_0$, $l_0 = 2S_0$ and $j_0 = 3S_0$. This gives the four exponents

$$\begin{aligned} &1/2+3S_0 = 2/3 - 2222179/15000000, && 2/3-9S_0+5S_0 = 2/3-30869/1250000,\\ &1/2+4S_0+2S_0 = 2/3-972179/7500000, && 2/3-9S_0+6S_0+2S_0 = 2/3-S_0. \end{aligned} \tag{5.8}$$

This proves (5.2). □

A determinant estimate for the short-shift rows. Write

$$e = S_0 = \epsilon_0 = k_0 = \frac{30869}{5000000}, \quad a_0 = \frac{6}{125}, \quad \nu = \frac{1}{64}, \quad \xi = \frac{1}{10000},$$

$$q_- = \frac{2}{3} - e, \quad q_+ = \frac{2}{3} + 2e, \quad \tau_0 = 1 - \frac{3}{2}e. \tag{5.9}$$

Thus $q_- = 9907393/15000000$, $q_+ = 5092607/7500000$, and $\tau_0 = 9907393/10000000$. The high-source inequalities give

$$m > \frac{Y^{\tau_0}}{\sqrt{3}}, \qquad Y^{q_-} < M < \kappa_Y^2 Y^{q_+}, \qquad \kappa_Y = 1 + Y^{-1}. \tag{5.10}$$

The first inequality follows from (2.8). The lower bound for M follows from failure of the O_M predicate in the first-match partition (3.1). Its upper bound follows from failure of O_R and (2.7).

We first give the determinant estimate for each fixed shift and coefficient box. Suppose $r_0, s_0 \ge 1$, $K_{\mathrm{det}} = (r_0+1)(s_0+1)$, and K_{det} source points lie in one cell. In both applications below, scale the two projective coordinates by the dyadic box centers. Then a determinant of the K_{det} monomial rows has the form

$$|\Delta_{\mathrm{det}}| \le C_{r_0,s_0} (H_1H_2)^{r_0K_{\mathrm{det}}/2} (EF)^{s_0K_{\mathrm{det}}/2} \max_{\substack{n_\ell \ge 0 \\ \sum_\ell n_\ell = K_{\mathrm{det}}}} \vartheta^{\sum_\ell \ell n_\ell} \lambda^{\sum_\ell n_\ell(n_\ell-1)/2}. \tag{5.11}$$

Here E, F are coefficient scales and H_1, H_2 are corner scales. The cell width in normalized coordinates is λ, and ϑ bounds the thickness of the source curve. In the u-chart, the remaining functions in a level-ℓ minor are x^{i-2j}, $x \in [1/2, 2]$. Here ϑ bounds θ in $(1-\theta)^j$. In the t-chart, put

$$\sigma_t = H_B/H_C, \qquad x = (B/C)/\sigma_t, \qquad \kappa_t = \sigma_t^2 E/F, \qquad y = (E/F)t/(\gamma C^2).$$

Then $x \in [1/2, 2]$, $1/16 < \kappa_t < 8$, and the normalized coefficient ratio is

$$(E/F)(\delta/\gamma) = \kappa_t x^2 + y.$$

The level-ℓ functions are therefore

$$x^i(\kappa_t x^2 + y)^j = \sum_{\ell=0}^{j} \binom{j}{\ell} \kappa_t^{j-\ell} y^\ell x^{i+2j-2\ell}.$$

Fix the determinant degree. The derivatives in x of these finitely many monomials are then uniformly bounded, since $x \in [1/2, 2]$ and $\kappa_t \in [1/16, 8]$. The thickness is the normalized term y. It is comparable to Ξ_t^{-1} in the $3t$ application below. This normalization gives the same level and cell-width gains in (5.11). To verify (5.11), expand each monomial column as a sum of level columns and use determinant multilinearity in the columns. In each resulting determinant, group

columns with the same level and apply Laplace expansion over row subsets of the corresponding sizes. In a level-ℓ minor, factor the ℓ-th power of the thickness coordinate from each evaluation row. The resulting thickness exponent is $\sum_\ell \ell n_\ell$. If two evaluation points or two powers agree, its remaining determinant is zero. Otherwise take n evaluation points in an interval of length λ. Take n functions with bounded derivatives through order $n-1$. Successive divided differences factor out the Vandermonde product. The remaining determinant is at most a constant times the derivative bounds. This gives $O(\lambda^{n(n-1)/2})$. Thus this group contributes $C_{r_0,s_0}\lambda^{n_\ell(n_\ell-1)/2}$. Here all derivatives of the finitely many powers are bounded on the fixed compact interval. Row and column scaling gives the first factor in (5.11).

Put $b_{\mathrm{cell}} = \log_Y(1/\lambda)$ and $\mu_{\mathrm{th}} = \log_Y(1/\vartheta)$. In the source boxes these exact quantities equal their limiting exponents up to $O(1/\log Y)$. Then

$$\sum_{\ell\ge 0}\left(\mu_{\mathrm{th}}\ell n_\ell + \frac{b_{\mathrm{cell}}}{2}n_\ell(n_\ell-1)\right) \ge \frac{2\sqrt{2}}{3}\sqrt{b_{\mathrm{cell}}\mu_{\mathrm{th}}}\,K_{\mathrm{det}}^{3/2} - \left(\mu_{\mathrm{th}} + \frac{b_{\mathrm{cell}}}{2}\right)K_{\mathrm{det}}. \tag{5.12}$$

For every $\Lambda > 0$, termwise completion of the square gives

$$\sum_{\ell}\left(\mu_{\mathrm{th}}\ell n_\ell + \frac{b_{\mathrm{cell}}}{2}n_\ell^2\right) \ge \Lambda K_{\mathrm{det}} - \frac{1}{2b_{\mathrm{cell}}}\sum_{\ell\ge 0}(\Lambda - \mu_{\mathrm{th}}\ell)_+^2 \ge \Lambda K_{\mathrm{det}} - \frac{\Lambda^2}{2b_{\mathrm{cell}}} - \frac{\Lambda^3}{6b_{\mathrm{cell}}\mu_{\mathrm{th}}}.$$

Taking $\Lambda = (2b_{\mathrm{cell}}\mu_{\mathrm{th}}K_{\mathrm{det}})^{1/2}$ gives the leading term in (5.12). The square-completion loss is $\Lambda^2/(2b_{\mathrm{cell}}) = \mu_{\mathrm{th}}K_{\mathrm{det}}$. Replacing n_ℓ^2 by $n_\ell(n_\ell-1)$ costs a separate $(b_{\mathrm{cell}}/2)K_{\mathrm{det}}$. For $r_0 = \lceil s_0q/p\rceil$ or $r_0 = \lfloor s_0q/p\rfloor$,

$$\frac{r_0p + s_0q}{2} \le s_0q + \frac{p}{2}, \qquad K_{\mathrm{det}}^{1/2} \ge (r_0s_0)^{1/2}.$$

Here p and q are the logarithmic scale exponents of the corner product and coefficient product in the chosen chart. The $3u$ and $3t$ applications below specify their normalizations. Hence a fixed positive margin in

$$b_{\mathrm{cell}}\mu_{\mathrm{th}} > \frac{9}{8}pq \tag{5.13}$$

makes every integerized determinant have absolute value less than 1. Here p, q, μ_{th} range over fixed compact intervals in both applications. The strict margin is uniform over the full source boxes. Fix that margin and those compact intervals, and then choose s_0. The degree-rounding and lower-order terms are then smaller than the margin. This one fixed degree works for every sufficiently large Y. The integerized matrix has rank below K_{det}. Let $r < K_{\mathrm{det}}$ be the largest order of a nonzero minor. If $r = 0$, a coordinate vector is a kernel vector. If $r > 0$, choose a maximal nonzero $r \times r$ minor and one additional column. Cramer's rule gives a nonzero integral kernel vector. Its coordinates are minors of order at most r. The selected rows span the row space, so this vector annihilates every row. Thus the kernel vector has polynomially bounded height and gives a nonzero integral auxiliary polynomial. If a cell contains fewer than K_{det} points, those points are counted directly. The degrees are fixed. The coefficient height of the polynomial is a fixed power of Y. This holds because it is a minor of a matrix whose entries have fixed polynomial height.

Heath–Brown's Lemma 2 [4, Lemma 2, p. 8], stated in Appendix E, has no size restriction on the second pair. In $3u$ and $3t$, the primitive coefficient pair is the first pair and the primitive corner pair is the second. For an irreducible factor, let a_c be its degree in the corner pair. Let b_q be its degree in the coefficient pair. In the theorem's notation this is $(a, b) = (b_q, a_c)$. Thus a factor with $a_c \ge 2$ has second-pair degree at least two. Any application of Heath–Brown below has $b_q \ge 1$. So both bidegrees are positive. The theorem's exponent is $2/a_c \le 1$. Here $T_1 + 1$ and the factor height are bounded by fixed powers of Y. Choose the theorem's ε after those height exponents. This gives $O_\eta(Y^\eta(T_1+1))$.

Let a bounded-degree integer polynomial have height Y^C. Then each primitive integer factor has height $Y^{O_C(1)}$. For the proof, apply a fixed Kronecker substitution with base larger than the total degree. Then use Gauss's lemma and the univariate Landau–Mignotte factor bound. The substitution has no collisions among the bounded-degree monomials.

The short-u row $3u$. In the u-chart put $\alpha = ac$, $\beta = bd$. The source identities (2.2) give

$$\beta D^2 - \alpha A^2 = u, \qquad \alpha A^2 = (1-\theta)\beta D^2, \qquad \theta = \frac{m+n+1}{(m+1)(n+1)} < \frac{2}{m+1}. \tag{5.14}$$

The residual factors are pairwise coprime, and so are the corner factors. Hence

$$\gcd(\alpha, \beta) = \gcd(A, D) = 1. \tag{5.15}$$

Also

$$\frac{um}{3} \le \alpha A^2 < um, \qquad \frac{um}{3} \le \beta D^2 \le u(m+1), \tag{5.16}$$

because $n/(m+n+1) \ge 1/3$. Multiply these two scale identities and use $\alpha\beta = M$. This gives

$$AD \ge \frac{um}{3\sqrt{M}} \gg uY^{2/3-5e/2}. \tag{5.17}$$

The implied constant is absolute for sufficiently large Y. The unit conditions needed below are also source consequences. If a prime divides A, it divides m, n. So it divides neither $b \mid m+1$ nor $d \mid n+1$. Hence $\gcd(\beta, A) = 1$. Also $uBC = m+n+1$ and $\gcd(A, BC) = 1$, so $\gcd(u, A) = 1$. Therefore $\gcd(\beta u, A) = 1$. Every corner is at most $Y+1$, and $\alpha, \beta \le u(Y+1)$.

Fix a positive shift u and a dyadic sector. The counted tuple is (α, A, D), with A, D in that sector. The conic determines the second coefficient β, subject to the integrality filter. Fix $u \le Y^{a_0}$ and dyadic scales

$$A \asymp H_A, \quad D \asymp H_D, \quad \alpha \asymp E, \quad \beta \asymp F.$$

There are $O((\log Y)^4)$ such boxes. Put

$$\chi = \frac{AH_D}{DH_A}, \quad \mu_b = \log_Y \frac{EH_A^2}{u}, \quad p = \log_Y(H_A H_D), \quad q = \log_Y(EF), \quad z = \log_Y u.$$

The box and (5.16) imply, uniformly,

$$\tau_0 - O(1/\log Y) \le \mu_b \le 1 + O(1/\log Y), \quad q_- - O(1/\log Y) \le q \le q_+ + O(1/\log Y),$$

$$p \le z + \mu_b - \frac{q}{2} + O(1/\log Y), \qquad 0 \le z \le a_0. \tag{5.18}$$

They also give

$$\frac{q_-}{a_0 + 2/3 + e/2} - O(1/\log Y) \le \frac{q}{p} \le \frac{q_+}{2/3 - 5e/2} + O(1/\log Y). \tag{5.19}$$

The exact endpoints in (5.19) are 19814786/21532607 and 20370428/19536965, both strictly between 0.9 and 1.1.

Set

$$B_u = \frac{9}{8} q_+ \left(1 + a_0 - \frac{q_+}{2}\right) = \frac{54121135983551}{100000000000000}, \qquad N_u = \left\lceil Y^{B_u + \xi} \right\rceil. \tag{5.20}$$

Let $f(q, z, \mu) = q(1 + (z - q/2)/\mu)$. Its derivatives in z, μ, q have signs $+, +, +$ on (5.18): for the last sign,

$$1 + \frac{z-q}{\mu} \ge \frac{9351751}{29722179} + O(1/\log Y) > 0,$$

and for the μ-derivative use $q_-/2 - a_0 > 0$. Hence

$$\frac{9}{8}\frac{pq}{\mu_b} \le B_u + O(1/\log Y). \tag{5.21}$$

Put $\delta_0 = \xi/4$. From (5.18), $\mu_b \geq \tau_0 - O(1/\log Y)$, and $p, q < 1 + O(1/\log Y)$. Therefore (5.21) gives the direct margin

$$(B_u + \xi)\mu_b - \frac{9}{8}(1+\delta_0)pq \geq \xi\tau_0 - \frac{9\xi}{32} - O(1/\log Y) > \frac{\xi}{2}$$

for all sufficiently large Y. The thickness in (5.14) satisfies $\theta \leq 2Y^{-\mu_b}$. The strict margin absorbs the factor 2.

Split $\chi \in (1/2, 2)$ into N_u half-open intervals of length $3/(2N_u)$. For $K_{\mathrm{det}} = (r_0+1)(s_0+1)$ source points in one interval, form the integer matrix whose $k, (i,j)$-entry is

$$A_k^i D_k^{r_0-i} \alpha_k^j \beta_k^{s_0-j},$$

where $0 \leq i \leq r_0$ and $0 \leq j \leq s_0$. Choose a fixed integer s_0 sufficiently large in terms of δ_0 and the compact ranges in (5.19). Put $r_0 = \lceil s_0 q/p \rceil$. By (5.19), the possible r_0 form a finite set and $r_0/s_0 = q/p + O(1/s_0) + O(1/\log Y)$ uniformly. The leading terms in (5.12) have a uniform positive margin by (5.21). The ceiling and all remaining terms in (5.12) are uniformly $O(1/s_0)+O(1/\log Y)$ relative to those leading terms. First choose s_0, then the common threshold for Y. Then these errors use less than half the margin. Apply the level estimate (5.11)–(5.13) with $b_{\mathrm{cell}} = B_u+\xi+O(1/\log Y)$ and $\mu_{\mathrm{th}} = \mu_b+O(1/\log Y)$. It makes every K_{det}-row determinant zero for one common sufficiently large Y. Each entry is an integer of size at most

$$(Y+1)^{r_0+(21/20)s_0},$$

since $u \leq Y^{a_0} < Y^{1/20}$. Hence cofactors give a nonzero integer polynomial of degree at most (r_0, s_0). Its coefficient height is Y^{C_ξ} with fixed C_ξ. This gives the required integer auxiliary polynomial.

We next count the lattice radii over the moving intervals. Write

$$\rho = \frac{\alpha/E}{\beta/F}, \qquad \rho = \kappa(1-\theta)\chi^{-2}, \qquad \kappa = \frac{FH_D^2}{EH_A^2}.$$

Since each normalized coefficient lies in $[1,2)$, actual ρ lies in $[1/2, 2]$. For each interval, crop its image in ρ to $(1/2, 2)$. Empty intervals contain no source. The map $\chi \mapsto \kappa\chi^{-2}$ has derivative bounded above and below on $[1/2, 2]$, uniformly in every nonempty box. We have $1/8 \leq \kappa \leq 16$ by the dyadic box inequalities and (5.14). Also $\theta \leq 2Y^{-\mu_b} < 1/N_u$. Hence the retained coefficient directions have angular width $O(1/N_u)$. Any angular window of width w meets $O(1 + N_u w)$ of them.

For a retained direction with unit vector ℓ and unit normal n, define

$$\mathcal{A}(\alpha,\beta) = \left(\ell \cdot (\alpha/E, \beta/F), \frac{N_u}{c_0} n \cdot (\alpha/E, \beta/F)\right),$$

where the fixed c_0 makes the images under $\mathcal{A}$ of the source coefficient vectors lie in the disk of radius 5. This linear map has determinant $N_u/(c_0 EF)$. Its image of the integer coefficient lattice has a Euclidean reduced basis b_1, b_2, with angle at least $60°$. The inverse images form a unimodular integer basis. For each source pair, write (v_1, v_2) for its coordinates in this reduced basis. These are distinct from the original coefficient coordinates (α, β). Put $T_i = 5\sqrt{2}/|b_i|$. Then $T_2 \leq T_1$, every source coefficient pair has lattice coordinates $|v_i| \leq T_i$, where $\mathbf{v} = (v_1, v_2)$, and

$$T_1 T_2 \gg \frac{EF}{N_u}, \qquad T_1^2 \gg \frac{EF}{N_u}. \tag{5.22}$$

The determinant of the lattice and the operator norms of $\mathcal{A}$ and $\mathcal{A}^{-1}$ are bounded by fixed powers of Y. Every nonzero integer preimage has image length at least $\|\mathcal{A}^{-1}\|^{-1}$. Minkowski's theorem bounds the shortest image vector by a constant times the square root of the covolume. The reduced-basis angle and determinant then bound the second basis vector by a fixed power

of Y as well. Apply $\mathcal{A}^{-1}$. Then both unimodular preimage vectors have polynomially bounded coordinates. Thus the coefficient-lattice change keeps the auxiliary polynomial's height at $Y^{O_\xi(1)}$.

We now bound the complete interval sum. Put $L_* = \max(1, (EF/N_u)^{1/2})$. Intervals with $T_1 \le 2L_*$ contribute $O(N_u + (EFN_u)^{1/2})$. For larger T_1, place it in a dyadic range $(T, 2T]$. Let $\mathbf{x}_{\min} = (x_1, x_2)$ be the primitive shortest integer preimage. The definition of $\mathcal{A}$ gives

$$|(x_1/E, x_2/F)| \ll T^{-1}, \qquad \operatorname{dist}((x_1/E, x_2/F), \ell) \ll (N_u T)^{-1}.$$

A nonaxis dyadic annulus of normalized radius r contains $O(r^2EF)$ integer vectors, with $r \ge \max(E^{-1}, F^{-1})$. A fixed vector is compatible with $O(1+(Tr)^{-1})$ interval directions. Sum $EF(r^2 + r/T)$ over the annuli up to $r \ll T^{-1}$. This gives $O(EF/T^2)$ intervals in this T-range. Their contribution to $\sum(T_1+1)$ is $O(EF/T)$. The vertical and horizontal primitive vectors are the two axes. The retained source directions have coordinate ratio $\rho \in [1/2, 2]$. So they stay a fixed positive angle from both axes. The corresponding vertical and horizontal cases cost $O(F + N_u)$ and $O(E + N_u)$, respectively. Finally $T_1 \gg (EF/N_u)^{1/2}$ by (5.22). Summing the dyadic ranges gives

$$\sum_{\text{intervals}} (T_1 + 1) \ll_\eta Y^\eta \{N_u + \sqrt{EFN_u}\log Y + E + F\}. \tag{5.23}$$

For each interval, factor its auxiliary polynomial over $\mathbb{Q}$. There are $O_\xi(1)$ factors, each of fixed degree and height $Y^{O_\xi(1)}$. Make the unimodular coefficient-lattice change. Then let an irreducible factor have degree a_c in the corner pair and b_q in the coefficient pair. Write the original coefficients $\alpha(\mathbf{v}), \beta(\mathbf{v})$ as homogeneous linear forms in the primitive lattice coordinates $\mathbf{v} = (v_1, v_2)$. The following cases account for every factor.

(1) If it is not absolutely irreducible, its distinct conjugate components have no common curve component. By bihomogeneous Bezout they have $O_\xi(1)$ rational common points. Each rational point gives at most four signed primitive pairs.

(2) If $a_c = 0$, the factor fixes $O_\xi(1)$ primitive coefficient ratios. For each, (5.14) is a square-fibre equation. It has positive coefficients α, β and nonzero shift u. The coefficients and shift are at most $(Y+1)^{21/20}$, and $A \le Y+1$. Theorem B.1 with $x = 2Y$ and fixed height exponent $A_{\mathrm{ht}} = 2$ gives Y^η solutions uniformly.

(3) If $b_q = 0$, the factor fixes $O_\xi(1)$ primitive corner pairs. The solutions in (α, β) to $\beta D^2 - \alpha A^2 = u$ form a progression with step (D^2, A^2). Its length in the α-box is at most

$$1 + \frac{E}{D^2} \ll 1 + \frac{uY}{A^2D^2} \ll 1 + \frac{Y^{-1/3+5e}}{u} = O(1)$$

uniformly for $u \ge 1$, by (5.16)–(5.17).

(4) If $a_c \ge 2$ and $b_q \ge 1$, apply Heath–Brown's Lemma 2. The first pair is the primitive coefficient-lattice pair, bounded by $T_1 + 1$. The second is the primitive corner pair. Its degree in the second pair is $a_c \ge 2$. Choose the theorem's epsilon after the fixed polynomial height exponent. Hence the count is $O_\eta(Y^\eta(T_1+1))$.

(5) If $a_c = 1$ and $b_q \ge 1$, write the factor as $AQ(\mathbf{v}) + DP(\mathbf{v})$, with coprime nonzero homogeneous forms P, Q. The source equation forces $P(\mathbf{v}) = -A\zeta$, $Q(\mathbf{v}) = D\zeta$ for a nonzero integer ζ. The Sylvester identities put $\operatorname{Res}(P,Q)v_i^{2b_q-1}$ in the ideal (P, Q). Primitivity of $\mathbf{v}$ therefore gives $\zeta \mid \operatorname{Res}(P, Q)$. The resultant has fixed polynomial height, so its divisors cost Y^η. Fix a divisor $\zeta \mid \operatorname{Res}(P, Q)$. Then the source conic gives the nonzero homogeneous level equation

$$\beta(\mathbf{v})Q(\mathbf{v})^2 - \alpha(\mathbf{v})P(\mathbf{v})^2 = u\zeta^2.$$

The left side is a nonzero binary form of degree $d_\Phi = 2b_q + 1$. If it vanished identically, coprimality of P, Q would force P^2 to divide the linear form $\beta(\mathbf{v})$. This is impossible

for $b_q \geq 1$. Denote this form by Φ. Its level is $h = u\zeta^2 \neq 0$. Define $\mathcal{E} = \{v_1 \in \mathbb{Z} : \Phi(v_1, v_2) - h \text{ is identically zero in } v_2\}$. Writing $\Phi(v_1, 0) = a_\Phi v_1^{d_\Phi}$, every $v_1 \in \mathcal{E}$ satisfies $a_\Phi v_1^{d_\Phi} = h$, so $|\mathcal{E}| \leq d_\Phi$. If $a_\Phi = 0$, then $\mathcal{E} = \varnothing$. This includes the pure-power case. There every slice is constant, and at most d_Φ slices equal the nonzero target h. There are at most $2\lfloor T_1 \rfloor + 1$ other first coordinates. For each, the nonzero polynomial in v_2 has degree at most d_Φ. Hence it has at most d_Φ integer roots. Each exceptional coordinate contributes at most $2\lfloor T_2 \rfloor + 1$ points. Thus

$$\begin{aligned}\#\{(v_1, v_2) \in \mathbb{Z}^2 : |v_i| \leq T_i,\ \Phi(v_1, v_2) = h\} &\leq d_\Phi(2\lfloor T_1 \rfloor + 1) + d_\Phi(2\lfloor T_2 \rfloor + 1) \\ &\leq 4d_\Phi(T_1 + 1), \qquad T_2 \leq T_1.\end{aligned}$$

The corners are then forced.

Thus every full interval contributes $Y^\eta(T_1 + 1)$. The underfull intervals cost $O_\xi(N_u)$. Fix the ordered residual factorizations $\alpha = ac,\ \beta = bd$. The inverse formulas (2.2) determine B, C, then $m = ABa, n = ACc$. The positivity, integrality, gcd, mask, shell and first-match filters only discard candidates. The inverse multiplicity is at most $\tau_{\mathrm{div}}(\alpha)\tau_{\mathrm{div}}(\beta) \ll_\eta Y^\eta$, since $\alpha, \beta \leq (Y + 1)^{21/20}$. The opposite shift is recovered from $ut = L$. Combine this with (5.23). For one fixed u and one box the count is

$$\ll_\eta Y^\eta\{N_u + \sqrt{EFN_u}\log Y + E + F\}. \tag{5.24}$$

Split the $3u$ population counted by (5.24) into $\min(A, D) \geq Y^{1/3-\nu}$ and its complement. In the first part, (5.16) gives

$$E + F \ll uY^{1/3+2\nu}. \tag{5.25}$$

In the second part, suppose $H_A \leq H_D$. The other case is symmetric. For fixed A, β, the congruence $\beta D^2 \equiv u \pmod{A^2}$ has $O(2^{\omega(A)}) = Y^\eta$ roots modulo A^2, since $\gcd(\beta u, A) = 1$. Each root class contributes $O(H_D/A^2 + 1)$ values of D. Also $\beta \ll uY/H_D^2$. Summing over A, β gives

$$\ll_\eta Y^\eta \left\{ \frac{uY}{H_A H_D} + \frac{uYH_A}{H_D^2} \right\}. \tag{5.26}$$

For odd prime powers the unit-square congruence has at most two roots. At powers of 2 it has at most four. The absolute factor 4 is absorbed in $O(2^{\omega(A)})$. By (5.17) and $H_A \leq Y^{1/3-\nu}$, the two terms in (5.26) are at most

$$Y^{1/3+5e/2+o(1)}, \qquad \frac{Y^{2/3+5e-3\nu+o(1)}}{u}. \tag{5.27}$$

The second term sums harmonically over u. The first costs $\sum_{u \leq Y^{a_0}} 1 \ll Y^{a_0}$.

Finally $EF \leq M < \kappa_Y^2 Y^{q_+}$. Sum (5.24)–(5.27) over $u \leq Y^{a_0}$, all boxes and the logarithmic factors. Before the final loss allocation, this gives five exponents and their savings from 2/3

term and exponent	saving from 2/3
square-root term: $a_0 + (q_+ + B_u + \xi)/2$	$\dfrac{5102312049347}{600000000000000}$ $(0.00850385\ldots)$
interval term: $a_0 + B_u + \xi$	$\dfrac{23206592049347}{300000000000000}$
core axes: $1/3 + 2\nu + 2a_0$	$\dfrac{2473}{12000}$
small corner, first: $a_0 + 1/3 + 5e/2$	$\dfrac{1619393}{6000000}$
small corner, second: $2/3 + 5e - 3\nu$	$\dfrac{8003}{500000}$

(5.28)

The smallest saving is $5102312049347/600000000000000 > S_0$. Fix the degree and height constants. Then all fixed-height, divisor, dyadic and logarithmic losses can be assigned arbitrarily small exponents. Therefore, for every $\varepsilon > 0$,

$$\#3u \ll_\varepsilon Y^{2/3-5102312049347/600000000000000+\varepsilon}. \tag{5.29}$$

The next proposition records this display as the complete first-match row estimate.

Proposition 5.3 ($3u$ row estimate)**.** *For every $\varepsilon > 0$, the complete first-match row satisfies*

$$\#3u \ll_\varepsilon Y^{2/3-5102312049347/600000000000000+\varepsilon}.$$

Proof. The full row count is (5.29). Its five additive terms are bounded in (5.24)–(5.28). The least saving is the one displayed there. □

The short-t row $3t$. In the t-chart use $\gamma = ad, \delta = bc$. The exact source equations are

$$\delta C^2 - \gamma B^2 = t, \qquad \frac{\delta}{\gamma} = \left(\frac{B}{C}\right)^2 (1+\beta_{\mathrm{rem}}), \qquad \beta_{\mathrm{rem}} = \frac{n-m}{m(n+1)} \in (0, 1/m). \tag{5.30}$$

By (5.10), $\beta_{\mathrm{rem}} \ll Y^{-\tau_0}$. The source heights and the O_R, O_M, $Q1$ and $Q2t$ complements give

$$\frac{t}{4} Y^{q_-} < BC \le 3tY^{c_t}, \quad \gamma, \delta \ll tY^{g_t}, \quad \frac{\gamma}{C^2} \le \frac{6Y^{-d_0}}{t} < 1 \quad \text{for all sufficiently large } Y, \tag{5.31}$$

where

$$c_t = \frac{2}{3} + l_0 = q_+, \quad g_t = \frac{1}{3} + e + l_0 + 2j_0 = \frac{5833463}{15000000}, \quad d_0 = \frac{1}{3} - 2e > 0.$$

For the γ/C^2 bound, start from the exact source identity

$$n - m = \frac{ZL}{m+n+1}.$$

Then use $BC = (m+n+1)t/L$, $m+n+1 > Y/2$, and $L < \kappa_Y Y^{1/3+e}$. Since $2\kappa_Y \le 3$, this gives $BC > (t/3)Y^{2/3-e} > (t/4)Y^{q_-}$. For the upper bound use $L \ge Y^{1/3-l_0}$ and $m+n+1 \le 2Y+1 \le 3Y$. This gives $BC \le 3tY^{2/3+l_0}$. Also

$$\gamma B^2 = \frac{t\, m(n+1)}{n-m}, \qquad \gamma/C^2 = \frac{m(n+1)}{ZBC};$$

using $Z > Y^{5/3-e}$ and the lower bound for BC proves the last inequality in (5.31). The bound for δ is symmetric. It uses $\delta C^2 = t\, n(m+1)/(n-m)$ and $B, C \ge Y^{1/3-j_0}$.

Fix $t \le Y^{a_0}$ and a dyadic box

$$B \asymp H_B, \quad C \asymp H_C, \quad \gamma \asymp E, \quad \delta \asymp F.$$

Put $\mathcal{L} = \ln Y$, $p = \log_Y(8H_BH_C)$, $q = \log_Y(128EF)$, $z = \log_Y t$, and

$$\Xi_t = \frac{H_BH_C\sqrt{EF}}{8t}, \quad \mu_0 = \log_Y \Xi_t, \quad \ell_1 = \ln(8H_BH_C), \quad \ell_2 = \ln(128EF).$$

From (5.10)–(5.31), with $\kappa_Y \le 3/2$,

$$q_-\mathcal{L} - \ln 2 < \ell_1 \le (c_t + a_0)\mathcal{L} + \ln 24, \quad q_-\mathcal{L} + \ln 32 < \ell_2 < q_+\mathcal{L} + \ln 288,$$

$$\tau_0\mathcal{L} - \ln 256 < \ln \Xi_t < (1 + e + l_0)\mathcal{L}, \qquad p - z \ge q_- - O(1/\mathcal{L}). \tag{5.32}$$

Dyadicity gives $BC/4 < H_BH_C \le BC$ and $EF \le M < 4EF$. With (5.31) these imply $8H_BH_C > tY^{q_-}/2$, $8H_BH_C \le 24tY^{c_t}$, $128EF > 32Y^{q_-}$, and $128EF < 288Y^{q_+}$. They also give

$$\frac{Y^{\tau_0}}{256} < \Xi_t \le \frac{3\kappa_Y}{8} Y^{3q_+/2} \le \frac{9}{16} Y^{1+e+l_0},$$

which proves the two Ξ_t bounds in (5.32). In particular $\ell_1, \ell_2, \ln \Xi_t$ lie in $[\mathcal{L}/5, 2\mathcal{L}]$ for $Y \ge 2048$.

Set $\sigma_t = H_B/H_C$ and $x = (B/C)/\sigma_t$. The source has $x \in (1/2, 2)$. Partition $x \in (1/2, 2)$ into

$$I_k = \left(\frac{k}{N}, \frac{k+1}{N}\right].$$

Equivalently, these are the intervals $(k\sigma_t/N, (k+1)\sigma_t/N]$ for B/C. Here $N/2 - 1 \le k \le 2N - 1$, with $N \ge 40$ to be fixed below. Let $x_4 = \lfloor k^2/N \rfloor$ and $r_k = x_4\sigma_t^2/N$. For $x \in I_k$, the width of the interval and (5.30) give

$$|\delta - r_k\gamma| \le 18\sigma_t^2 E/N, \qquad 0 < \gamma < 2E. \tag{5.33}$$

The square of B/C differs from r_k by at most $5\sigma_t^2/N$. The additional term $(B/C)^2\beta_{\mathrm{rem}}$ is at most $4\sigma_t^2/m$, and the choice below has $N < m$. The endpoint $k \le 2N - 1$ follows from $x < 2$. This is the range used in the $5\sigma_t^2/N$ bound. A nonempty box also gives

$$F/16 < \sigma_t^2 E < 8F. \tag{5.34}$$

Define the linear map

$$\mathcal{A}_k(y_1, y_2) = \left(\frac{N}{\sigma_t^2 E}(y_1 - r_k y_2), \frac{y_2}{E}\right).$$

Its image of $\mathbb{Z}^2$ has covolume $N/(\sigma_t^2 E^2) \asymp N/(EF)$. By (5.33), coefficient pairs (δ, γ) in the box map into the disk of radius 19. Choose a Euclidean reduced basis b_1, b_2 for this lattice. Let h_1, h_2 be its unimodular integer preimages. Put $T_i = 19\sqrt{2}/|b_i|$. The reduced-basis angle is at least $60°$. Hence $T_2 \le T_1$, every source pair has coordinates $|v_i| \le T_i$, where $\mathbf{v} = (v_1, v_2)$, and (5.34) gives

$$T_1T_2 = \frac{722}{|b_1||b_2|} \ge \frac{361\sqrt{3}}{\mathrm{covol}(\mathcal{A}_k(\mathbb{Z}^2))} = \frac{361\sqrt{3}\,\sigma_t^2E^2}{N} > \frac{361\sqrt{3}}{16}\frac{EF}{N} > 39\frac{EF}{N}, \qquad T_1^2 \ge T_1T_2. \tag{5.35}$$

The matrix $\mathcal{A}_k$, its inverse, and the covolume have polynomial height. Every nonzero integer preimage has image length at least $\|\mathcal{A}_k^{-1}\|^{-1}$. Minkowski's theorem bounds the shortest image vector by a constant times the square root of the covolume. The reduced-basis angle and determinant then give a polynomial upper bound for the second vector. Apply $\mathcal{A}_k^{-1}$. Then both unimodular integer preimages have polynomially bounded coordinates.

We now count the moving grid. The integer x_4 lies in $[N/5, 4N]$. Each x_4 comes from at most two k, since

$$\sqrt{Nx_4} \le k < \sqrt{N(x_4 + 1)}$$

has length at most $\sqrt{5}/2 < 2$ for $x_4 \ge N/5$. Therefore it contains at most two integers. Take a dyadic range $T_1 \in (T, 2T]$. Write $\mathcal{A}_k(y) = b_1$ for the primitive shortest vector $y = (y_1, y_2)$. Then

$$|y_1 - r_k y_2| \ll \sigma_t^2 E/(NT), \qquad |y_2| \ll E/T. \tag{5.36}$$

If $y_2 = 0$, primitivity gives $y_1 = \pm 1$. The contribution over all k is then $O(F + N)$. If $y_1 = 0$, then $y_2 = \pm 1$. Since $x_4 \ge N/5$, (5.36) bounds its total contribution by $O(E + N)$. For $y_1y_2 \ne 0$, (5.36) gives $T \ll E$ and $|y_1| \ll F/T$. Multiply the first inequality in (5.36) by N/σ_t^2. Then, for each fixed y_1, the nonzero integer y_2x_4 lies in an interval of length $O(E/T)$. Since $E/T \gg 1$, there are $O(E/T)$ such integer values. Their absolute values satisfy $|y_2x_4| \ll NE/T$, a fixed power of Y. So each has Y^η factorizations as y_2x_4. Each x_4 has at most two k. Thus the number of intervals in this T-range is $O_\eta(Y^\eta EF/T^2)$. This includes the endpoint cases $|y_1| \le CF/T$ and $E/T \ge c > 0$. Their contribution to $\sum_k(T_1(k) + 1)$ is $O_\eta(Y^\eta EF/T)$. By (5.35), $T \gg (EF/N)^{1/2}$. Sum the dyadic T-ranges and pay the $+1$ terms. This gives

$$\sum_k (T_1(k) + 1) \ll_\eta Y^\eta\{N + \sqrt{EFN}\log Y + E + F\}. \tag{5.37}$$

For each interval I_k, apply the determinant construction (5.11) to the normalized monomials $x^i\zeta^j$, where $\zeta = (E/F)(\delta/\gamma) = \kappa_t x^2 + y$. Scaling rows and columns back to the integer monomials

gives (5.41). We choose the grid size N used above as follows. Put

$$B_t = \frac{3}{4}(q_+ + a_0) = \frac{5452607}{10000000}, \quad \nu_t = B_t + \xi = \frac{5453607}{10000000}, \quad N_t = \lceil Y^{\nu_t} \rceil. \tag{5.38}$$

The exact gap

$$\tau_0 - \nu_t = \frac{2226893}{5000000} > 0$$

shows that $N_t < m$ for all sufficiently large Y. Here $m > Y^{\tau_0}/\sqrt{3}$ and $N_t \le 2Y^{\nu_t}$. In (5.33)–(5.37) take $N = N_t$. Since $EF > M/4 > Y^{q_-}/4$, we have

$$\frac{EF}{N_t} \gg Y^{q_- - \nu_t} = Y^{690793/6000000}.$$

Thus (5.35) puts every dyadic radius T above $cY^{690793/12000000} > 1$ for all sufficiently large Y. The separately counted $+1$ per grid interval is already included in the $O(N_t)$ term of (5.37).

The determinant thickness ϑ in the normalized coefficient ratio is comparable to Ξ_t^{-1}. The conic gives

$$\frac{\delta}{\gamma} = \left(\frac{B}{C}\right)^2 + \frac{t}{\gamma C^2}.$$

Normalize the coefficient ratio by E/F. Then the thickness is $(E/F)t/(\gamma C^2) \asymp t/(FH_C^2)$. From (5.34), $FH_C^2 \asymp EH_B^2 \asymp H_B H_C \sqrt{EF}$. Hence this thickness is $\asymp \Xi_t^{-1}$. More precisely, the dyadic bounds $E \le \gamma < 2E$, $H_C \le C < 2H_C$, and $1/16 < \sigma_t^2 E/F < 8$ imply

$$\frac{\Xi_t^{-1}}{256} \le \vartheta \le \Xi_t^{-1}.$$

Thus $0 \le \mu_{\mathrm{th}} - \mu_0 \le \ln 256/\mathcal{L}$. The normalized cell width is exactly $1/N_t$, so $\nu_t \le b_{\mathrm{cell}} = \log_Y N_t \le \nu_t + \ln 2/\mathcal{L}$.

The box bounds also give

$$\frac{q_-}{c_t + a_0} + O(1/\mathcal{L}) \le \frac{\ell_2}{\ell_1} \le \frac{q_+}{q_-} + O(1/\mathcal{L}).$$

The limiting endpoints are $9907393/10905214 > 0.9$ and $10185214/9907393 < 1.1$. So this ratio belongs to $[0.9, 1.1]$ for all sufficiently large Y. Choose a fixed integer s_0 sufficiently large in terms of the strict margin below and this compact range. Set

$$r_0 = \left\lfloor s_0 \frac{\ell_2}{\ell_1} \right\rfloor.$$

The possible r_0 form a finite set. The degree rounding and the lower-order terms in (5.12) are uniformly $O(1/s_0)$ relative to the leading terms. Choose s_0 first. Then enlarge the threshold for Y to absorb the $O(1/\mathcal{L})$ errors.

The normalized scale parameters in (5.11) are $p = \ell_1/\mathcal{L}$, $q = \ell_2/\mathcal{L}$, and $\mu_{\mathrm{th}} = p + q/2 - z + O(1/\mathcal{L})$. Precisely,

$$\mu_0 = p + q/2 - z - \frac{\log(64\sqrt{128})}{\mathcal{L}}.$$

The displayed constant and the additional $O(1/\mathcal{L})$ from the thickness comparison are among the finite errors accounted for below. The limiting function

$$(p, q, z) \longmapsto \frac{9}{8} \frac{pq}{p + q/2 - z}$$

is increasing in p, q, z on the source range. The relevant derivative numerators are $q(q/2-z)$, $p(p-z)$ and pq. With $D = p + q/2 - z$, its three partial derivatives are

$$\frac{9}{8}\frac{q(q/2-z)}{D^2}, \qquad \frac{9}{8}\frac{p(p-z)}{D^2}, \qquad \frac{9}{8}\frac{pq}{D^2}.$$

On the stated source domain each has absolute value at most $(9/8)\cdot 4\cdot 25 = 112.5 < 120$. Here

$$q_-/2 - a_0 = \frac{8467393}{30000000} > 0, \qquad q_- - a_0 = \frac{9187393}{15000000} > 0.$$

The bound $BC \le 3tY^{c_t}$ gives the one-sided endpoint estimates

$$p \le c_t + z + \frac{\ln 24}{\mathcal{L}}, \qquad q \le q_+ + \frac{\ln 288}{\mathcal{L}}.$$

First put $p = c_t + z + O(1/\mathcal{L})$. Then the denominator is $c_t + q/2 + O(1/\mathcal{L})$, and $q/(c_t + q/2)$ increases. Thus the maximum is $B_t + O(1/\mathcal{L})$, attained at $q = q_+$, $z = a_0$. For $\delta_0 = \xi/4 = 1/40000$,

$$\nu_t - (1+\delta_0)B_t = \xi - \delta_0 B_t = \frac{34547393}{400000000000} > 0. \tag{5.39}$$

Now the uniform error. The upper endpoints in (5.32) change p, q from $c_t + z, q_+$ by at most $\ln 24/\mathcal{L}, \ln 288/\mathcal{L}$. The exact normalization of μ_0 contributes $\ln(64\sqrt{128})/\mathcal{L}$. The thickness comparison contributes at most $\ln 256/\mathcal{L}$. Take the source range $0 \le p, q, z \le 2$ and $p + q/2 - z \ge 1/5$. There every first derivative of $\Phi(p, q, z) = (9/8)pq/(p + q/2 - z)$ has absolute value at most 120. Since $\ln 24 < 4$, $\ln 288 < 6$, $\ln(64\sqrt{128}) < 7$, and $\ln 256 < 6$, the total perturbation is at most $2760/\mathcal{L}$. We may therefore take $C_t = 4000$. Hence, whenever

$$(\xi - \delta_0 B_t)\ln Y > 2C_t(1+\delta_0), \tag{5.40}$$

the determinant condition has the strict surplus required in (5.13). Take one interval with $K_{\det} = (r_0+1)(s_0+1)$ source rows. Use columns indexed by $0 \le i \le r_0,\ 0 \le j \le s_0$, and multiply each row by $C^{r_0}\gamma^{s_0}$. The resulting entry is

$$C^{r_0}\gamma^{s_0}(B/C)^i(\delta/\gamma)^j = B^i C^{r_0-i}\delta^j\gamma^{s_0-j} \in \mathbb{Z}. \tag{5.41}$$

We have $B, C \le Y + 1$ and $\gamma, \delta \ll Y^{a_0+g_t}$, and the degrees are fixed. So every matrix entry and cofactor has height $Y^{O(1)}$. By (5.11)–(5.13), every integerized $K_{\det} \times K_{\det}$ determinant has absolute value less than one. Hence it vanishes. If an interval has fewer than $K_{\det}$ source rows, count those points directly. Otherwise the integer matrix has a nonzero rational kernel vector. Clearing its denominator gives a nonzero integral auxiliary polynomial on that interval. It has bounded bidegree and height $Y^{O_\xi(1)}$.

In this t-chart, $\gcd(\gamma, \delta) = \gcd(B, C) = 1$. The transformed auxiliary factors have height $Y^{O_\xi(1)}$ by the fixed-degree factor-height argument above. These are the primitivity and height hypotheses used in the square-fibre and Heath–Brown counts. The coefficient-lattice coordinates $\mathbf{v} = (v_1, v_2)$ are primitive, since (γ, δ) is primitive and the basis change is unimodular. The original coefficients $\gamma(\mathbf{v}), \delta(\mathbf{v})$ are homogeneous linear forms in these coordinates. Factor an auxiliary form over $\mathbb{Q}$. Let a_c and b_q be its degrees in the corner and coefficient pairs. There are five branches. The conjugate-component Bezout argument above handles a factor that is not absolutely irreducible. If $a_c = 0$, it fixes $O_\xi(1)$ primitive coefficient pairs. Then the square-fibre theorem (Theorem B.1 in Appendix B) applies to $\delta C^2 - \gamma B^2 = t$, with $x = 2Y$ and fixed height exponent $A_{\mathrm{ht}} = 2$. If $b_q = 0$, it fixes a primitive corner pair (B, C). Reducing the conic modulo C^2 fixes γ. Also $0 < \gamma < C^2$ by (5.31). So there is at most one coefficient pair. If $a_c \ge 2$ and $b_q \ge 1$, apply Heath–Brown's Lemma 2 with the primitive coefficient pair first. It gives $O_\eta(Y^\eta(T_1 + 1))$. Finally, if $a_c = 1$ and $b_q \ge 1$, write the factor as $BQ(\mathbf{v}) + CP(\mathbf{v})$, with coprime nonzero homogeneous forms P, Q. The

conic forces $P(\mathbf{v}) = -B\zeta$, $Q(\mathbf{v}) = C\zeta$. The Sylvester identities and primitivity give $\zeta \mid \mathrm{Res}(P, Q)$. There are Y^η choices for ζ. For each, the conic gives

$$\delta(\mathbf{v})Q(\mathbf{v})^2 - \gamma(\mathbf{v})P(\mathbf{v})^2 = t\zeta^2.$$

The left side is a nonzero binary form of degree $2b_q + 1$. An identity would force P^2 to divide the linear form $\delta(\mathbf{v})$, contrary to $b_q \geq 1$. The preceding complete slice count applies with the nonzero target $h = t\zeta^2$. It counts $O_\eta(T_1 + 1)$ coefficient pairs. Then B, C are forced. Thus every full interval contributes $Y^\eta(T_1 + 1)$.

Fix the ordered factorizations $\gamma = ad, \delta = bc$. The inverse formulas (2.2) determine

$$A = (dB - bC)/t, \qquad D = (cC - aB)/t,$$

then $m = ABa, n = ACc$. Also $u = (cd - ab)/t$. These are forced, not summed. The factor multiplicity is $\tau_{\mathrm{div}}(\gamma)\tau_{\mathrm{div}}(\delta) \ll_\eta Y^\eta$. Thus, for fixed t and box, (5.37) gives

$$\ll_\eta Y^\eta\{N_t + \sqrt{EFN_t}\log Y + E + F\}. \tag{5.42}$$

The row $3t$ is contained in this larger population with $1 \leq t \leq Y^{a_0}$, $L \geq Y^{1/3-l_0}$ and $B, C \geq Y^{1/3-j_0}$. The restrictions $t < u$ and $K_{\mathrm{res}} \leq Y^{\Delta_{\exp}}$ only reduce it. There are $O((\log Y)^4)$ boxes. By (5.31), $EF \leq M \ll Y^{q_+}$ and $E + F \ll tY^{g_t}$. Sum (5.42) once over the positive shifts $t \leq Y^{a_0}$. This gives

$$\#3t \ll_\eta Y^\eta \left\{Y^{a_0+\nu_t} + Y^{a_0+(q_+ +\nu_t)/2} + Y^{g_t+2a_0}\right\}. \tag{5.43}$$

The three exponents and their savings are

(5.44)

term	exponent	saving from 2/3
interval count	5933607/10000000	2199179/30000000
square-root count	39611249/60000000	388751/60000000 (0.00647918…)
coefficient axes	7273463/15000000	2726537/15000000.

The least saving is $388751/60000000 > S_0$. Fix the uniform degree and height constants. Then the fixed loss terms, divisor factors and logarithms fit within any prescribed $\varepsilon > 0$. Hence

$$\#3t \ll_\varepsilon Y^{2/3-388751/60000000+\varepsilon}. \tag{5.45}$$

Proposition 5.4 ($3t$ row estimate)**.** *For every $\varepsilon > 0$, the complete first-match row satisfies*

$$\#3t \ll_\varepsilon Y^{2/3-388751/60000000+\varepsilon}.$$

Proof. The complete row is bounded in (5.43). Its three contributions and their savings are listed in (5.44)–(5.45). □

6. The intermediate rows

Throughout this section abbreviate

$$e = \epsilon_0, \qquad k = k_0, \qquad l = l_0, \qquad j = j_0, \qquad p = p_0, \qquad \mu_{\mathrm{HP}} = \frac{16}{25}, \qquad S_0 = \frac{30869}{5000000}.$$

The first five and S_0 are the fixed parameters from (2.5); $\mu_{\mathrm{HP}} = 16/25$ is set here for the 4P grid.

In this section the chart and all source predicates are those of Section 3. If the selected chart is the u-chart, then $(J, K; f, g; s) = (A, D; ac, bd; u)$. If it is the t-chart, then $(J, K; f, g; s) = (B, C; ad, bc; t)$. In either case

$$gK^2 - fJ^2 = s > 0, \qquad fg = M, \qquad ut = L. \tag{6.1}$$

The chart choice is unique, including equality $u = t$. Fix the residual factors, the chart, the shift and the two selected corners. Then the inverse formulas in (2.2) determine the original pair. All

counts below keep the positivity, integrality, gcd, mask, shell and first-match tests when a relaxed tuple is reconstructed.

6.1. **The supported-part row** $4E$**.** Let f_J be the full prime-power part of f supported on primes dividing J. Let g_K be the full prime-power part of g supported on primes dividing K. Write

$$\begin{aligned} f = d_f v, \qquad g = d_g w, \qquad d_f = f_J, \qquad d_g = g_K, \\ \mathrm{rad}(d_f) \mid J, \qquad \mathrm{rad}(d_g) \mid K. \end{aligned} \tag{6.2}$$

Thus d_f, d_g contain all powers of their supported primes. For fixed (f, J) and (g, K), each is uniquely determined.

Explicitly, d_f is the unique largest divisor of f all of whose prime divisors divide J, and $v = f/d_f$. Similarly, d_g is the unique largest divisor of g supported on K, and $w = g/d_g$. Once g and K are recovered in the v-tag, d_g is determined by (g, K). The same holds for d_f in the w-tag. The mask assigns every supported prime power to these determined parts.

In the u-chart, $JK = AD = (n - m)/t$, so $JK \le Yu/L$. In the t-chart, $JK = BC = (m+n+1)/u$, so $JK \le 2Yt/L$. We have $s = u$ in the first chart, $s = t$ in the second, and $ut = L$. Hence both inequalities give $JK \le 2Ys/L$. The failed $Q1$ and selected-corner $Q2$ tests give the weak lower bounds below. Thus

$$JK \le \frac{2Ys}{L}, \qquad L \ge Y^{1/3-l}, \qquad J, K \ge Y^{1/3-j}. \tag{6.3}$$

Hence

$$J, K \le 2sY^{1/3+l+j}. \tag{6.4}$$

The row has $Y^{a_0} < s < Y^{b_0}$ and

$$\min(v, w) \le Y^p. \tag{6.5}$$

It has at most four disjoint tags. They are the selected chart and either $v \le Y^p$, or $v > Y^p$ and $w \le Y^p$. By (6.5), the second alternative covers the complement of the first.

Each tag uses a uniform supported-power bound. For fixed $C, D, \alpha > 0$, if $J \le Y^C$, then

$$\#\{d \le Y^D : \mathrm{rad}(d) \mid J\} \ll_{C,D,\alpha} Y^\alpha. \tag{6.6}$$

To prove it, put $\sigma = \alpha/(2D)$. Rankin's inequality gives

$$\#\{d \le Y^D : \mathrm{rad}(d) \mid J\} \le Y^{D\sigma} \prod_{q \mid J} (1 - q^{-\sigma})^{-1}.$$

Fix any $\delta > 0$. All primes above a fixed threshold satisfy $-\log(1 - q^{-\sigma}) \le \delta \log q$. The product over the finitely many smaller primes is a constant. The remaining product is at most J^δ. Taking $\delta = \alpha/(2C)$ proves (6.6), uniformly in the moving label J.

In the tag $v \le Y^p$, enumerate (s, v, J, d_f) and put

$$N = v d_f J^2 + s = f J^2 + s.$$

For an actual source, $N = gK^2 > 0$. Thus $K^2 \mid N$, and each divisor choice for K^2 forces $g = N/K^2$. There are at most $\tau_{\mathrm{div}}(N)$ choices. In the other tag enumerate (s, w, K, d_g) and put

$$N = w d_g K^2 - s = g K^2 - s.$$

Keep only $N > 0$. On an actual source $N = fJ^2$, so $J^2 \mid N$ and $f = N/J^2$ is forced. Discarding $N \le 0$ is safe, since an actual source has $N = fJ^2 > 0$.

Once f, g are known, the ordered residual splittings are

$$(a, c) \text{ with } ac = f, \quad (b, d) \text{ with } bd = g \quad \text{or} \quad (a, d) \text{ with } ad = f, \quad (b, c) \text{ with } bc = g,$$

according to the chart. Their total number is at most $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g)$. Each splitting and (s, J, K) determine the remaining corners, and then (m, n) by (2.2). Candidates failing an original-source predicate are discarded. Once the ordered residuals are fixed, the reconstruction has multiplicity at most one. Then $ut = L$ determines the opposite shift.

Before the divisor estimates, discard formal candidates that violate the original source-height conditions. Every retained source has $J, K \le Y+1$, $f, g, d_f, d_g \le (Y+1)^2$, and $1 \le N \le (Y+1)^4$. Apply (6.6) to the one supported part. Apply the standard estimate $\tau_{\mathrm{div}}(q_0) \ll_\rho q_0^\rho$ to each of N, f, g. Fix a target loss $\eta > 0$ and put $T = Y+1$. Apply (6.6) with T in place of its displayed base variable Y, and take $C = 1$, $D = 2$, and output exponent $\alpha = \eta/4$. Its Rankin parameters may be $\sigma = \eta/16$ and $\delta = \eta/8$. For the divisor bounds, take $\rho_N = \eta/16$ for $\tau_{\mathrm{div}}(N)$ and $\rho_f = \rho_g = \eta/8$ for $\tau_{\mathrm{div}}(f), \tau_{\mathrm{div}}(g)$. Since $N \le T^4$ and $f, g \le T^2$, each of the four multiplicative factors is $O_\eta(T^{\eta/4})$. Their product is $O_\eta(T^\eta) = O_\eta(Y^\eta)$, uniformly in the retained source labels. Thus each fixed (s, v, J), or its counterpart (s, w, K), contributes at most Y^η complete source records. Sum over the small endpoint, then over the selected corner using (6.4), then over $s < Y^{b_0}$. This gives

$$\#4E \ll_\eta Y^{p+1/3+l+j+\eta} \sum_{1 \le s < Y^{b_0}} s \ll_\eta Y^{1/3+2b_0+l+j+p+\eta}. \tag{6.7}$$

The direct sum satisfies $\sum_{s<Y^{b_0}} s \ll Y^{2b_0}$. At the parameter values in (2.5)

$$\frac{1}{3} + 2b_0 + l + j + p = \frac{9907393}{15000000} = \frac{2}{3} - S_0. \tag{6.8}$$

This equality, with η arbitrary, is the claimed $4E$ bound.

Proposition 6.1 (4E row estimate)**.** *For every $\eta > 0$, the complete first-match row satisfies*

$$\#4E \ll_\eta Y^{2/3-S_0+\eta}.$$

Proof. The full count is (6.7), and (6.8) gives its exponent. □

6.2. **The opposite-shift cubic row $4H$.** For each prime dividing a residual, the radical mask places that prime in one of the four corners. The adjacent coprimalities determine which corner. Primes in a, b, c, d go respectively to one of $(A, B), (C, D), (A, C), (B, D)$. These choices are unique, since the two corners in each listed pair are coprime. Apply the same assignment to the full prime powers. This gives four pairwise coprime parts, assigned to corners, whose product is M:

$$(ac)_A, \quad (bd)_D, \quad (ad)_B, \quad (bc)_C.$$

Set

$$G_u = \mathrm{rad}((ac)_A(bd)_D), \qquad G_t = \mathrm{rad}((ad)_B(bc)_C).$$

Then $G_uG_t = \mathrm{rad}(M)$. In the u-chart, the supported prime-power parts are $(ac)_A$ and $(bd)_D$. Their radicals have product G_u. In the t-chart the corresponding parts are $(ad)_B$ and $(bc)_C$. Their radicals have product G_t. Put

$$\Pi = \min(\mathrm{rad}(d_f), v_{\mathrm{sel}}) \min(\mathrm{rad}(d_g), w_{\mathrm{sel}}).$$

In $4H$, $v_{\mathrm{sel}}, w_{\mathrm{sel}} > Y^p$ and $\Pi \le Y^p$. Hence neither minimum can equal v_{sel} or w_{sel}. So $\Pi = \mathrm{rad}(d_f)\,\mathrm{rad}(d_g) \le Y^p$. This is the selected chart's assigned-corner radical product. Since $\mathrm{rad}(M) > Y^{2/3-k}$, the opposite assigned-corner product satisfies

$$G_{\mathrm{opp}} > R_0 := Y^{2/3-k-p}. \tag{6.9}$$

The two-sided prime assignment in the mask makes G_{opp} the complementary assigned-corner product.

We use the following published theorem of Bombieri and Schmidt [5]. Let $F \in \mathbb{Z}[X, Z]$ be an irreducible binary form of degree $r \ge 3$, and let $N \ge 1$. Then the number of primitive integer pairs

(x,z) with $|F(x,z)|=N$ is at most $C_{\mathrm{BS}}r^{1+\omega(N)}$, with an absolute C_{BS}. Here primitive means $\gcd(x,z)=1$. This is the unnumbered theorem on pp. 69–70 of *Inventiones Mathematicae* 88 (1987), 69–81, with the correction on p. 445 of *Inventiones Mathematicae* 97 (1989) [6]. Its bound is uniform in the coefficients. It has no discriminant-coprimality or coefficient-height hypothesis.

Hence, for each fixed $C_0>0$ and every $\eta>0$, uniformly for positive integers $c_1,c_2\le Y^{C_0}$ and $0<|h|\le Y^{C_0}$,

$$\#\{x,z>0: c_1x^3-c_2z^3=h\}\ll_{C_0,\eta} Y^{\eta}, \tag{6.10}$$

For the proof, let $c=\gcd(c_1,c_2)$. If $c\nmid h$ there are no solutions. Otherwise divide by c. This gives coprime positive coefficients c_1',c_2' and $h'=h/c\neq 0$. For each $g=\gcd(x,z)$, write $x=gx_0, z=gz_0$. Then $g^3\mid h'$, $(x_0,z_0)=1$, and $c_1'x_0^3-c_2'z_0^3=h'/g^3$. If this cubic form is irreducible, Bombieri–Schmidt gives at most $C3^{1+\omega(|h'|/g^3)}$ primitive pairs for that g. There are at most $\tau_{\mathrm{div}}(|h'|)$ choices of g, and $3^{\omega(n)}\le \tau_{\mathrm{div}}(n)^{\log_2 3}$. The divisor bound $\tau_{\mathrm{div}}(n)\ll_\rho n^\rho$, with ρ chosen after η, makes their total $O_{C_0,\eta}(Y^\eta)$.

If the normalized cubic is reducible, c_1'/c_2' is a rational cube. Coprimality then forces $c_1'=A_{\mathrm{cube}}^3, c_2'=B_{\mathrm{cube}}^3$ for coprime positive integers $A_{\mathrm{cube}},B_{\mathrm{cube}}$. Put $X=A_{\mathrm{cube}}x, Z=B_{\mathrm{cube}}z$. Then

$$(X-Z)(X^2+XZ+Z^2)=h'.$$

The nonzero integer $X-Z$ is a signed divisor of h'. For each such divisor the resulting quadratic has at most two integer roots. The divisor bound absorbs the content, gcd and reducible branches into Y^η. Here $h'\neq 0$, so the right side is nonzero.

For the opposite-chart conic count, write

$$\begin{aligned} r_{\mathrm{opp}}&=\mathrm{rad}(d_{f,\mathrm{opp}}), & q_{\mathrm{opp}}&=\mathrm{rad}(d_{g,\mathrm{opp}}),\\ d_{1,\mathrm{opp}}&=d_{f,\mathrm{opp}}/r_{\mathrm{opp}}, & d_{2,\mathrm{opp}}&=d_{g,\mathrm{opp}}/q_{\mathrm{opp}},\\ J_{\mathrm{opp}}&=r_{\mathrm{opp}}j_1, & K_{\mathrm{opp}}&=q_{\mathrm{opp}}z_1. \end{aligned}$$

The actual conic becomes the integral equation

$$(w_{\mathrm{opp}}d_{2,\mathrm{opp}}z_1^2)q_{\mathrm{opp}}^3-(v_{\mathrm{opp}}d_{1,\mathrm{opp}}j_1^2)r_{\mathrm{opp}}^3=s>0. \tag{6.11}$$

Here s is the shift of the opposite, counted chart. The selected shift is then L/s. The opposite chart has data

$$(d_{f,\mathrm{opp}},d_{g,\mathrm{opp}},v_{\mathrm{opp}},w_{\mathrm{opp}})$$

These are its supported parts and complementary factors from (2.4). They differ from $(v_{\mathrm{sel}},w_{\mathrm{sel}})$ in the $4H$ predicate. Suppose the endpoint squares are at most H_{4H} and $r_{\mathrm{opp}}q_{\mathrm{opp}}\ge R_0$. Fix $s,v_{\mathrm{opp}},w_{\mathrm{opp}},d_{1,\mathrm{opp}},d_{2,\mathrm{opp}},j_1,z_1$. Apply (6.10) to dyadic boxes $r_{\mathrm{opp}}\asymp R$, $q_{\mathrm{opp}}\asymp Q$. There are only $O(\log^2 Y)$ such boxes, and $RQ\ge R_0/4$. The endpoint conditions yield

$$j_1\le\sqrt{H_{4H}/(v_{\mathrm{opp}}d_{1,\mathrm{opp}}R^3)},\qquad z_1\le\sqrt{H_{4H}/(w_{\mathrm{opp}}d_{2,\mathrm{opp}}Q^3)},$$

$$v_{\mathrm{opp}}w_{\mathrm{opp}}d_{1,\mathrm{opp}}d_{2,\mathrm{opp}}r_{\mathrm{opp}}q_{\mathrm{opp}}<X_{\mathrm{cap}}.$$

The displayed coefficient product is $v_{\mathrm{opp}}w_{\mathrm{opp}}d_{1,\mathrm{opp}}d_{2,\mathrm{opp}}r_{\mathrm{opp}}q_{\mathrm{opp}}=M$. Failure of the O_R cut and (2.7) give $M<\kappa_Y^2Y^{q_+}=\kappa_Y^2Y^{2/3+2e}$. Define $X_{4H}:=\kappa_Y^2Y^{2/3+2e}$ here and take $X_{\mathrm{cap}}=X_{4H}$. If either upper bound is below one, that class is empty. Otherwise put $N_{\mathrm{opp}}=v_{\mathrm{opp}}w_{\mathrm{opp}}$. Sum over its at most $\tau_{\mathrm{div}}(N_{\mathrm{opp}})$ factorizations. The number of (j_1,z_1) is at most the product of their upper bounds, so

$$\frac{H_{4H}}{\sqrt{d_{1,\mathrm{opp}}d_{2,\mathrm{opp}}}(RQ)^{3/2}}\sum_{N_{\mathrm{opp}}<X_{\mathrm{cap}}/(d_{1,\mathrm{opp}}d_{2,\mathrm{opp}}RQ)}\frac{\tau_{\mathrm{div}}(N_{\mathrm{opp}})}{\sqrt{N_{\mathrm{opp}}}}\ll\frac{H_{4H}\sqrt{X_{\mathrm{cap}}}\log(2X_{\mathrm{cap}})}{d_{1,\mathrm{opp}}d_{2,\mathrm{opp}}(RQ)^2}.$$

Here we use $\sum_{N\le Z}\tau_{\mathrm{div}}(N)N^{-1/2}\ll\sqrt{Z}\log(2Z)$. It follows from $\tau_{\mathrm{div}}(N)=\sum_{ab=N}1$ by summing first over b. Summing $(d_{1,\mathrm{opp}},d_{2,\mathrm{opp}})$ harmonically and over the dyadic radical boxes costs only Y^{η}. Since $RQ\ge R_0/4$, the fixed-shift count is

$$N_s\ll_{\eta} H_{4H}\sqrt{X_{4H}}R_0^{-2}Y^{\eta}. \tag{6.12}$$

Fix an outer tuple $s,v_{\mathrm{opp}},w_{\mathrm{opp}},d_{1,\mathrm{opp}},d_{2,\mathrm{opp}},j_1,z_1$. Then the cubic coefficients in (6.11) are the fixed positive integers $c_1=w_{\mathrm{opp}}d_{2,\mathrm{opp}}z_1^2$ and $c_2=v_{\mathrm{opp}}d_{1,\mathrm{opp}}j_1^2$. The cubic unknowns are q_{opp} and r_{opp}. So (6.10) applies uniformly in every dyadic radical box. The coefficients may vary between outer tuples. The uniformity of (6.10) permits the preceding sum over those tuples and boxes. All canonical-source filters are kept on reconstruction.

In the following displays, J,K,s denote the counted chart and its shift; this chart may be opposite to the first-match chart. The variables in (6.11) come from the canonical source. In the u-chart they are

$$(f,g,J,K,s)=(ac,bd,A,D,u),\qquad (d_{f,\mathrm{opp}},d_{g,\mathrm{opp}},v_{\mathrm{opp}},w_{\mathrm{opp}})=(f_J,g_K,f/f_J,g/g_K),$$

and in the t-chart they are

$$(f,g,J,K,s)=(ad,bc,B,C,t),\qquad (d_{f,\mathrm{opp}},d_{g,\mathrm{opp}},v_{\mathrm{opp}},w_{\mathrm{opp}})=(f_J,g_K,f/f_J,g/g_K).$$

The two ordered residual factorizations are the only inverse choices. In the u-chart, fix a solution of (6.11) and choose $ac=f$ and $bd=g$. Then

$$B=\frac{cA+bD}{u},\quad C=\frac{aA+dD}{u},\quad t=\frac{cd-ab}{u},\quad m=ABa,\quad n=ACc. \tag{6.13}$$

In the t-chart choose $ad=f$ and $bc=g$. Then

$$A=\frac{dB-bC}{t},\quad D=\frac{cC-aB}{t},\quad u=\frac{cd-ab}{t},\quad m=ABa,\quad n=ACc. \tag{6.14}$$

Only positive integral reconstructions passing the original mask, gcd, shell and first-match predicates are kept. Hence $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g)$ bounds the inverse multiplicity, and $ut=L$ determines the other shift.

For the selected chart, $ut=L$ determines the opposite shift. The source height and $R>Y^{5/3-e}$ give

$$U=\kappa_Y Y^{1/3+e-a_0},\qquad H_{4H}=12\kappa_Y^2Y^{4/3+e-a_0},\quad R_0=Y^{2/3-k-p},$$

where $\kappa_Y=1+Y^{-1}\le 3/2$ for $Y\ge 2$. If $t<u$ is selected, count the opposite u chart with $H=uY$. Then sum the actual $u<U$ once. If $u\le t$ is selected, count the opposite t chart with endpoint cap H_{4H}. Then sum the actual $t<U$ once; the other shift is forced. For the endpoint bound in the t-chart, the canonical identities give

$$(ad)B^2=\frac{m(n+1)}{AD},\qquad (bc)C^2=\frac{n(m+1)}{AD},\qquad (bc)C^2-(ad)B^2=t. \tag{6.15}$$

The high-height condition $ABCD>Y^{5/3-e}$ and $n(n+1)\le\kappa_Y Y^2$ give

$$cd=\frac{n(n+1)}{ABCD}<\kappa_Y Y^{1/3+e}.$$

Now $BC=(m+n+1)/u$ and $ABCD\,cd=n(n+1)$. Also $n>Y/2$ and $m+n+1\le 3Y$. Hence

$$AD=\frac{n(n+1)u}{cd(m+n+1)}>\frac{uY^{2/3-e}}{12\kappa_Y}. \tag{6.16}$$

Thus both t-chart endpoint squares in (6.15) are at most $12\kappa_Y^2Y^{4/3+e}/u$. On the subrow $u>Y^{a_0}$ this is at most H_{4H}. So this endpoint estimate uses both the high-height cut and the subrow lower bound for u. The two counts are

$$\#4H_t\ll U^2Y\sqrt{X_{4H}}R_0^{-2}Y^{\eta},\qquad \#4H_u\ll UH_{4H}\sqrt{X_{4H}}R_0^{-2}Y^{\eta}. \tag{6.17}$$

Here $4H_t$ is the population selected in the t-chart ($t < u$) and counted through the opposite u-chart. Similarly, $4H_u$ is selected in the u-chart ($u \le t$) and counted through the opposite t-chart. The factor $\sqrt{X_{4H}} = \kappa_Y Y^{1/3+e}$ is the square-root factor in the fixed-shift cubic count (6.12). Substitute U, H_{4H} and R_0 from above into either expression. Both exponents equal

$$\frac{2}{3} - 2a_0 + 3e + 2k + 2p,$$

so the saving is

$$r_H = 2a_0 - 3e - 2k - 2p = \frac{140441}{5000000} > S_0. \tag{6.18}$$

The content split, reducible cubic count, residual divisor factors and inverse formulas each take an arbitrary power loss. Their number is fixed. Summation over the shifts is the factor displayed in (6.17).

Proposition 6.2 (4H row estimate)**.** *For every $\eta > 0$, the complete opposite-shift row satisfies*

$$\#4H \ll_\eta Y^{2/3-140441/5000000+\eta}.$$

Proof. The two endpoint counts in (6.17) have the common exponent of (6.18). Their finite sum keeps that exponent. □

6.3. **The simultaneous-shift row $4P$.** The count in this row is simultaneous in both shifts. Its source keeps $v, w > Y^p$ and the strict defect bound $\Pi > Y^p$. It also keeps the full canonical source and all three high cuts. It keeps the mask, unit and gcd conditions and the selected-chart rule. It keeps the outer and $Q1/Q2$ complements and the open middle-shift interval. The 4P estimate below applies to this source. Section 7 counts the later $5W$ row separately.

In a dyadic box write

$$\begin{gathered} f \asymp E, \qquad g \asymp F, \qquad J \asymp H_J, \qquad K \asymp H_K, \\ S_h \le s < 2S_h, H_{4P} = 2S_h, \quad X_{4P} = H_J^2 E, \quad V_* = X_{4P}/H_K^2, \\ N = \lceil Y^{16/25} \rceil. \end{gathered}$$

Here E and F are integer powers of two. So EF is an integer common denominator for the coefficient-lattice map below. We also put

$$\begin{aligned} &q = \frac{2}{3} + 2e, \qquad h_{\mathrm{HP}} = \frac{2}{3} + l + b_0, \\ &c = 1 + e + l + b_0 - 2(1/3 - j), \qquad \tau = \tau_0 - \frac{1}{10000}, \qquad \tau_0 = 1 - \frac{3e}{2}. \end{aligned} \tag{6.19}$$

The same enlarged source gives $EF \ll Y^q$, $H_J H_K \ll Y^{h_{\mathrm{HP}}}$, $E, F \ll Y^c$, and thickness $Y^{-\tau_0}$. For $z = J/K$, $w = g/f$, the conic is

$$gK^2 - fJ^2 = s.$$

In particular $H_{4P}/X_{4P} \ll Y^{-\tau_0}$, uniformly in the dyadic box. This is the thickness estimate used below. For fixed nonnegative degrees $A_{\mathrm{det}}, B_{\mathrm{det}}$, use the bidegree-$(A_{\mathrm{det}}, B_{\mathrm{det}})$ columns

$$\{x^r y^j : 0 \le r \le A_{\mathrm{det}},\ 0 \le j \le B_{\mathrm{det}}\}, \qquad x = f/g, \quad y = K/J. \tag{6.20}$$

Clearing each row by $g^{A_{\mathrm{det}}} J^{B_{\mathrm{det}}}$ gives the integer entries in (6.24). The normalized determinant has the strict criterion

$$\mu_{\mathrm{HP}} > \frac{9}{8}\frac{h_{\mathrm{HP}} q}{\tau}, \qquad \mu_{\mathrm{HP}} = \frac{16}{25}. \tag{6.21}$$

The statement of Lemma 7.1 fixes the 5W parameters in (7.2)–(7.6). For 4P we use the determinant calculation in its proof, with the separate 4P parameters. The 4P grid exponent is $a = \mu_{\mathrm{HP}} = 16/25$, and

$$\frac{3}{5} < a < \min(q_{\min}, \tau_0), \qquad q_{\max} < \tau_0, \qquad b_0 > \frac{3}{2}e.$$

The bounds $H_J \le Y+1$ and $E \le (Y+1)^2$ follow from the source identities (2.2). The 4P row itself has $H_{4P} > Y^{a_0}$. The failed $Q1$ and $Q2$ cuts and the shell lower bound (2.8) supply the remaining endpoint restrictions in the 4P boxes. In particular these source bounds give $X_{4P}/H_{4P} \le Y^4$ for sufficiently large Y. The Taylor order is seven because $8\mu_{\mathrm{HP}} = 8(16/25) = 128/25 > 4$. With

$$p_1 = \frac{q}{\mu_{\mathrm{HP}}} = \frac{5092607}{4800000} > 1, \qquad p_2 = \frac{h_{\mathrm{HP}}}{\tau} = \frac{24536965}{29719179},$$

we have $2/3 < p_2 < 1$, $p_1 p_2 < 8/9$, and $p_1 - 1 < 3(1-p_2)^2$. These include the strict trapezoid-branch hypotheses in the proof of the general determinant lemma. They give its 4P specialization $(a, p_1, p_2) = (16/25, 5092607/4800000, 24536965/29719179)$. The determinant criterion needs these strict range and strip-weight hypotheses as well as its displayed inequality. The exact collared determinant margin is positive; it is recorded just below. Here $\rho = p_2 = h_{\mathrm{HP}}/\tau > 2/3$, so 4P uses the trapezoid branch. The stronger sufficient condition $\mu_{\mathrm{HP}} > (9/8)q\rho$ implies its criterion for $2/3 \le \rho < 1$, because

$$\frac{9}{8}\rho\big(1 + 3(1-\rho)^2\big) - 1 = \frac{(1 - 3(1-\rho))^3}{8} \ge 0.$$

At (2.5), the collared gap $\mu_{\mathrm{HP}} - (9/8)q\rho$ equals $368942136449/39625572000000 > 0$. Fix half this strict gap, then choose the finite degree family. All maximal-size integer minors have absolute value less than one. Hence they vanish, and the integerized matrix $\mathcal{M}$ has rank $r < D$, where D is its number of columns. If an interval contains fewer than D source rows, count those rows directly; otherwise the vanishing D-minors imply rank below D. If $r = 0$, a coordinate vector is in its kernel. Otherwise choose r pivot columns and r rows giving a nonsingular minor A, and one free column j. Set the j-coordinate to $\det A$, the other free coordinates to zero, and the pivot coordinates to $-\operatorname{adj}(A)\mathcal{M}_{I,j}$. This nonzero integer vector annihilates the selected rows. They span the row space, so it annihilates every row. Its entries are minors of bounded order and have polynomial height. The same argument covers the fixed anchor, the inverse monomials and every endpoint branch.

For the normalization, use in each box the exact half-open ratio cells

$$r_i = \frac{i}{2N}\frac{H_J}{H_K}, \qquad I_i = (r_i, r_{i+1}], \qquad i = N, \dots, 4N-1. \tag{6.22}$$

For a point in I_i, put $x = f/g$, $y = K/J$, $y_0 = r_i^{-1}$, $\delta_{\mathrm{cell}} = s/(fJ^2)$, and $\alpha_{\mathrm{cell}} = x/y_0^2 - 1$. The conic gives the exact identities

$$x = \frac{y^2}{1+\delta_{\mathrm{cell}}}, \qquad \frac{y}{y_0} = \sqrt{1+\alpha_{\mathrm{cell}}}\sqrt{1+\delta_{\mathrm{cell}}}.$$

The cell width and $s \le H_{4P}$ give

$$|\alpha_{\mathrm{cell}}| \le \frac{3}{N}, \qquad 0 \le \frac{y}{y_0} - \sqrt{1+\alpha_{\mathrm{cell}}} \le \frac{H_{4P}}{2X_{4P}}. \tag{6.23}$$

Here $r_{i+1}/r_i = 1 + 1/i \le 1 + 1/N$ and $1 - (1+\delta_{\mathrm{cell}})^{-1/2} \le \delta_{\mathrm{cell}}/2$. Since $X_2 = X_{4P}/(2H_{4P})$, the conic error in (6.23) is $H_{4P}/(2X_{4P}) = X_2^{-1}/4$. Also $NH_{4P}/X_{4P} \le Y^{\mu_{\mathrm{HP}} - \tau_0 + o(1)} < 1$. Choose a fixed Taylor degree T and write $\sqrt{1+\alpha_{\mathrm{cell}}} = 1 + P_T(\alpha_{\mathrm{cell}}) + \varepsilon_T$. With $X_1 = N/3$, (6.23) gives $|\alpha_{\mathrm{cell}}| \le X_1^{-1}$. Choose T large enough that the Taylor remainder is at most $X_2^{-1}/2$, where

$X_2 = X_{4P}/(2H_{4P})$. Then the combined remainder $\beta = (y/y_0) - 1 - P_T(\alpha_{\mathrm{cell}})$ satisfies $|\beta| \le X_2^{-1}$, uniformly over all boxes and shifts.

Take columns $x^r y^j$, $0 \le r \le A_{\det}$, $0 \le j \le B_{\det}$, and multiply each row by $g^{A_{\det}} J^{B_{\det}}$. The resulting entries are the integers

$$f^r g^{A_{\det}-r} K^j J^{B_{\det}-j}. \tag{6.24}$$

Write the polynomial in (6.20) as

$$C(x,y) = \sum_{r=0}^{A_{\det}} \sum_{j=0}^{B_{\det}} c_{rj} x^r y^j.$$

With $z = J/K = 1/y$ and $w = g/f = 1/x$, its reciprocal is

$$F(z,w) = z^{B_{\det}} w^{A_{\det}} C(1/w, 1/z) = \sum_{r,j} c_{rj} z^{B_{\det}-j} w^{A_{\det}-r}.$$

It has degrees at most $(B_{\det}, A_{\det})$ in (z,w) and the same coefficient height. Its coefficient vector permutes the full rectangular coefficient array. The column map is $(r,j) \mapsto (a,b) = (B_{\det} - j, A_{\det} - r)$. In particular,

$$K^{B_{\det}} f^{A_{\det}} F(J/K, g/f) = J^{B_{\det}} g^{A_{\det}} C(f/g, K/J),$$

so the two integer matrices agree after clearing denominators, up to that column permutation. After extracting the common anchor, the normalized entries are $(1+\alpha_{\mathrm{cell}})^r (1 + P_T(\alpha_{\mathrm{cell}}) + \beta)^j$. If $D_c = (A_{\det}+1)(B_{\det}+1)$, the anchor and row-clearing factors have logarithmic cost at most

$$\frac{D_c}{2}\{A_{\det}\Lambda_1 + B_{\det}\Lambda_2\}, \qquad \Lambda_1 = \log(324EV_*), \quad \Lambda_2 = \log(8H_J H_K), \tag{6.25}$$

The column-anchor factor is y_0^{2r+j}. Its exponent sum is

$$\sum_{\substack{0 \le r \le A_{\det} \\ 0 \le j \le B_{\det}}} (2r+j) = D_c \left(A_{\det} + \frac{B_{\det}}{2} \right).$$

The other anchor bound is

$$J y_0^{1/2} < 2H_J (2H_K/H_J)^{1/2} = (8H_J H_K)^{1/2}.$$

Also $V_* = EH_J^2/H_K^2$, so

$$18 V_* \frac{H_K}{H_J} = 18\sqrt{EV_*}, \qquad (18\sqrt{EV_*})^2 = 324 EV_*.$$

These inequalities give the constants 8 and 324 in the uniform anchor bounds. The conic and the thickness give $EV_* \asymp EF$. So replacing one by the other changes only an absolute logarithmic constant. The displayed cost follows from $g < 9V_*$, $y_0 \le 2H_K/H_J$, and $J < 2H_J$. It is independent of i and s.

The proof of Lemma 7.1 derives the finite Cauchy–Binet expansion, the strip-weight lower bound, the branch integrals and the integer-minor conclusion. It does so for general logarithmic parameters satisfying its strict inequalities. We apply that calculation with the 4P parameters and effective scales below. For this row put

$$X_1 = N/3, \qquad X_2^{\mathrm{eff}} = Y^\tau,$$

and choose fixed constants C_1, C_2 such that, uniformly in every box,

$$\Lambda_1^{\mathrm{eff}} = q \log Y + C_1 \ge \log(324EV_*), \qquad \Lambda_2^{\mathrm{eff}} = h_{\mathrm{HP}} \log Y + C_2 \ge \log(8H_J H_K).$$

Such constants exist by the source bounds $EV_* \ll EF \ll Y^q$ and $H_J H_K \ll Y^{h_{\mathrm{HP}}}$. Also $H_{4P}/X_{4P} \ll Y^{-\tau_0}$, where $\tau_0 > \tau$. So for all sufficiently large Y, $X_2 = X_{4P}/(2H_{4P}) \geq X_2^{\mathrm{eff}}$. Thus the actual Taylor error $|\beta| \leq X_2^{-1}$ is at most $(X_2^{\mathrm{eff}})^{-1}$. The arithmetic cost in (6.25) is bounded by the one obtained from $\Lambda_1^{\mathrm{eff}}, \Lambda_2^{\mathrm{eff}}$. With

$$\ell_1 = \log X_1, \quad \ell_2^{\mathrm{eff}} = \tau \log Y, \quad P_{\mathrm{eff}} = \frac{\Lambda_1^{\mathrm{eff}}}{\ell_1}, \quad R_{\mathrm{eff}} = \frac{\Lambda_2^{\mathrm{eff}}}{\ell_2^{\mathrm{eff}}}, \quad \ell_{\mathrm{det}} = \frac{\ell_1}{\ell_2^{\mathrm{eff}}},$$

we have

$$P_{\mathrm{eff}} = \frac{q}{\mu_{\mathrm{HP}}} + O(1/\log Y), \quad R_{\mathrm{eff}} = \frac{h_{\mathrm{HP}}}{\tau} + O(1/\log Y), \quad \ell_{\mathrm{det}} = \frac{\mu_{\mathrm{HP}}}{\tau} + O(1/\log Y).$$

At the limiting 4P vector,

$$\mu_{\mathrm{HP}} = \frac{16}{25}, \qquad p_1 = \frac{5092607}{4800000}, \qquad p_2 = \frac{24536965}{29719179}, \qquad \ell_{\mathrm{det}} = \frac{6400000}{9906393}.$$

The corresponding normalized determinant margins are

$$\frac{8}{9} - p_1 p_2 = \frac{368942136449}{28530411840000} > 0, \qquad 3(1-p_2)^2 - (p_1 - 1) = \frac{14253084474184329457}{471055786898155200000} > 0.$$

Hence the strict margin in (6.21), with the fixed collar $\tau = \tau_0 - 1/10000$, applies to these uniform majorants for all sufficiently large Y. The finite determinant calculation uses only the displayed upper bounds for the two errors and the arithmetic cost. So it gives one degree choice and one polynomial-height bound, uniformly in the cell, shift and box.

After bihomogenization and factorization, we count each factor. A factor that is not absolutely irreducible meets a distinct conjugate in only boundedly many rational projective points at fixed degree. For corner-degree-zero factors, first fix the primitive coefficient pair, then its ordered residual factors. These fix $L = cd - ab$. Discard $L \leq 0$ and enumerate the divisor choices $s \mid L$. For each fixed (f, g, s) apply Theorem B.1 with $(a, b, h, n, m) = (f, g, s, J, K)$, $x = 2Y$, and fixed height exponent $A_{\mathrm{ht}} = 2$. The residuals are at most $Y + 1$, whence $f, g, s \leq (Y+1)^2$ and $J, K \leq Y + 1$. This costs Y^η per coefficient label. The selected shift is enumerated among the divisors of L. Explicitly, this cost consists of three parts. They are the ordered residual splitting factors $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g)$, the selected-shift choices $\tau_{\mathrm{div}}(L)$, and the uniform square-fibre bound for each fixed (f, g, s). Give each operation its own small enough internal exponent. Then their product is Y^η.

For a coefficient-degree-zero factor, fix its primitive positive corner pair (J, K). For two source coefficient pairs the conic gives

$$K^2(g_1 f_2 - g_2 f_1) = s_1 f_2 - s_2 f_1, \qquad |g_1 f_2 - g_2 f_1| \leq \frac{4H_{4P}E}{H_K^2} \ll EF\frac{H_{4P}}{X_{4P}} \ll Y^{q-\tau_0} = o(1). \tag{6.26}$$

Here $s_i < H_{4P}$, $f_i < 2E$, $K \geq H_K$, $X_{4P} = H_J^2 E$, and $V_* = X_{4P}/H_K^2 \ll F$ by the source box bounds. The cross determinant is an integer. It vanishes for all sufficiently large Y, since $\tau_0 - q = 9351751/30000000 > 0$. Primitivity of (f_i, g_i) then forces the coefficient pairs to agree, and the conic fixes the shift. This branch contributes $O(N)$ primitive conic-coordinate tuples across the primary cells. To return to complete canonical source records, include the ordered residual factorizations. They cost $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g) \ll_\eta Y^\eta$, so this branch is $O_\eta(Y^\eta N)$.

Let the coefficient degree be $b_q \geq 1$ and the corner degree $a_c \geq 2$. Then apply Heath–Brown's bihomogeneous Lemma 2 [4] to the factor after the unimodular coefficient-lattice change of basis. Its bidegree is (b_q, a_c). The first pair is the primitive transformed coefficient pair of radius $O(T_1 + 1)$. The second pair is the original primitive corner pair (J, K), with no size hypothesis. The factor height is Y^C for a fixed C. Thus the count is $O((T_1+1)^{2/a_c+\varepsilon} Y^{C\varepsilon})$. Here $2/a_c \leq 1$ and $T_1 + 1 \leq Y^{C'}$ for a fixed C'. Choose the theorem's internal $\varepsilon < \eta/(C + C' + 1)$; this

bounds the count by $Y^{\eta}(T_1+1)$. For corner degree one write the factor as $KP(f,g)+JQ(f,g)$ with coprime homogeneous P,Q. Their nonzero resultant and homogeneous Bezout identities give $d_0=\gcd(P(f,g),Q(f,g))\mid \mathrm{Res}(P,Q)$, since $\gcd(f,g)=1$. The factor equation forces $(J,K)=\pm(P(f,g)/d_0,-Q(f,g)/d_0)$, with at most one positive sign. Write $f_J=d_f$, $g_K=d_g$ for the supported parts and $v=f/d_f$, $w=g/d_g$. Write ℓ_f,ℓ_g for the endpoint labels in the four patterns $(P(0,1),Q(1,0))$:

$P(0,1)$	$Q(1,0)$	ℓ_f	ℓ_g
$\neq 0$	$\neq 0$	$\mathrm{rad}(d_f)$	$\mathrm{rad}(d_g)$
$\neq 0$	$=0$	$\mathrm{rad}(d_f)$	w
$=0$	$\neq 0$	v	$\mathrm{rad}(d_g)$
$=0$	$=0$	v	w.

(6.27)

If $P(0,1)\neq 0$, reducing $d_0J=P(f,g)$ modulo $\mathrm{rad}(d_f)$ gives $\mathrm{rad}(d_f)\mid P(0,1)$. If $P(0,1)=0$, then $f\mid P(f,g)$, and $\gcd(v,J)=1$ gives $v\mid d_0\mid \mathrm{Res}(P,Q)$. The same argument with $Q(1,0)$ proves the two g-side alternatives. The chosen endpoint labels satisfy $\ell_f\ell_g\geq \Pi>Y^p$, since each is at least the corresponding minimum in the definition of Π. Each label divides a fixed nonzero endpoint coefficient or resultant of polynomial height. So there are $\ll_\eta Y^\eta$ possible pairs of endpoint labels, for any prescribed $\eta>0$. Also $\gcd(\ell_f,\ell_g)=1$, since $\gcd(f,g)=1$ and the labels are supported on the respective parts of f and g. Take two distinct primitive coefficient pairs with the same labels. Their divisibilities combine, and the nonzero cross determinant is divisible by $\ell_f\ell_g$. Take a coefficient cell of width $O(F/(EN))$ in g/f. There the ratios are separated by at least $\ell_f\ell_g/(4E^2)$, since $f_i<2E$. Thus the number of pairs in one cell is

$$O\left(1+\frac{EF}{N\ell_f\ell_g}\right).$$

Summing over the $O(N)$ coefficient cells and the divisor labels gives

$$\ll_\eta Y^\eta\left(N+\frac{EF}{\ell_f\ell_g}\right)\ll_\eta Y^\eta\left(N+\frac{EF}{Y^p}\right),$$

including unit labels and endpoint equalities. A factor equal to a source coordinate vanishes only when that coordinate is zero, so it has no points in the positive source. This is separate from the lattice shortest-vector axes, which are retained. This branch counts the primitive coefficient and corner tuples. Recovering complete source pairs still needs the ordered residual splittings of f and g, at most $\tau_{\mathrm{div}}(f)\tau_{\mathrm{div}}(g)$. That multiplicity is absorbed in the displayed Y^η.

For the high-degree factors use a second, coefficient-ratio grid. Write

$$r_{i_c}=\frac{F}{E}\frac{i_c}{N},\qquad C_{i_c}=(r_{i_c},r_{i_c+1}],\qquad \lfloor N/2\rfloor\leq i_c<2N. \tag{6.28}$$

The dyadic source has $g/f>F/(2E)$, since $g\geq F$ and $f<2E$. Thus these $O(N)$ cells cover its full range $F/(2E)<g/f<2F/E$. Now use

$$\frac{g}{f}=\left(\frac{J}{K}\right)^2+\frac{s}{fK^2},\qquad \frac{s}{gK^2}\ll\frac{H_{4P}}{X_{4P}}\ll Y^{-\tau_0}, \tag{6.29}$$

and $\tau_0>\mu_{\mathrm{HP}}$. The relative displacement from the square map is then $o(1/N)$. Each grid has relative width $O(1/N)$, and squaring preserves relative widths on positive intervals. Hence each coefficient cell meets $O(1)$ primary J/K cells, and conversely. Thus the cell polynomials from (6.22) give $O(1)$ factors per coefficient cell.

For a coefficient cell define the rational linear map

$$\mathcal{T}_{i_c}(g,f)=\left(\frac{N}{F}(g-r_{i_c}f),\frac{f}{E}\right). \tag{6.30}$$

For a point in this cell, $f/E \in [1,2)$ and

$$0 \le \frac{N}{F}(g - r_{i_c} f) < \frac{N}{F}\, 2E\, \frac{F}{EN} = 2,$$

so its image lies in $[0,2] \times [1,2]$. The matrix of $\mathcal{T}_{i_c}$ is

$$\begin{pmatrix} N/F & -i_c/E \\ 0 & 1/E \end{pmatrix}, \qquad \det \mathcal{T}_{i_c} = N/(EF).$$

Let $\Lambda_{i_c} = \mathcal{T}_{i_c}\mathbb{Z}^2$. Choose a Lagrange-reduced basis v_1, v_2 of this image lattice. Its lengths are $\lambda_1 \le \lambda_2$, and its angle is bounded away from zero. Write $v_j = \mathcal{T}_{i_c} h_j$. Since v_1, v_2 are a lattice basis, $h_1, h_2 \in \mathbb{Z}^2$ form a unimodular basis of $\mathbb{Z}^2$. Hence every primitive coefficient pair has a primitive transformed pair (x_1, x_2) in $(g, f) = x_1 h_1 + x_2 h_2$. Put $T_1 = 1/\lambda_1$, $T_2 = 1/\lambda_2$. The determinant and reduction give

$$T_1 \ge T_2, \qquad T_1 T_2 \asymp \frac{EF}{N}, \qquad T_1 \gg \sqrt{EF/N}, \qquad \max(|x_1|, |x_2|) \ll T_1 \tag{6.31}$$

for the transformed coordinates (x_1, x_2). For the last bound, the bounded image rectangle and the angle bound imply $|x_j| \ll 1/\lambda_j \le T_1$. The source has $EF \gg Y^{2/3-e}$ and $N \asymp Y^{16/25}$, so EF/N grows as a positive power. A common denominator for the matrix of $\mathcal{T}_{i_c}$ is EF. So every nonzero image lattice vector has length at least $(EF)^{-1}$. Also $\lambda_1 \ll \sqrt{N/(EF)}$ by the two-dimensional Minkowski bound, and $\lambda_1\lambda_2 \asymp N/(EF)$ by reduction. Hence $\lambda_2 \ll N$. The inverse matrix is

$$\mathcal{T}_{i_c}^{-1}(y_1, y_2) = \big((F/N)(y_1 + i_c y_2), E y_2\big).$$

Since $i_c \asymp N$ and $E, F, N \le Y^C$ for a fixed C, both v_j and their integer preimages h_j have polynomial height. So substitution by the unimodular matrix $[h_1\ h_2]$ keeps every fixed-degree factor at polynomial coefficient height. The unimodular change uses no moving denominator.

The census below includes axes and first points. At scale $D_{\mathrm{lat}} \le T_1 < 2D_{\mathrm{lat}}$, let $(g_{\mathrm{lat}}, f_{\mathrm{lat}})$ be the primitive preimage of the shortest basis vector. From (6.30),

$$|f_{\mathrm{lat}}| \ll E/D_{\mathrm{lat}}, \quad |g_{\mathrm{lat}}| \ll F/D_{\mathrm{lat}}, \quad |i_c f_{\mathrm{lat}} - (NE/F) g_{\mathrm{lat}}| \ll E/D_{\mathrm{lat}}. \tag{6.32}$$

If $f_{\mathrm{lat}}, g_{\mathrm{lat}} \ne 0$, then E/D_{lat} and F/D_{lat} are at least a fixed positive constant. There are $O(F/D_{\mathrm{lat}})$ choices for g_{lat}. For each, the integer product $i_c f_{\mathrm{lat}}$ lies in an interval of length $O(E/D_{\mathrm{lat}})$. This gives that many possible values. Each nonzero value has at most $\tau_{\mathrm{div}}(|i_c f_{\mathrm{lat}}|) \ll_\eta Y^\eta$ factorizations into $i_c f_{\mathrm{lat}}$, since all factors have polynomial height. Thus

$$\#\{i_c : T_1 \asymp D_{\mathrm{lat}},\ f_{\mathrm{lat}} g_{\mathrm{lat}} \ne 0\} \ll_\eta Y^\eta \frac{EF}{D_{\mathrm{lat}}^2}. \tag{6.33}$$

If $f_{\mathrm{lat}} = 0$, primitivity gives $g_{\mathrm{lat}} = \pm 1$ and $T_1 \asymp F/N$. So the sum over $O(N)$ cells is $O(F+N)$. If $g_{\mathrm{lat}} = 0$, then $f_{\mathrm{lat}} = \pm 1$, $i_c \asymp N$, and $T_1 \asymp E/N$, giving $O(E+N)$. Cells with $T_1 < 1$ contribute $O(N)$. These cells, both axes and every $+1$ are kept. Weight (6.33) by $T_1 \asymp D_{\mathrm{lat}}$ and sum geometrically from $D_{\mathrm{lat}} \gg \sqrt{EF/N}$. This yields

$$\sum_{i_c} (T_1(i_c) + 1) \ll_\eta Y^\eta \{N + \sqrt{EFN} + E + F\}. \tag{6.34}$$

This count factors the exact integer product $i_c f_{\mathrm{lat}}$. The number of cells in this dyadic range is $O_\eta(Y^\eta EF/D_{\mathrm{lat}}^2)$.

For a factor with $b_q \ge 1$ and $a_c \ge 2$, the Heath–Brown bound above is $Y^\eta(T_1 + 1)$. Summing over coefficient cells by (6.34) gives $Y^\eta(N + \sqrt{EFN} + E + F)$. Short matrices, nonabsolute factors and the two zero-degree branches contribute $Y^\eta N$. The corner-linear branch contributes $Y^\eta(N + EF/Y^p)$, by the endpoint-label spacing argument above. These cases exhaust the fixed-degree factorization.

Combine the lattice census (6.34) with the corner-linear count. The complete bound then has the form

$$\#4P \ll_\eta Y^\eta \left(N + \sqrt{EFN} + E + F + \frac{EF}{Y^p} \right).$$

These local bounds are summed over the $O(\log^4 Y)$ choices of dyadic scales and the $O(\log Y)$ shift scales. The resulting logarithmic factor is absorbed into the arbitrary Y^η loss. The four distinct exponent contributions, in the order EF/Y^p, $\sqrt{EFN}$, N, and $E+F$, are

$$q - p = \frac{9907393}{15000000}, \quad \frac{q+\mu}{2} = \frac{9892607}{15000000}, \quad \mu = \frac{16}{25}, \quad c = \frac{5277821}{10000000}. \tag{6.35}$$

Their savings relative to $2/3$ are respectively

$$S_0, \qquad \frac{107393}{15000000}, \qquad \frac{2}{75}, \qquad \frac{4166537}{30000000}. \tag{6.36}$$

The minimum is $S_0 = 30869/5000000$. We combine the uniform lattice census (6.34), the finite factor family and the polynomial-height estimates. With the divisor losses and the one simultaneous source inversion, this gives

$$\#4P \ll_\eta Y^{2/3 - S_0 + \eta}. \tag{6.37}$$

Proposition 6.3 (4P row estimate)**.** *For every $\eta > 0$, the complete simultaneous-shift row satisfies*

$$\#4P \ll_\eta Y^{2/3 - S_0 + \eta}.$$

Proof. Equations (6.34)–(6.36) bound every factor branch and its exponent. Then (6.37) sums them with the source-uniform lattice and inverse bounds. □

The $4P$ estimate applies to its enlarged source. Section 7 counts $5W$ from its original-cell source and geometric envelope.

7. The long-shift row and its geometric cap

The row $5W$ is the exact first-match remainder after $4P$. Its source keeps every earlier complement. In particular,

$$s \geq Y^{b_0}, \quad K_{\mathrm{res}} \leq Y^{\Delta_{\exp}}, \quad \Delta_{\exp} = 2\epsilon_0 + k_0 = 3S_0. \tag{7.1}$$

The selected corners and both shifts keep their lower bounds from Section 3. The remainder predicate includes the exact bound $K_{\mathrm{res}} \leq Y^{\Delta_{\exp}}$.

With $e = \epsilon_0$, $k = k_0$, $l = l_0$, $j = j_0$, and $p = p_0$, write

$$\begin{aligned} S_0 = e = k = \frac{30869}{5000000}, \qquad l = 2S_0, \qquad j = p = 3S_0, \\ b_0 = \frac{4166537}{30000000}, \quad a_0 = \frac{6}{125}, \qquad \mu_{\mathrm{HP}} = \frac{16}{25}, \\ \nu = \frac{1}{64}, \qquad \xi = \frac{1}{10000}. \end{aligned} \tag{7.2}$$

Set

$$\begin{aligned} q_{\min} = \frac{2}{3} - S_0, \qquad q_{\max} = \frac{2}{3} + 2S_0, \\ \tau_0 = 1 - \frac{3e}{2}, \qquad \lambda = \log_Y(1 + Y^{-1}), \qquad d_2 = \log_Y 2, \\ \ell_C = \log_Y(1 + \sqrt{3}Y^{-\tau_0}), \qquad \ell_{\mathrm{end}} = \log_Y\big((1 + \sqrt{3}Y^{-\tau_0})(1 + 2/Y)\big). \end{aligned} \tag{7.3}$$

The source inequalities imply $q_{\min} < \log_Y M < q_{\max} + 2\lambda$. They also imply all endpoint comparisons used below. Each chart keeps its own shift interval throughout the count.

7.1. **A polynomial on each ratio cell.** In a local chart box write

$$D_0 \le J < 2D_0, \quad E_0 \le K < 2E_0, \quad U_0 \le f < 2U_0, \quad S_h \le s < 2S_h, \quad H_0 = 2S_h,$$

$$X = D_0^2 U_0, \qquad V_* = \frac{X}{E_0^2}. \tag{7.4}$$

These symbols are local box scales. On the actual population the source identities give, uniformly,

$$\frac{X}{H_0} > \frac{Y^{\tau_0}}{96}, \qquad \frac{M}{18} < U_0 V_* \le 4M, \qquad U_0 \le f, \quad V_* \le 4g. \tag{7.5}$$

The selected chart stays fixed as part of the first-match source label. Independently, either counting orientation $(J, K; f, g; s) = (A, D; ac, bd; u)$ or $(B, C; ad, bc; t)$ may be used on that retained population. All constants in (7.5) are uniform in this orientation and its moving shift box. Here $T = fJ^2/s$ equals $mn/(m+n+1) \ge m/2$ or $m(n+1)/(n-m) > m$, respectively. The dyadic definitions give $T < 16X/H_0$. Hence $X/H_0 > m/32 > Y^{\tau_0}/96$. Also $X/H_0 < (Y+1)^4/Y^{b_0} < Y^4$ for sufficiently large Y. The conic gives the other comparisons in (7.5) in both orientations. It needs no lower bound for the unselected corners.

Put $R_{\mathrm{box}} = D_0/E_0$, $L_{\mathrm{grid}} = 2N$, where

$$\bar{m} = \frac{q_{\max}}{1+3b_0^2} = \frac{203704280000000}{317360030572369}, \qquad a_{\mathrm{grid}} = \bar{m} + \xi, \qquad N = \lceil Y^{a_{\mathrm{grid}}} \rceil. \tag{7.6}$$

The original grid consists of exactly the $3N$ half-open cells

$$r_i = \frac{i}{2N} R_{\mathrm{box}}, \qquad I_i = (r_i, r_{i+1}], \qquad i = N, \dots, 4N-1, \tag{7.7}$$

for the ratio J/K. As $D_0 \le J < 2D_0$ and $E_0 \le K < 2E_0$, these cells cover the whole local source ratio. For $r_0 = r_i$, the relative width satisfies $r_{i+1}/r_i = 1 + 1/i \le 1 + 1/N$.

The exact determinant cap used to choose N is

$$\phi(\rho) = \begin{cases} 9\rho/8, & 0 < \rho \le 2/3, \\ \big(1 + 3(1-\rho)^2\big)^{-1}, & 2/3 \le \rho < 1. \end{cases} \tag{7.8}$$

Write $q_M = \log_Y M$, $c_0 = \log_Y(D_0E_0)$, and $\tau = \log_Y(X/H_0)$. The limiting comparison ratio is $\rho = c_0/\tau$. The comparison $\log_Y N > q_M\phi(\rho) + o(1)$ is the leading-order sufficient condition. The proof below gives the uniform finite-Y margin after the source and arithmetic collars. For the source comparison put $\sigma = \log_Y m$, $z_s = \log_Y s$, and $\ell_L = \log_Y L$. In the second chart also put $z_u = \log_Y u$. The exact identities from (2.1)–(2.2) used below are

$$AD = \frac{n-m}{t}, \quad BC = \frac{m+n+1}{u}, \quad L = \frac{(n-m)(m+n+1)}{Z}, \quad Z^2M = m(m+1)n(n+1),$$

$$A^2(ac) = \frac{umn}{m+n+1}, \quad B^2(ad) = \frac{tm(n+1)}{n-m}, \quad C^2(bc) = \frac{tn(m+1)}{n-m}. \tag{7.9}$$

In the first chart, $s = u$. The limiting source coordinates are

$$(c^*, \tau^*) = (\sigma + z_s - q_M/2, \sigma), \qquad \rho_1^* = \frac{c^*}{\tau^*} = 1 + \frac{z_s - q_M/2}{\sigma}. \tag{7.10}$$

Before the change to local dyadic scales,

$$\log_Y(AD) - c^* = \log_Y \frac{\sqrt{m(m+1)n(n+1)}}{m(m+n+1)}, \quad \log_Y(A^2 f/u) - \sigma = \log_Y \frac{n}{m+n+1}. \tag{7.11}$$

The first ratio factors as $(1+1/m)^{1/2}[n/(m+n+1)](1+1/n)^{1/2}$. The middle factor lies in $[1/(2+2/Y),1]$. The source gives $\tau_0 - 2d_2 < \sigma \le 1$. Both shifts satisfy $u,t \ge Y^{b_0}$, and $L = ut < \kappa_Y Y^{1/3+e}$. So for either chart shift $s \in \{u,t\}$,

$$z_s \le q_{\max}/2 - b_0 + \lambda. \tag{7.12}$$

On the selected chart, $s = \min(u,t)$. So $s^2 \le L$ gives the stronger individual estimate

$$z_s \le \frac{1}{6} + \frac{e}{2} + \frac{\lambda}{2}.$$

The alternative chart uses the other shift and the common bound (7.12). The stronger estimate holds only for the selected smaller shift. It is recorded for that source coordinate. It is not used in (7.15). Both chart caps use the common shift bound, which applies to either shift. The limiting gap in this comparison is

$$\frac{q_{\max}}{2} - b_0 - \left(\frac{1}{6} + \frac{e}{2}\right) = \frac{30869}{1000000} > 0. \tag{7.13}$$

The common shift bound minus the selected-shift bound is exactly

$$\frac{30869}{1000000} + \frac{\lambda}{2} > 0.$$

Thus their ordering needs no extra restriction on λ. Small λ is used below for the other strict endpoint comparisons. Its terms tend to zero and are absorbed by their fixed margins. Since $q_M > q_{\min}$,

$$q_M/2 - z_s \ge b_0 - 3e/2 - \lambda > 0$$

for all sufficiently large Y. Therefore

$$\frac{\partial \rho_1^*}{\partial \sigma} = \frac{q_M/2 - z_s}{\sigma^2} > 0, \qquad \frac{\partial \rho_1^*}{\partial z_s} = \frac{1}{\sigma} > 0.$$

Hence the maximum in σ is at $\sigma = 1$. Fix z_s. On the triangle branch

$$\frac{d}{dq_M}\left[\frac{9}{8} q_M (1 - q_M/2 + z_s)\right] = \frac{9}{8}(1 - q_M + z_s) > 0.$$

On the trapezoid branch, with $r = q_M/2 - z_s$,

$$\frac{d}{dq_M} \frac{q_M}{1 + 3(q_M/2 - z_s)^2} = \frac{1 - 3q_M^2/4 + 3z_s^2}{(1 + 3(q_M/2 - z_s)^2)^2} > 0,$$

since $q_M < 1$. So this chart's maximum is at $q_M = q_{\max} + 2\lambda$. At the joint upper endpoints $\sigma = 1$, $z_s = q_{\max}/2 - b_0 + \lambda$, the ratio is $\rho_1^* = 1 - b_0$. This is the alternative u-chart bound, even on a source whose selected chart is t. The comparison follows from the two lower shift bounds and $ut = L$.

In the second chart, $s = t$. So the opposite shift satisfies $z_u \ge b_0$ on the entire $5W$ source, and $\ell_L \le 1/3 + e + \lambda$. Its limiting coordinates are

$$(c^*, \tau^*) = (1 - z_u, 1 - \ell_L + q_M/2), \qquad \rho_2^* = \frac{1 - z_u}{1 - \ell_L + q_M/2}. \tag{7.14}$$

At $\lambda = 0$, increasing ℓ_L and decreasing z_u increase the ratio. So $\rho_2^* \le (1 - b_0)/(q_{\min} + q_M/2)$. For finite Y, the denominator is at least $D_\lambda = q_{\min} + q_M/2 - \lambda$, where $D = q_{\min} + q_M/2$. With $c = 1 - b_0$, the triangle-branch derivative of $q_M \phi(c/D_\lambda)$ is

$$\frac{9}{8} \frac{c(q_{\min} - \lambda)}{D_\lambda^2} > 0.$$

Here $9/8$ is the value of $\phi(\rho)/\rho$ on this branch. On the trapezoid branch the derivative has the sign of

$$1+3(1-\rho_2)^2-3(q_M/D_\lambda)\rho_2(1-\rho_2)>1-3\rho_2(1-\rho_2)\ge 1/4,$$

since $q_M<D_\lambda$ for all sufficiently large Y. The maximum is at $q_M=q_{\max}+2\lambda$. There $D_\lambda=1$ and $\rho_2^*=1-b_0$. The same endpoint applies to the first chart above. Thus both chart comparisons give the finite bound

$$q_M\phi(\rho_j^*)\le (q_{\max}+2\lambda)\phi(1-b_0)\le \bar m+2\lambda. \tag{7.15}$$

Here $j=1,2$ indexes the two local counting orientations and their starred comparison ratios. Either orientation is available for every source, whatever its selected-chart tag. The collar below compares the local ratios with them. Their common limiting cap as $Y\to\infty$ is $q_{\max}/(1+3b_0^2)=\bar m$. They are alternative counting routes for the same full $5W$ source. Each source box is counted once, by its better route. The choice is made separately for each box.

It remains to bound the finite collars. For $\rho\in(0,1)$, $\phi'\le 9/8$. On the triangle branch $\phi'=9/8$. On the trapezoid branch, with $r=1-\rho\in[0,1/3]$, $\phi'=6r/(1+3r^2)^2\le 9/8$. Thus the partial derivatives of $q_M\phi(c_0/\tau)$ in (q_M,c_0,τ) are bounded by $(1,9/4,9/4)$ whenever $q_M<1$ and $\tau>1/2$. In the two charts, (7.9) and the local scale definitions give

$$|c_0-c^*|\le B_Y,\quad |\tau-\tau^*|\le B_Y,\qquad B_Y=6d_2+\tfrac12\ell_{\mathrm{end}}+\lambda. \tag{7.16}$$

For chart one the exact residuals are (7.11). For chart two they are

$$\begin{aligned}\log_Y(BC)-(1-z_u)&=\log_Y\frac{m+n+1}{Y},\\ \tau^*-\log_Y(B^2f/t)&=\log_Y\left(\frac{Y}{m+n+1}\sqrt{\frac{m+1}{m}}\sqrt{\frac{n}{n+1}}\right).\end{aligned} \tag{7.17}$$

Here $Y/2<m+n+1\le 2Y$. So the logarithm of $(m+n+1)/Y$, and also its negative, has magnitude at most d_2. In (7.11), the middle factor $n/(m+n+1)$ lies between $1/(2+2/Y)$ and 1. Its logarithm has magnitude at most $d_2+\lambda$. The other factors contribute at most $(h_m+h_n)/2$. The local dyadic replacement of AD by D_0E_0 costs at most $2d_2$. The replacement of A^2f/u by X/H_0 costs at most $4d_2$. In the second chart, the first residual in (7.17) costs at most d_2, and the second at most $d_2+(h_m+h_n)/2$. Its local scale changes have the same respective costs $2d_2$ and $4d_2$. Also $h_m+h_n\le\ell_{\mathrm{end}}$, where $h_m=\log_Y(1+1/m)$, $h_n=\log_Y(1+1/n)$. The bounds $m>Y^{\tau_0}/\sqrt3$ and $n>Y/2$ imply

$$h_m+h_n\le\log_Y\big((1+\sqrt3Y^{-\tau_0})(1+2/Y)\big)=\ell_{\mathrm{end}}.$$

The larger of these bounds gives (7.16), with room for the extra λ term in the first chart. This keeps one d_2 unit of slack beyond the largest bound above.

Put

$$E_Y=\rho_Y+\log_Y(96\cdot18\cdot324\cdot8\cdot3\cdot2),\qquad \rho_Y=11d_2+2\ell_{\mathrm{end}}+2\lambda. \tag{7.18}$$

The six factors pay for fixed comparisons. The factor 96 is used in $X/H_0>Y^{\tau_0}/96$. The factor 18 is used in $M<18U_0V_*$. The factor 324 is used in the first anchor $\Lambda_1=\log(324U_0V_*)$, together with $U_0V_*\le 4M$. The factor 8 is used in the second anchor $\Lambda_2=\log(8D_0E_0)$. The factors 3 and 2 are used in $X_1=N/3$ and in $X_2=X/(2H_0)$. The source collars contribute at most 2λ in q_M. Then (7.16) contributes at most $(1+9/4+9/4)B_Y+2\lambda\le 6\rho_Y$. With the fixed logarithmic comparisons above, these terms are bounded by $6E_Y$. The upper comparison $U_0V_*\le 4M$ contributes $\log_Y 4=2d_2$. This lies in the $11d_2$ part of ρ_Y, hence in the $6\rho_Y$ collar. Also

$$\frac{\Lambda_1}{\log Y}=q_M+\log_Y(324U_0V_*/M),$$

so the coefficient of $q_M = \log_Y M$ in this anchor is exactly one. Hence the full finite cap is at most $\bar{m} + 6E_Y$. Since $E_Y \to 0$, we may impose $6E_Y < \xi/2$. This leaves a grid margin of at least $\xi/2$ in (7.6).

On the trapezoid branch the strict determinant surplus at the limiting source comparison is at least

$$\begin{aligned} \frac{q_{\min}\xi}{\bar{m}(\bar{m}+\xi)} &= \frac{9978467548053735534965717685839862 73}{6225284767495687836225589800000000000000} \\ &= 0.0001602893\ldots > 0. \end{aligned} \tag{7.19}$$

The triangle branch has its own strict surplus. The fixed margin ξ is charged in (7.6), before the degree family is chosen. The closed source-parameter ranges are compact, and (7.19) gives a positive surplus. Together they give a fixed $\delta_T > 0$ below the minimum trapezoid-branch gap. The inequalities

$$q_{\min} - a_{\text{grid}} > 0, \quad \tau_0 - a_{\text{grid}} > 0, \quad \tau_0 - q_{\max} = \frac{9351751}{30000000} > 0, \quad b_0 - \frac{3e}{2} = \frac{972179}{7500000} > 0 \tag{7.20}$$

put all ratios in compact subintervals of their required ranges. Thus one finite set of degrees, one Taylor order, and one threshold work for every cell and shift.

Lemma 7.1 (Ratio-cell determinant)**.** *Fix the parameters in* (7.2)–(7.6)*, with* $3/5 < a_{\text{grid}} < \min(q_{\min}, \tau_0)$*,* $q_{\max} < \tau_0$*, and* $b_0 > 3e/2$*. Let* Y *be sufficiently large. Keep the exact first-match* $5W$ *population* (3.1)*, with its source masks, original selected-chart labels, and all preceding-row complements. Take either counting orientation, any local box satisfying* (7.4)–(7.5) *and* $X/H_0 \le Y^4$*, and any cell in* (7.7)*. Then there is a nonzero polynomial* $F_i \in \mathbb{Z}[X, Z]$ *of degree at most* D_θ *and coefficient height at most* Y^{H_θ}*. Here* D_θ, H_θ *depend only on the fixed parameter vector. It satisfies* $F_i(J/K, g/f) = 0$ *for every retained source point in that cell and every admissible shift in the box. The bounds are uniform in the counting orientation, original selected chart, box, cell, labels and shift.*

Proof. The proof is an integer calculation. We give the form used below. Put

$$x = \frac{f}{g}, \quad y = \frac{K}{J}, \quad y_0 = r_i^{-1}, \quad \delta_{\text{cell}} = \frac{s}{fJ^2}, \quad \alpha_{\text{cell}} = \frac{x}{y_0^2} - 1.$$

The conic gives $x = y^2/(1 + \delta_{\text{cell}})$ and $y/y_0 = \sqrt{1 + \alpha_{\text{cell}}}\sqrt{1 + \delta_{\text{cell}}}$. As $J/K \in (r_i, r_{i+1}]$, we have $i/(i+1) \le y/y_0 < 1$, with $i \ge N$. Also

$$0 < \delta_{\text{cell}} \le \frac{H_0}{U_0 D_0^2} = \frac{H_0}{X} \le \frac{1}{N}$$

for sufficiently large Y, since $X/H_0 > Y^{\tau_0}/96$, $N \le 2Y^{a_{\text{grid}}}$, and $a_{\text{grid}} < \tau_0$. Therefore

$$|\alpha_{\text{cell}}| = 1 - \frac{(y/y_0)^2}{1 + \delta_{\text{cell}}} \le \frac{2}{N} + \frac{H_0}{X} \le \frac{3}{N}.$$

Next,

$$0 \le \frac{y}{y_0} - \sqrt{1 + \alpha_{\text{cell}}} = \sqrt{1 + \alpha_{\text{cell}}}\big(\sqrt{1 + \delta_{\text{cell}}} - 1\big) \le \frac{H_0}{2X},$$

using $\sqrt{1 + \alpha_{\text{cell}}} \le 1$ and $\sqrt{1 + \delta} - 1 \le \delta/2$. Thus

$$|\alpha_{\text{cell}}| \le 3/N, \qquad 0 \le y/y_0 - \sqrt{1 + \alpha_{\text{cell}}} \le H_0/(2X). \tag{7.21}$$

As $X_2 = X/(2H_0)$, the conic error is $H_0/(2X) = X_2^{-1}/4$. Set $T_{\text{tay}} = 7$. Let $P_{T_{\text{tay}}}$ be the degree-seven Taylor polynomial of $\sqrt{1 + \alpha_{\text{cell}}} - 1$. For $c_k = |\binom{1/2}{k}|$, we have $c_{k+1}/c_k = (k - 1/2)/(k+1) < 1$

for $k \geq 1$, and $c_8 = 429/32768$. So, for $|\alpha_{\mathrm{cell}}| \leq 1/2$, the remainder is at most

$$\frac{c_8|\alpha_{\mathrm{cell}}|^8}{1-|\alpha_{\mathrm{cell}}|} \leq \frac{429}{16384}|\alpha_{\mathrm{cell}}|^8 < 2|\alpha_{\mathrm{cell}}|^8.$$

The source gives $X/H_0 \leq Y^4$ and $a_{\mathrm{grid}} > 3/5$. Hence, once $4 \cdot 3^8 Y^{-8a_{\mathrm{grid}}} \leq Y^{-4}$, the Taylor remainder is at most $X_2^{-1}/2$, where $X_1 = N/3$, $X_2 = X/(2H_0)$. To check this, $X/H_0 \leq Y^4$ gives $X_2^{-1} \geq 2Y^{-4}$. So the Taylor bound is at most $Y^{-4}/2 \leq X_2^{-1}/2$. Add the conic error $X_2^{-1}/4$. Then $\beta_{\mathrm{cell}} = y/y_0 - 1 - P_{T_{\mathrm{tay}}}(\alpha_{\mathrm{cell}})$ satisfies $|\alpha_{\mathrm{cell}}| \leq X_1^{-1}$, $|\beta_{\mathrm{cell}}| \leq X_2^{-1}$, uniformly.

Take columns $x^r y^j$, $0 \leq r \leq A_{\mathrm{det}}$, $0 \leq j \leq B_{\mathrm{det}}$. Multiply the row for (f, g, J, K) by $g^{A_{\mathrm{det}}} J^{B_{\mathrm{det}}}$. The entries are the integers

$$f^r g^{A_{\mathrm{det}}-r} K^j J^{B_{\mathrm{det}}-j}. \tag{7.22}$$

After the anchor is removed the entries are $(1+\alpha_{\mathrm{cell}})^r(1 + P_{T_{\mathrm{tay}}}(\alpha_{\mathrm{cell}}) + \beta_{\mathrm{cell}})^j$. With $D_c = (A_{\mathrm{det}}+1)(B_{\mathrm{det}}+1)$, $g < 9V_*$, $y_0 \leq 2E_0/D_0$, and $J < 2D_0$, the total row and column arithmetic cost is at most

$$\frac{D_c}{2}\{A_{\mathrm{det}}\Lambda_1 + B_{\mathrm{det}}\Lambda_2\}, \qquad \Lambda_1 = \log(324U_0V_*), \quad \Lambda_2 = \log(8D_0E_0). \tag{7.23}$$

This includes the factor $y_0^{D_c(A_{\mathrm{det}}+B_{\mathrm{det}}/2)}$. To verify $g < 9V_*$, multiply the conic $g/f = (J/K)^2 + s/(fK^2)$ by f:

$$g = f(J/K)^2 + s/K^2 < 8U_0D_0^2/E_0^2 + H_0/E_0^2 = 8V_* + H_0/E_0^2 < 9V_*.$$

Here $f < 2U_0$, $J/K < 2D_0/E_0$, $s < H_0$, $K \geq E_0$, and $X/H_0 > Y^{\tau_0}/96 > 1$ for sufficiently large Y.

For the finite Cauchy–Binet calculation let $\ell_1 = \log X_1$, $\ell_2 = \log X_2$, $p_1 = \Lambda_1/\ell_1$, $p_2 = \Lambda_2/\ell_2$, and $\ell_{\mathrm{det}} = \ell_1/\ell_2$. Set $\ell = \ell_{\mathrm{det}}$. Here $X_1 = N/3$, $X_2 = X/(2H_0)$, $\ell_1 = \log(N/3)$, and $\ell_2 = \log(X/(2H_0))$. So the finite parameters in the strip comparison are exactly

$$p_1 = \frac{\log(324U_0V_*)}{\log(N/3)}, \qquad p_2 = \frac{\log(8D_0E_0)}{\log(X/(2H_0))}, \qquad \ell_{\mathrm{det}} = \frac{\log(N/3)}{\log(X/(2H_0))}.$$

Define the strip weight

$$W_{\mathrm{strip}} = \min_{\substack{\mathcal{S} \subset \mathbb{Z}_{\geq 0} \times \{0,\dots,B_{\mathrm{det}}\} \\ |\mathcal{S}| = D_c}} \sum_{(i,k)\in\mathcal{S}} (i\ell_1 + k\ell_2).$$

After column normalization the entries are polynomials in $(\alpha_{\mathrm{cell}}, \beta_{\mathrm{cell}})$ of β_{cell}-degree at most B_{det}. The Taylor coefficients of $\sqrt{1+\alpha_{\mathrm{cell}}} - 1$ have absolute sum at most one. So each column's coefficient ℓ^1-norm is at most $2^{A_{\mathrm{det}}}3^{B_{\mathrm{det}}}$. Cauchy–Binet, Hadamard, and the at most $2^{(A_{\mathrm{det}}+1+B_{\mathrm{det}}T_{\mathrm{tay}})(B_{\mathrm{det}}+1)}$ subsets of the finite monomial set then give

$$\begin{aligned} \log|\Delta_1| &\leq \log K_{A_{\mathrm{det}},B_{\mathrm{det}},T_{\mathrm{tay}}} - W_{\mathrm{strip}}, \\ K_{A_{\mathrm{det}},B_{\mathrm{det}},T_{\mathrm{tay}}} &= 2^{(A_{\mathrm{det}}+1+B_{\mathrm{det}}T_{\mathrm{tay}})(B_{\mathrm{det}}+1)}(D_c!)^2(2^{A_{\mathrm{det}}}3^{B_{\mathrm{det}}})^{D_c}. \end{aligned} \tag{7.24}$$

Here $K_{A_{\mathrm{det}},B_{\mathrm{det}},T_{\mathrm{tay}}}$ is fixed once the degree and Taylor order are fixed. It is independent of Y, the box, the cell, and the shift.

We give the strip lower bound explicitly. Put $A' = A_{\mathrm{det}} + 1$, $B' = B_{\mathrm{det}} + 1$, so $D_c = A'B'$. Suppose $A'\ell_1 \geq B'\ell_2/2$. To each lattice point (i,k) attach its unit square $[i, i+1) \times [k, k+1)$. For $\Phi(x,y) = x\ell_1 + y\ell_2$, the integral over that square is $i\ell_1 + k\ell_2 + (\ell_1+\ell_2)/2$. The union E of the D_c squares has area D_c and lies in the strip $x \geq 0, 0 \leq y \leq B'$. Among all sets of this area in that strip, the integral of Φ is smallest on the sublevel set $E_* = \{\Phi \leq w\} \cap \{x \geq 0,\ 0 \leq y \leq B'\}$. Points outside E_* have $\Phi \geq w$, and points removed from E_* have $\Phi \leq w$. So exchanging equal

areas cannot lower the integral. The condition above makes E_* a trapezoid with right edge $x = (w - y\ell_2)/\ell_1$. Its area is

$$\frac{wB' - \ell_2 B'^2/2}{\ell_1} = A'B', \qquad w = A'\ell_1 + B'\ell_2/2.$$

Integrating first in x, then in y, gives

$$\int_{E_*} \Phi = \int_0^{B'} \frac{w^2 - y^2\ell_2^2}{2\ell_1}\, dy = \frac{1}{2}A'^2B'\ell_1 + \frac{1}{2}A'B'^2\ell_2 - \frac{B'^3\ell_2^2}{24\ell_1}. \tag{7.25}$$

Subtracting the unit-square correction gives

$$W_{\mathrm{strip}} \geq \frac{1}{2}A'^2B'\ell_1 + \frac{1}{2}A'B'^2\ell_2 - \frac{B'^3\ell_2^2}{24\ell_1} - \frac{1}{2}A'B'(\ell_1 + \ell_2). \tag{7.26}$$

The integer row clearing and the anchor factors have total logarithmic cost at most $C_{\mathrm{arith}} = D_c(A_{\det}\Lambda_1 + B_{\det}\Lambda_2)/2$, as in (7.23). As $A_{\det} \leq A'$ and $B_{\det} \leq B'$, this is at most $D_c(A'\Lambda_1 + B'\Lambda_2)/2$. Put $z = A'/B'$. Division by $B'^3\ell_2/2$ gives, on the trapezoid branch,

$$W_{\mathrm{strip}} - C_{\mathrm{arith}} \geq \frac{B'^3\ell_2}{2}\left[-\ell_{\det}(p_1 - 1)z^2 + (1 - p_2)z - \frac{1}{12\ell_{\det}} - \frac{z(\ell_{\det} + 1)}{B'}\right]. \tag{7.27}$$

The first three terms form the concave quadratic

$$G(z) = -\ell_{\det}(p_1 - 1)z^2 + (1 - p_2)z - \frac{1}{12\ell_{\det}}.$$

If $p_2 \geq 2/3$, $p_1 > 1$, and $p_1 - 1 \leq 3(1 - p_2)^2 - \delta_T$, then

$$z_* = \frac{1 - p_2}{2\ell_{\det}(p_1 - 1)} \geq \frac{1}{2\ell_{\det}}, \qquad G(z_*) = \frac{3(1 - p_2)^2 - (p_1 - 1)}{12\ell_{\det}(p_1 - 1)} > 0.$$

Here we use $3(1 - p_2)^2 \leq 1 - p_2$ for $p_2 \geq 2/3$. On the compact source range, z_*, ℓ, and $p_1 - 1$ are uniformly bounded, and $\ell(p_1 - 1)$ is bounded away from zero. Choose one fixed B' so large that the rounding loss $\ell(p_1 - 1)/B'^2$ and the boundary term $(z_* + 1/B')(\ell + 1)/B'$ sum to less than half the minimum of $G(z_*)$. Then set $A' = \lceil z_*B' \rceil$. Since $G(z) = G(z_*) - \ell(p_1 - 1)(z - z_*)^2$, the bracket in (7.27) is positive uniformly. The strict source margin in (7.19) gives the positive minimum. Choose the fixed lower threshold so that the total $O(1/\log Y)$ collar is at most half that minimum. The finite comparison then keeps at least half the limiting gap uniformly.

For $p_2 \leq 2/3$, use the whole quadrant in place of the strip. The sublevel set $\Phi \leq w$ in $x, y \geq 0$ is a triangle of area

$$\frac{w^2}{2\ell_1\ell_2} = D_c, \qquad \int_{E_*} \Phi = \frac{2\sqrt{2}}{3}D_c^{3/2}\sqrt{\ell_1\ell_2}.$$

The unit-square error is $D_c(\ell_1 + \ell_2)/2$. Put $z = A'/B'$. The leading weight minus the upper bound for C_{arith}, divided by $B'^3\ell_2/2$, is

$$z\left(\frac{4\sqrt{2}}{3}\sqrt{\ell z} - p_1\ell z - p_2\right).$$

Set $v = \sqrt{\ell z}$. The parenthesis is the concave quadratic $(4\sqrt{2}/3)v - p_1v^2 - p_2$. Its maximum is at $v = 2\sqrt{2}/(3p_1)$, which corresponds to

$$z_* = \frac{8}{9p_1^2\ell}, \qquad A' = \left\lceil \frac{8B'}{9p_1^2\ell} \right\rceil,$$

and the maximum value is $(8/9 - p_1p_2)/p_1$. Thus the triangle branch has a uniform positive gap whenever $p_1p_2 < 8/9$. A fixed large enough B' absorbs the square and rounding errors. At

$p_2 = 2/3$, the two threshold conditions agree. In both branches the source inequality $N > Y^{q\phi(\rho)}$, with the fixed margin (7.19), gives these strict conditions for every cell.

After the fixed collar cutoff, the normalized bracket has a uniform positive lower bound on both branches. Also $\ell_2 = \log(X/(2H_0)) > \tau_0 \log Y - \log 192$. Hence $W_{\mathrm{strip}} - C_{\mathrm{arith}} \geq c_\theta \log Y$ for some fixed $c_\theta > 0$ and all sufficiently large Y. The degrees and Taylor order range over a fixed finite family. So

$$K_{\max} = \max K_{A_{\det}, B_{\det}, 7}$$

is fixed. Increase the determinant cutoff until $c_\theta \log Y > \log K_{\max}$. Then $W_{\mathrm{strip}} - C_{\mathrm{arith}} > \log K_{\max}$ uniformly. Every cleared D_c-minor is an integer of absolute value less than one, hence zero. The matrix has rank $r < D_c$. If $r = 0$, a coordinate vector gives a kernel polynomial. Otherwise choose r pivot columns and r rows giving a nonsingular minor A, then one free column j. Set the j-coordinate to $\det A$, the other free coordinates to zero, and the pivot coordinates to $-\operatorname{adj}(A) M_{I,j}$. The resulting nonzero integer vector annihilates the selected rows. They span the row space, so it annihilates every row. Its coordinates are bounded-order minors, so they have polynomial height. This gives a nonzero integer polynomial $C(x, y)$ vanishing at $(x, y) = (f/g, K/J)$. If a cell has fewer than D_c rows, then $r < D_c$ is automatic and its points are counted directly; this includes an empty cell. Define

$$F_i(R, S) = R^{B_{\det}} S^{A_{\det}} C(1/S, 1/R). \tag{7.28}$$

This is a nonzero integer polynomial of degrees at most $(B_{\det}, A_{\det})$, with the same coefficient height. It vanishes at $(R, S) = (J/K, g/f)$. The substitution permutes the coefficients, so the integral polynomial and its height are unchanged. This proves the ratio-cell lemma. □

7.2. **One reduced lattice and the complete cell census.** The auxiliary polynomial is bihomogenized in the two primitive pairs. For its high-degree factors use the single canonical coefficient lattice

$$j(i) = \left\lfloor \frac{i^2}{L_{\mathrm{grid}}} \right\rfloor, \quad t_i = \frac{j(i)}{L_{\mathrm{grid}}} \frac{V_*}{U_0}, \quad W_i(g, f) = \left(\frac{L_{\mathrm{grid}}}{V_*}(g - t_i f), \frac{f}{U_0} \right). \tag{7.29}$$

Its determinant is $L_{\mathrm{grid}}/(U_0 V_*)$. The map $i \mapsto j(i)$ is injective, because $((i+1)^2 - i^2)/L_{\mathrm{grid}} = (2i+1)/(2N) > 1$. Recall $R_{\mathrm{box}} = D_0/E_0$. So $V_*/U_0 = R_{\mathrm{box}}^2$, and

$$W_i = \begin{pmatrix} L_{\mathrm{grid}}/V_* & -j(i)/U_0 \\ 0 & 1/U_0 \end{pmatrix}.$$

The source conic gives $g/f = (J/K)^2 + s/(fK^2)$. Since $r_i = iR_{\mathrm{box}}/L_{\mathrm{grid}}$, $J/K \leq r_i(1 + 1/i)$, and $0 \leq r_i^2 - t_i < R_{\mathrm{box}}^2/L_{\mathrm{grid}}$, for $N \geq 4$ we get

$$0 < g/f - t_i \leq R_{\mathrm{box}}^2 \left(\frac{10}{N} + \frac{H_0}{X} \right).$$

As $f < 2U_0$, $L_{\mathrm{grid}} = 2N$, and $NH_0/X \leq 2^{-20}$, the first coordinate of $W_i(g, f)$ lies between zero and 41. Its second lies in $[1, 2]$. So all source images lie in one fixed bounded rectangle. Put $\Lambda_i = W_i \mathbb{Z}^2$. Choose a Lagrange-reduced basis $v_{1,i}, v_{2,i}$ of this image lattice, with lengths $\lambda_{1,i} \leq \lambda_{2,i}$ and angle bounded away from zero. Write $v_{j,i} = W_i h_{j,i}$. Since $v_{1,i}, v_{2,i}$ are a basis of Λ_i, their integer preimages $h_{1,i}, h_{2,i}$ form a unimodular basis of $\mathbb{Z}^2$. The vector $v_{1,i}$ is shortest in the Euclidean norm of the image lattice Λ_i. Define

$$T_{1,i} = \frac{32\sqrt{2}}{\lambda_{1,i}}, \qquad T_{2,i} = \frac{32\sqrt{2}}{\lambda_{2,i}}.$$

Then $T_{2,i} \leq T_{1,i}$. The determinant identity gives $T_{1,i} T_{2,i} \asymp U_0 V_*/L_{\mathrm{grid}}$. A unimodular change preserves primitivity. By the bounded rectangle and the reduced-basis angle, every source point

has coordinates $O(T_{1,i})$ in this basis. The map and its inverse are rational matrices of polynomial height. Explicitly,

$$W_i^{-1}(y_1, y_2) = \left(\frac{V_*}{L_{\mathrm{grid}}}(y_1 + j(i)y_2), U_0 y_2 \right).$$

A common denominator for W_i is at most a fixed power of Y. So every nonzero vector of Λ_i has length at least Y^{-C}. Its covolume is $L_{\mathrm{grid}}/(U_0V_*) \le Y^C$. Minkowski's bound and the reduced-angle determinant identity then give $\lambda_{1,i}, \lambda_{2,i} \le Y^C$. The inverse matrix gives $\|h_{j,i}\| \le Y^{C'}$. Thus the unimodular basis and every fixed-degree transformed factor have polynomial height, uniformly in the box, cell and labels. The first-coordinate radius $T_{1,i}$ satisfies

$$\sum_i (T_{1,i} + 1) \ll_\eta Y^\eta \{N + \sqrt{U_0 V_* N} + U_0 + V_*\}. \tag{7.30}$$

Take a dyadic range $D_{\mathrm{lat}} < T_{1,i} \le 2D_{\mathrm{lat}}$. Write the primitive shortest preimage in source coordinates as (ω_1, ω_2). It satisfies

$$|\omega_1| \ll V_*/D_{\mathrm{lat}}, \quad |\omega_2| \ll U_0/D_{\mathrm{lat}}, \quad \omega_2 j(i) = \frac{L_{\mathrm{grid}}U_0}{V_*}\omega_1 + O(U_0/D_{\mathrm{lat}}). \tag{7.31}$$

The first coordinate bound follows from the first component of W_i, the bound $j(i)/L_{\mathrm{grid}} = O(1)$, and $|\omega_2| \ll U_0/D_{\mathrm{lat}}$. Suppose both coordinates are nonzero. Fix ω_1 first. There are $O(V_*/D_{\mathrm{lat}})$ choices. For each such ω_1, (7.31) places the integer product $w_i = \omega_2 j(i)$ in an interval of length $O(U_0/D_{\mathrm{lat}})$. As $\omega_2 \neq 0$, $U_0/D_{\mathrm{lat}} \gg 1$. So this interval contains $O(U_0/D_{\mathrm{lat}})$ integers. Also $w_i \neq 0$, because $j(i) > 0$. For each fixed w_i, the divisor bound gives Y^η choices of $(\omega_2, j(i))$. Injectivity of j gives at most one index i for each $j(i)$. So there are $O_\eta(Y^\eta U_0 V_*/D_{\mathrm{lat}}^2)$ indices in this dyadic range. Weighting by D_{lat} and summing from the lattice minimum gives $O_\eta(Y^\eta\sqrt{U_0V_*N})$. If $\omega_2 = 0$, primitivity gives $\omega_1 = \pm 1$, and the total is $O(V_* + N)$. If $\omega_1 = 0$, it is $O(U_0 + N)$. The cells with $T_{1,i} < 1$ and all endpoint $+1$ terms are included in N. This proves (7.30).

The nonzero product is at most a fixed power of Y. So its divisor estimate is uniform over the boxes and cells. The lower limit $D_{\mathrm{lat}} \gg \sqrt{U_0V_*/N}$ follows from $T_{1,i}T_{2,i} \asymp U_0V_*/N$ and $T_{1,i} \ge T_{2,i}$. Summing the unweighted number of non-axis indices over the same dyadic ranges gives $O_\eta(Y^\eta N)$. On the first axis, $\omega_2 = 0$. There primitivity gives $\omega_1 = \pm 1$, and the contribution is $V_* + N$. On the second axis, $\omega_1 = 0$. There it gives $\omega_2 = \pm 1$, and the contribution is $U_0 + N$. The $T_{1,i} < 1$ bins contribute another $O(N)$. The bound (7.30) keeps all these terms. The bins with $T_{2,i} < 1$ are included in the $+1$ terms.

A factor has degree (a_c, b_q): corner degree first, coefficient degree second. If $a_c \ge 2$ and $b_q \ge 1$, apply Heath–Brown's published Lemma 2. Put the primitive coefficient-lattice pair first and the primitive corner pair second. The bound is

$$O_{a_c,b_q,\eta}\big((T_{1,i} + 1)^{2/a_c+\eta}\|F\|^\eta\big), \qquad 2/a_c \le 1, \tag{7.32}$$

and the second pair is unrestricted. The statement in Appendix E, cited as [4, Lemma 2, p. 8], applies with the reduced coefficient-lattice pair first and the corner pair second. Reversing the degree order gives second-pair degree a_c. The coefficient pair is bounded by $X \asymp T_{1,i} + 1$, while the corner pair is unrestricted. The source gcd conditions make both pairs primitive. The factor height is $Y^{O(1)}$ before η is chosen. The stated bound therefore gives (7.32). Polynomial factor heights and the finite degree family let the loss be made arbitrary.

The remaining factors are as follows.

(1) A rational factor that is not absolutely irreducible has rational points only on intersections with distinct conjugate components. Resultants in the four affine and infinity charts give $O(1)$ points at fixed degree.

(2) If $a_c = 0$, a coefficient pair fixes the polynomial but not yet the shift. First choose its ordered residual factors. These fix the nonzero integer $L = cd - ab$. Then enumerate $s \mid L$. Apply the uniform square-fibre theorem of Appendix B with $(a, b, h, n, m) = (f, g, s, J, K)$, $x = 2Y$, and fixed height exponent $A_{\mathrm{ht}} = 2$. This costs Y^{η}, including the shift divisor count.

(3) If $b_q = 0$, fix (J, K) and compare two coefficient pairs:

$$\begin{aligned} K^2(g_1 f_2 - g_2 f_1) &= s_1 f_2 - s_2 f_1, \\ |g_1 f_2 - g_2 f_1| &< \frac{3S_h U_0}{E_0^2} = \frac{3U_0 V_* H_0}{2X} \le \frac{6MH_0}{X} \\ &< 576 Y^{q_{\max} + 2\lambda - \tau_0} = o(1). \end{aligned} \tag{7.33}$$

The determinant is an integer, so it vanishes for all sufficiently large Y. Primitivity forces the coefficient pairs to agree. Then the conic fixes the shift. The strict gap is $\tau_0 - q_{\max} = 9351751/30000000$. Since $s_i f_j \in [S_h U_0, 4S_h U_0)$, the difference of two such products has absolute value less than $3S_h U_0$. Then (7.5) gives the bound.

(4) For $b_q \ge 1$ and $a_c = 1$, write the factor as $K\mathcal{P}(f, g) + J\mathcal{Q}(f, g)$, with coprime homogeneous $\mathcal{P}, \mathcal{Q}$. Their resultant is nonzero. By its Bezout identities, $\gcd(\mathcal{P}(f, g), \mathcal{Q}(f, g))$ divides a fixed polynomial-height integer. Write $d_0 = \gcd(\mathcal{P}(f, g), \mathcal{Q}(f, g))$. Homogeneous Bezout identities and $\gcd(f, g) = 1$ give $d_0 \mid \mathrm{Res}(\mathcal{P}, \mathcal{Q})$. Primitivity then determines the positive corner pair $(J, K) = \pm(\mathcal{P}(f, g)/d_0, -\mathcal{Q}(f, g)/d_0)$.

Let $d_f = f_J$ be the full prime-power part of f supported on primes dividing J. Let $d_g = g_K$ be the full prime-power part of g supported on primes dividing K. Put $v = f/d_f$, $w = g/d_g$. As d_f, d_g contain the full powers of their supported primes, d_f, d_g, v, w are pairwise coprime, and $d_f d_g v w = fg = M$. The four endpoint patterns use the labels

$\mathcal{P}(0,1)$	$\mathcal{Q}(1,0)$	ℓ_f	ℓ_g
$\neq 0$	$\neq 0$	$\mathrm{rad}(d_f)$	$\mathrm{rad}(d_g)$
$\neq 0$	$= 0$	$\mathrm{rad}(d_f)$	w
$= 0$	$\neq 0$	v	$\mathrm{rad}(d_g)$
$= 0$	$= 0$	v	w.

(7.34)

Suppose $\mathcal{P}(0, 1) \neq 0$. Every prime dividing d_f divides J and does not divide K or g. The factor equation modulo that prime gives $\mathcal{P}(f, g) \equiv 0$. Homogeneity then gives $\mathcal{P}(0, 1) \equiv 0$. Hence $\mathrm{rad}(d_f) \mid \mathcal{P}(0, 1)$. If $\mathcal{P}(0, 1) = 0$, then $f \mid \mathcal{P}(f, g)$. As $\gcd(v, J) = 1$, the factor equation gives $v \mid \mathcal{Q}(f, g)$. So $v \mid d_0 \mid \mathrm{Res}(\mathcal{P}, \mathcal{Q})$. The same argument with $g, K, w, \mathcal{Q}(1, 0)$ proves the two g-side entries. These divisibilities include full prime powers in v, w. The zero endpoint cases are handled by the matching table entries. So each label divides a fixed nonzero endpoint coefficient or resultant of polynomial height. Hence there are $\ll_{\eta} Y^{\eta}$ possible pairs of endpoint labels for any fixed $\eta > 0$.

For each of the four patterns, the sum shown is the lower-corner base-Y exponent of the matching full assigned-part product. In chart 1 these sums are

$$P + Q, \quad P + (q - c_1 - Q), \quad (c_1 - P) + Q, \quad (c_1 - P) + (q - c_1 - Q),$$

and in chart 2 they are

$$R_{\mathrm{LP}} + T, \quad R_{\mathrm{LP}} + (q - c_2 - T), \quad (c_2 - R_{\mathrm{LP}}) + T, \quad (c_2 - R_{\mathrm{LP}}) + (q - c_2 - T),$$

in the same row order. Here P, Q, R_{LP}, T are the logarithmic sums in (7.36). The forms are $\mathcal{P}, \mathcal{Q}$. Put $k(x) = x/\mathrm{rad}(x)$. Pairwise coprimality gives $k(d_f)k(d_g)k(v)k(w) = M/\mathrm{rad}(M) = K_{\mathrm{res}}$. A label is either the full assigned part or its radical. So in every

row

$$\ell_f\ell_g \geq \frac{\text{corresponding assigned-part product}}{K_{\mathrm{res}}}.$$

By $K_{\mathrm{res}} \leq Y^{\Delta_{\exp}}$, each matching actual product is at least $Y^{E_{\mathrm{pat}}}$, where E_{pat} is its lower-corner exponent. Thus $\ell_f\ell_g \geq Y^{E_{\mathrm{pat}}-\Delta_{\exp}}$. These are exactly the four chart pattern terms with loss $-\Delta_{\exp}$ in (7.37). Each of the eight assigned factors is below twice its dyadic lower endpoint. So $M \leq 2^8 Y^q$. Also $U_0V_* \leq 4M$. Hence

$$U_0V_* \leq 2^{10}Y^q, \qquad \frac{U_0V_*}{\ell_f\ell_g} \leq 2^{10}Y^{q-E_{\mathrm{pat}}+\Delta_{\exp}}.$$

For the spacing estimate, the conic gives

$$\frac{g}{f} = \left(\frac{J}{K}\right)^2 + \frac{s}{fK^2}.$$

Over a cell of width $O(D_0/(E_0N))$ in J/K, the first term varies by $O(D_0^2/(E_0^2N)) = O(V_*/(U_0N))$. The second term has size $O(H_0/(U_0E_0^2))$, smaller by the factor

$$\frac{H_0/(U_0E_0^2)}{V_*/(U_0N)} = \frac{NH_0}{X} = o(1)$$

by (7.5)–(7.6) and $a_{\mathrm{grid}} < \tau_0$. Hence the ratios g/f in one cell lie in an interval of length $O(V_*/(U_0N))$. Take two distinct primitive coefficient pairs with the same labels. Then $\ell_f\ell_g \mid f_1g_2 - f_2g_1$. This coefficient determinant is a nonzero integer, so its absolute value is at least $\ell_f\ell_g$. As $U_0 \leq f_1, f_2 < 2U_0$, their ratios differ by at least $\ell_f\ell_g/(4U_0^2)$. Therefore

$$\#\{\text{coefficient pairs in one cell}\} \ll 1 + \frac{U_0V_*}{N\ell_f\ell_g}. \tag{7.35}$$

The graph equation gives at most one positive primitive corner pair from each coefficient pair. Fix an endpoint pattern. Summing over the $3N$ cells and the divisor-bounded labels gives

$$O_\eta\left(Y^\eta\left(N + \frac{U_0V_*}{\ell_f\ell_g}\right)\right).$$

By the bound above for this pattern, the count is

$$O_\eta\left(Y^\eta\left(N + Y^{q-E_{\mathrm{pat}}+\Delta_{\exp}}\right)\right).$$

The $+1$ term, unit labels, and endpoint equalities are kept. The eight chart-pattern classes have fixed cardinality. So their sum adds no further power of Y.

The source corner coordinates are positive. The lattice census in (7.30) keeps both shortest-vector axes, which contribute $U_0 + V_*$. In either counting orientation take $c = c_1$ or $c = c_2$ respectively, as defined below. The four assigned factors of f and the four complementary factors of g give

$$U_0 \leq f < 2^4Y^c, \qquad V_* \leq 4g < 2^6Y^{q-c}.$$

These two terms are charged through the first two savings in (7.37). The four graph patterns give its other four savings. The grid and square-root terms are kept separately.

7.3. **The lower-corner linear program and lattice-point count.** The exact finite optimization below controls the source's choice of chart or corner route. In each source box, let $x_0, \dots, x_7$ be the lower endpoints of the base-Y logarithms of the eight uniquely assigned corner factors, in the order

$$(a_A, a_B, b_C, b_D, c_A, c_C, d_B, d_D).$$

Each factor is the full prime-power part of its residual supported on the eligible corner set. Supported complementary parts are fixed by the complete coefficient-corner labels and are charged through (6.6). Write

$$\begin{aligned} x_{AB} &= x_0 + x_1, & x_{AC} &= x_4 + x_5, & q &= \sum_{i=0}^{7} x_i, \\ x_{ABCD} &= x_0 + x_1 + x_2 + x_3, & c_1 &= x_{AB} + x_{AC}, & & \\ c_2 &= x_0 + x_1 + x_6 + x_7, & P &= x_0 + x_4, & Q &= x_3 + x_7, \\ R_{\mathrm{LP}} &= x_1 + x_6, & T &= x_2 + x_5. & & \end{aligned} \tag{7.36}$$

The source constraints include the canonical radical, shell, unit, chart, and corner-capacity inequalities. Their limiting affine logarithmic forms are listed in Appendix D. Their finite source collars are recorded below. The source log coordinates μ, z_u, z_t also use their box lower endpoints. Thus the eleven source coordinates are binned, and the auxiliary W is evaluated by (7.39) at those endpoints. With ρ_Y as in (7.18), every source-row inequality has lower-corner slack at most ρ_Y. The 22 affine source rows in Appendix D have coefficient ℓ^1-norm at most 8. So replacing the eleven source coordinates by reference values costs at most $8d_2$. In deriving these rows from (7.9), the remaining finite terms are logarithms of n/Y, $(m+n+1)/Y$, $(m+1)/m$, $(n+1)/n$, and $\kappa_Y = 1 + 1/Y$. The first two have absolute logarithm at most d_2. The next two together are at most ℓ_{end}. Each source row uses at most three further d_2 units, two ℓ_{end} units, and two powers of κ_Y. This gives the stated ρ_Y. The count is applied after these source-row comparisons.

For chart 1 put $(c, F, G) = (c_1, P, Q)$, and for chart 2 put $(c, F, G) = (c_2, R_{\mathrm{LP}}, T)$. Its six nominal savings are

$$\begin{aligned} &\frac{2}{3} - c, \quad \frac{2}{3} - q + c, \\ &\frac{2}{3} - q + F + G - \Delta_{\exp}, \quad \frac{2}{3} - q + F + (q - c - G) - \Delta_{\exp}, \\ &\frac{2}{3} - q + (c - F) + G - \Delta_{\exp}, \quad \frac{2}{3} - q + (c - F) + (q - c - G) - \Delta_{\exp}. \end{aligned} \tag{7.37}$$

Let A_j be their minimum. The four assigned-corner savings are

$$\begin{aligned} G_A &= \frac{2}{3} - \left(\frac{\mu + z_u - c_1}{2} + x_1 + x_5 + z_u \right), \\ G_D &= \frac{2}{3} - \left(\frac{\mu + z_u - q + c_1}{2} + x_2 + x_6 + z_u \right), \\ G_B &= \frac{2}{3} - \left(\frac{1}{2} - \frac{z_u}{2} + \frac{q}{4} - \frac{c_2}{2} + x_0 + x_7 + z_t \right), \\ G_C &= \frac{2}{3} - \left(\frac{1}{2} - \frac{z_u}{2} - \frac{q}{4} + \frac{c_2}{2} + x_3 + x_4 + z_t \right). \end{aligned} \tag{7.38}$$

At the lower corner set

$$W = \max(0, A_1, A_2, G_A, G_D, G_B, G_C). \tag{7.39}$$

The exact linear program has the twelve nonnegative coordinates $(x_0,\dots,x_7,W,z_u,z_t,\mu)$, 22 source rows, two chart rows and four corner-route rows. The 72 branches come from the selected-chart choice and the two six-way choices of minimizing chart scores. Dropping the inequalities that enforce those choices enlarges each branch polytope. Exact primal and dual elimination on these rational polytopes gives

$$W \ge L_{5W} - 5\rho_Y, \qquad L_{5W} = \frac{1203113}{90000000}. \tag{7.40}$$

In every branch the counting-row dual mass is 1, the source-row dual mass is at most 5, and the total dual mass is at most 6. The finite certificate and its exact row matrix are given in Appendix D.

Next we tie these nominal rows to counts on the same filtered box. In chart 1 write $\alpha = ac$, $\beta = bd$. For G_A, record (A, a_B, c_C, u), with $a_A c_A = f_J$ paid through (6.6). The identity $u+\alpha A^2 = \beta D^2$ chooses D by a square-divisor choice and then fixes β. For G_D, record (D, b_C, d_B, u), with $b_D d_D = g_K$ paid in the same way. Then $\beta D^2 - u = \alpha A^2$ chooses A and then fixes α. In both cases split $\alpha = ac, \beta = bd$ and reconstruct

$$B = \frac{cA+bD}{u}, \quad C = \frac{aA+dD}{u}, \quad t = \frac{cd-ab}{u}, \quad m = ABa, \quad n = ACc.$$

In chart 2 write $\gamma = ad$, $\delta = bc$. For G_B, record (B, a_A, d_D, t), with $a_B d_B$ paid through (6.6). Then $t + \gamma B^2 = \delta C^2$ chooses C and then fixes δ. For G_C, record (C, b_D, c_A, t), with $b_C c_C$ paid through (6.6). Then $\delta C^2 - t = \gamma B^2$ chooses B and fixes γ. Split $\gamma = ad, \delta = bc$ and reconstruct

$$A = \frac{dB-bC}{t}, \quad D = \frac{cC-aB}{t}, \quad u = \frac{cd-ab}{t}, \quad m = ABa, \quad n = ACc.$$

Each square-divisor choice and supported-prime choice costs Y^η at the fixed polynomial heights. The residual products must also be split into their ordered corner factors. For example, splitting $\alpha = ac$ and $\beta = bd$ costs at most $\tau_{\mathrm{div}}(\alpha)\tau_{\mathrm{div}}(\beta) \ll_\eta Y^\eta$. The same bound holds for $\gamma = ad$, $\delta = bc$. Once these choices are fixed, the inverse formulas give the other corners, and $ut = L$ gives the other shift. Thus the total inverse multiplicity is Y^η.

Each encoding freely enumerates one corner, two individual assigned parts, and the shift of that route. These four data have upper dyadic endpoints costing $4d_2$. The supported parts are determined by the full coefficient-corner labels. They are not extra free variables. Supported parts use the full prime-power convention of (2.4). Their supported-prime choices are paid by the uniform estimate (6.6). Each sum of two assigned-part coordinates is a lower bound for the base-Y logarithm of their product. To compare the corner with its LP reference value, put

$$\nu_n = \log_Y n, \quad H = \log_Y(m+n+1), \quad h_m = \log_Y(1+1/m), \quad h_n = \log_Y(1+1/n).$$

Let subscript r denote the actual source values of μ, z_u, q, c_1, c_2. Define the lower-corner reference exponents

$$\begin{aligned} \mathsf{A} &= (\mu + z_u - c_1)/2, & \mathsf{D} &= (\mu + z_u - q + c_1)/2, \\ \mathsf{B} &= 1/2 - z_u/2 + q/4 - c_2/2, & \mathsf{C} &= 1/2 - z_u/2 - q/4 + c_2/2. \end{aligned}$$

Take logarithms in (7.9), and use $Z^2 M = m(m+1)n(n+1)$ and $ZL = (n-m)(m+n+1)$. This gives the exact formulas

$$\begin{aligned} \log_Y A &= (\mu_r + z_{u,r} - c_{1,r})/2 + (\nu_n - H)/2, \\ \log_Y D &= (\mu_r + z_{u,r} - q_r + c_{1,r})/2 + (\nu_n - H + h_m + h_n)/2, \\ \log_Y B &= 1/2 - z_{u,r}/2 + q_r/4 - c_{2,r}/2 + (H-1)/2 + (h_n - h_m)/4, \\ \log_Y C &= 1/2 - z_{u,r}/2 - q_r/4 + c_{2,r}/2 + (H-1)/2 + (h_m - h_n)/4. \end{aligned} \tag{7.41}$$

Each of the eleven independently binned source coordinates lies between its lower endpoint and that endpoint plus d_2. Apply this coordinatewise to the linear forms in (7.41), before combining the corner coordinates into q, c_1, c_2. This gives

$$\begin{aligned} &\log_Y A \le \mathsf{A} + d_2, && \log_Y D \le \mathsf{D} + 3d_2 + \ell_{\mathrm{end}}/2, \\ &\log_Y B \le \mathsf{B} + 3d_2/2 + \lambda/2, && \log_Y C \le \mathsf{C} + 3d_2/2 + \ell_C/2 + \lambda/2. \end{aligned} \tag{7.42}$$

The exact correction in the C formula is $(h_m - h_n)/4 \le \ell_C/4$. The $\ell_C/2$ shown is conservative. Here $0 < H - \nu_n \le d_2 + \lambda$, $H - 1 \le d_2 + \lambda$, $h_m + h_n \le \ell_{\mathrm{end}}$, and $0 \le h_m - h_n \le \ell_C$. The last bound follows from $m > Y^{\tau_0}/\sqrt{3}$, $n > Y/2$, and $m < n$. The coefficient variation from the lower endpoints is at most d_2 for A and $3d_2$ for D. The $3d_2$ bound for D is conservative. Cancellation between q and c_1 reduces the positive-coordinate variation in its lower-corner form to at most d_2. We keep the larger uniform allowance. In each of B, C, the positive coefficients among $q/4 - c_2/2$ (or its negative) sum to one across the original eight corner coordinates. So the upper variation is at most d_2, and the other coordinates decrease the form. The other terms in (7.41) give the corrections shown. Adding the $4d_2$ data endpoints gives the four route costs

$$5d_2, \qquad 7d_2 + \ell_{\mathrm{end}}/2, \qquad 11d_2/2 + \lambda/2, \qquad 11d_2/2 + \ell_C/2 + \lambda/2. \tag{7.43}$$

By (7.3), $\ell_C \le \ell_{\mathrm{end}}$. The two chart counts use the complete (7.30) and (7.35), with both axis terms included in the six chart costs. They apply to the same filtered source box. Refine it at once into the dyadic boxes of both counting orientations, keeping its original selected-chart label. The fixed power of $\log Y$ refinements is absorbed in Y^η. Their lower-corner loss is at most $10d_2$. This includes both axis terms and the four-part products in all graph patterns. For $Y \ge 2$, every route cost is at most ρ_Y, by its definition and $\ell_C \le \ell_{\mathrm{end}}$. These are alternative bounds for the same source box. Choose a route attaining (7.39). Its lattice-point count has geometric saving at least $W - \rho_Y$, before the arbitrary analytic loss allocated in Section 9. Then (7.40) gives $L_{5W} - 6\rho_Y$. The geometric, interval, and square-root terms enter the completed source count additively. The axis costs are already in the chart bound. The number of terms is fixed. So the largest term exponent, that is the minimum saving, controls their sum.

The interval and square-root terms in (7.30) have savings

$$\begin{aligned} s_{\mathrm{int}} &= \frac{2}{3} - a_{\mathrm{grid}} \\ &= \frac{235120131355662893}{9520800917171070000} \approx 0.0246954, \\ s_{\mathrm{HD}} &= \frac{2}{3} - \frac{q_{\max} + a_{\mathrm{grid}}}{2} \\ &= \frac{29390272487700347267}{4760400458585535000000} \approx 0.00617391. \end{aligned} \tag{7.44}$$

For

$$c_N = \log_Y(1 + Y^{-a_{\mathrm{grid}}}), \qquad E_q = 2\lambda + \log_Y 18,$$

the interval term has saving at least $s_{\mathrm{int}} - c_N$. For the square-root term, $M < Y^{q_{\max}+2\lambda}$ and $N < Y^{a_{\mathrm{grid}}+c_N}$ give

$$\sqrt{4MN} \le Y^{(q_{\max}+a_{\mathrm{grid}})/2+\lambda+c_N/2+d_2}.$$

This is a conservative uniform upper bound for the finite high-degree correction. It uses the largest allowed source product and the ceiling bound for N. It uses no correlation between those endpoints. So the high-degree saving is at least $s_{\mathrm{HD}} - (E_q + c_N)/2 - 12d_2$, since

$$\frac{E_q + c_N}{2} + 12d_2 = \lambda + \frac{\log_Y 18}{2} + \frac{c_N}{2} + 12d_2 \ge \lambda + \frac{c_N}{2} + d_2.$$

Thus all endpoint terms are kept. Let $\eta > 0$. Choose all finite determinant, factor, divisor, Rankin, and logarithmic losses to total at most $\eta/2$. Then enlarge one threshold. This gives

$$\sigma_{5W} = \min(L_{5W} - 6\rho_Y,\, s_{\mathrm{int}} - c_N,\, s_{\mathrm{HD}} - (E_q + c_N)/2 - 12d_2)\,.$$

Thus

$$\#\{5W \text{ sources in one box}\} \ll_\eta Y^{\,2/3-\sigma_{5W}+\eta/2}. \tag{7.45}$$

The number of boxes is a fixed power of $\log Y$. It is absorbed in the remaining $\eta/2$. The margins of the three limiting caps above S_0 are

$$\begin{aligned} L_{5W} - S_0 &= \underset{(0.007194)}{\frac{647471}{90000000}}, \\ s_{\mathrm{int}} - S_0 &= \underset{(0.018522)}{\frac{88170305326616070517}{4760400458585535000000}}, \\ s_{\mathrm{HD}} - S_0 &= \underset{(1.076\times10^{-7})}{\frac{128034121242821}{1190100114646383750000}} > 0. \end{aligned} \tag{7.46}$$

The three limiting caps are at least S_0 by (7.46). Their corrected versions bound the additive contributions in (7.45), whose minimum controls the sum exponent. For $Y \geq 4$, since $\tau_0 > 1/2$,

$$(1+\sqrt{3}Y^{-\tau_0})(1+2/Y)(1+1/Y) < 2\cdot\frac{3}{2}\cdot\frac{5}{4} = \frac{15}{4} < 4.$$

Thus $\ell_{\mathrm{end}} + \lambda \leq 2d_2$, hence $\rho_Y \leq 15d_2$ and $6\rho_Y \leq 90\log 2/\log Y$. The inequality $\log(1+t) \leq t$ for $t \geq 0$ and $a_{\mathrm{grid}} > 1/2$ give, for $Y \geq 2$,

$$c_N \leq \frac{Y^{-a_{\mathrm{grid}}}}{\log Y} < \frac{1}{\log Y}, \qquad \lambda + \frac{c_N}{2} \leq \frac{Y^{-1} + \frac{1}{2}Y^{-1/2}}{\log Y} < \frac{1}{\log Y}.$$

Therefore, for $Y \geq 4$,

$$\frac{E_q + c_N}{2} + 12d_2 \leq \frac{\frac{1}{2}\log 18 + 12\log 2 + 1}{\log Y} < \frac{90\log 2}{\log Y}.$$

Thus every correction in (7.45) is at most $90\log 2/\log Y$, and (7.46) gives

$$Y^{S_0-\sigma_{5W}} \leq 2^{90}.$$

The finite corrections therefore change only the implied constant. After the box sum, (7.45) implies the uniform estimate

$$\#5W \ll_\eta Y^{2/3-S_0+\eta}. \tag{7.47}$$

Proposition 7.2 ($5W$ row estimate)**.** *For every $\eta > 0$, the complete first-match remainder satisfies*

$$\#5W \ll_\eta Y^{2/3-S_0+\eta}.$$

Proof. The three additive source-box contributions are bounded in (7.45). Their limiting caps and positive margins are (7.46). This gives (7.47). □

The threshold in (7.47) depends on η and the fixed source parameters. It does not depend on any moving cell, shift, label, factor, or resultant value. For each box and cell the degree set and height exponents are fixed. For each selected factor the polynomial, its endpoint coefficients and its resultant are fixed. They may vary between boxes, cells and factors.

The exact global reserve is

$$g_0 = S_0 - \frac{1}{165} = \frac{18677}{165000000} > 0. \tag{7.48}$$

It is used in Section 9 to choose one common loss and one Y_{final}. The source-to-count argument above gives the uniform estimate used there.

8. Exact exponent summary

Put

$$\begin{gathered} S_0 = \epsilon_0 = k_0 = \frac{30869}{5000000}, \quad l_0 = 2S_0, \quad j_0 = p_0 = 3S_0, \quad \xi = \frac{1}{10000}, \\ a_0 = \frac{6}{125}, \quad b_0 = \frac{4166537}{30000000}, \quad \Delta_{\exp} = 3S_0. \end{gathered} \tag{8.1}$$

All entries below are limiting savings from 2/3 at this fixed rational parameter vector. Sections 7 and 9 handle the finite source collars and the arbitrary analytic losses. The row estimates in Sections 4–7 give the exact values in the second column. The symbolic counting exponents and their evaluations are displayed in Sections 5–6 (for $3u$, $3t$, $4H$, and $4P$). Appendix D collects the parameter vector, exact savings, and margins above S_0.

The table expands the savings used in the preceding row estimates. The minimum row and its margin are read from the exact fractions. The rounded decimals are annotations only.

Contribution	Limiting saving	Decimal
O_R	30869/5000000	–
O_M	30869/5000000	–
$O_{\mathrm{rad}\,M}$	30869/5000000	–
$Q0$	2/3	–
$Q1$	30869/5000000	–
$Q2t_{\mathrm{int}}$	972179/7500000	–
$Q2t_{\mathrm{harm}}$	30869/5000000	–
$Q2u_{\mathrm{int}}$	2222179/15000000	–
$Q2u_{\mathrm{harm}}$	30869/1250000	–
$3u_{\mathrm{sqrt}}$	5102312049347/600000000000000	–
$3u_{\mathrm{int}}$	23206592049347/300000000000000	–
$3u_{\mathrm{core}}$	2473/12000	–
$3u_{\mathrm{small},1}$	1619393/6000000	–
$3u_{\mathrm{small},2}$	8003/500000	–
$3t_{\mathrm{sqrt}}$	388751/60000000	–
$3t_{\mathrm{int}}$	2199179/30000000	–
$3t_{\mathrm{axes}}$	2726537/15000000	–
$4E$	30869/5000000	–
$4H$	140441/5000000	–
$4P_{\mathrm{graph}}$	30869/5000000	–
$4P_{\mathrm{HD}}$	107393/15000000	–
$4P_{\mathrm{cover}}$	2/75	–
$4P_{\mathrm{axes}}$	4166537/30000000	–
$5W_{\mathrm{HD}}$	$\frac{29390272487700347267}{4760400458585535000000}$	0.006174

Contribution	Limiting saving	Decimal
$5W_{\mathrm{int}}$	$\dfrac{235120131355662893}{9520800917171070000}$	0.024695
L_{5W}	1203113/90000000	0.013368

Some entries add up within one row estimate. These are the five $3u$ entries, the three $3t$ entries, the four $4P$ entries, the two $Q2$ terms per chart and the three $5W$ entries. In each row the largest exponent controls the sum. Equivalently, the smallest saving does. In each $5W$ source box, Section 7 selects the best chart and assigned-corner route. The three displayed terms are the geometric, interval and high-degree contributions that follow. In the $Q2$ entries, $Q2t_{\mathrm{int}}, Q2t_{\mathrm{harm}}$ are the first and second terms of the $Q2t$ estimate (5.2). Likewise, $Q2u_{\mathrm{int}}, Q2u_{\mathrm{harm}}$ are the terms of the $Q2u$ estimate. Each pair is summed within its own first-match population. The table has 26 scalar terms. After grouping the three $5W$ contributions, 24 effective terms remain.

The minimum row saving is S_0. It is attained by $O_R, O_M, O_{\mathrm{rad}\,M}, Q1, Q2t_{\mathrm{harm}}, 4E$, and $4P_{\mathrm{graph}}$. In the final row the smallest of the three savings is $5W_{\mathrm{HD}}$, and

$$5W_{\mathrm{HD}} - S_0 = \frac{128034121242821}{1190100114646383750000} \quad (\text{about } 1.076 \times 10^{-7}) > 0. \tag{8.2}$$

The 72-branch minimum used for the geometric cap is

$$L_{5W} = \frac{1203113}{90000000}, \qquad L_{5W} - S_0 = \frac{647471}{90000000} > 0. \tag{8.3}$$

Section 7 proves the branch construction and the comparison needed for $5W$. No LP value is used as a count by itself.

Proposition 8.1 (Reserve)**.** *Put*

$$g_0 = S_0 - \frac{1}{165} = \frac{18677}{165000000}, \qquad \beta_0 = \frac{g_0}{16} = \frac{18677}{2640000000}.$$

Then

$$S_0 - \beta_0 - \frac{g_0}{4} = \frac{1}{165} + \frac{11g_0}{16} = \frac{16205447}{2640000000} > \frac{1}{165}. \tag{8.4}$$

Proof. Put the displayed fractions over the common denominator 2640000000. The loss β_0 is charged once to each complete analytic row estimate. In the last row, the $g_0/4$ term covers the maximum of the geometric, interval, and high-degree corrections. □

9. One loss and one threshold

Fix the rational vector in (8.1) and the complete analytic loss $\beta_0 = g_0/16$ from (8.4). In every determinant application, the positive exponent gap is fixed first. The degree and the matrix dimensions are chosen after it. This gives a finite family of degrees and dimensions. A moving box may select one member of it. The coefficients, the polynomial and the resultant may vary with the box and with Y. Only their degrees and height exponents are uniform. Nonvanishing comes from the source determinant, from coprimality, or from the endpoint alternative in the relevant row. The height bound plays no part in it.

Fix a selected member. The cofactor solution and the explicit integerizer give a polynomial height Y^{C_i}. Here C_i depends only on the fixed degree, the dimension and the allocated loss. Hence a nonzero resultant is an integer of size at most $Y^{C_i'}$ for a fixed exponent C_i'. The integer itself may change as the source labels move.

We allocate the loss as follows. Suppose one complete row proof contains N_i divisor, Heath–Brown, square-fibre, or lattice estimates. Let the j-th estimate apply to a parameter bounded by $Y^{h_{ij}}$. Here N_i is the fixed finite number of estimate stages in the complete row proof. A stage repeated over moving boxes keeps its one uniform exponent choice. It does not incur a new loss

for each box. Choose h_{ij} at least as large as every polynomial-height exponent in that invocation. This covers the auxiliary form, its integerization, any nonzero resultant and the size exponent of the counted parameter. For a product of height factors, add their exponents when choosing h_{ij}. Then it bounds the total contribution of the product. The degrees and height exponents are uniform over the complete moving source family in that row. Choose the internal exponent ρ_{ij} so that

$$0 < \rho_{ij} \le \frac{\beta_0}{8N_i(1+h_{ij})}. \tag{9.1}$$

Then

$$\sum_{j=1}^{N_i} h_{ij}\rho_{ij} \le \frac{\beta_0}{8}.$$

Allocate the other fixed portions of β_0 to the finitely many dyadic sums, logarithms and divisor multiplicities in the same complete row proof. This is one allocation for the whole row. After a row estimate is stated, no factor or inverse count is charged again.

The thirteen first-match estimates are stated in Propositions 4.1 and 5.2, and in the row Propositions 5.3, 5.4, 6.1, 6.2, 6.3, and 7.2. They use the finite cutoff data summarized below. Each cutoff is fixed by the corresponding proof and depends only on β_0 and the rational vector. The table records conditions and the estimates that supply them; it does not give numerical values for all row thresholds or analytic implied constants. Equation (9.4) takes the maximum of the thresholds used in those proofs, including all displayed row conditions.

TABLE 1. Threshold conditions for the thirteen row estimates.

Row	Threshold conditions included in $Y_i(\beta_0)$
O_R	$Y \ge 64$, $H = Y/2 \ge 2$, $V = Y^{5/3-\epsilon_0} < H^2$; fixed-height partner and low-radical estimates
O_M	Ordered residual-factor divisor bound and the fixed-height quadratic-fibre estimate
$O_{\mathrm{rad}\,M}$	One fixed positive Rankin parameter and the same quadratic-fibre estimate
$Q0$	$Y \ge 256$ and $3Y^{q_{\max}-1} < 1$
$Q1$	Fixed-height bound for all (P, L) labels
$Q2u$	$1/3 - 5\epsilon_0 > 0$; unit-root count, one u-sum, and the dyadic sector count
$Q2t$	$1/3 - 6\epsilon_0 - l_0 > 0$; corresponding t-chart count and one t-sum
$3u$	Positive determinant gap, finite factor family, all-aspect lattice count, axes, and inverse bounds
$3t$	Determinant cutoff, fixed grid condition $N \ge 40$, and later lattice, factor, census, and inverse cutoffs
$4E$	Four supported-factor sums, their fixed-height bounds, and the single shift sum
$4H$	Reducible, content, and primitive cubic branches, including the primitive-form estimate
$4P$	Graph, high-degree, cover, and axis bounds, with simultaneous shifts counted once
$5W$	Lemma 7.1, finite chart comparisons, original-grid census, factor branches, endpoint bounds, assigned-corner cases, and the geometric envelope; in particular $6E_Y < \xi/2$ and $576Y^{q_{\max}+2\lambda-\tau_0} < 1$ in (7.33)

In the $5W$ row, the geometric cutoff is

$$T_{\mathrm{geom}} = \frac{252}{g_0} + 1. \tag{9.2}$$

For $\log Y > T_{\mathrm{geom}}$, $\log 2 < 7/10$ gives

$$\frac{90 \log 2}{\log Y} < \frac{63}{252/g_0} = \frac{g_0}{4}. \tag{9.3}$$

The high-degree correction in (7.45) is separate from the geometric correction. Since $a_{\mathrm{grid}} > 1/2$, for $Y \geq 2$,

$$\lambda + \frac{c_N}{2} \leq \frac{Y^{-1} + \frac{1}{2}Y^{-1/2}}{\log Y} < \frac{1}{\log Y}.$$

Also, $c_N = \log_Y(1 + Y^{-a_{\mathrm{grid}}}) \leq Y^{-a_{\mathrm{grid}}}/\log Y$, so the interval correction is smaller than $1/\log Y$. Consequently

$$\frac{E_q + c_N}{2} + 12d_2 \leq \frac{\frac{1}{2}\log 18 + 12\log 2 + 1}{\log Y} < \frac{90 \log 2}{\log Y} < \frac{g_0}{4}.$$

Here $\frac{1}{2}\log 18 + 12\log 2 = 9.763\ldots$. Thus each of the geometric correction $6\rho_Y$, the interval correction c_N, and the high-degree correction $(E_q + c_N)/2 + 12d_2$ is at most $g_0/4$. The final row estimate is charged their maximum, not their sum. We retain this conservative charge in the reserve (8.4); it is unnecessary for the exponent in Proposition 7.2, where the three finite corrections are absorbed into the implied constant. The determinant, census and geometric-envelope thresholds in that row are the conditions used in Section 7. The contribution 90 is the source-mass 5 plus the one geometric collar mass 1, each times the coefficient 15 in the row margin.

Let $Y_i(\beta_0)$ be the maximum of the finitely many thresholds used in the proof of row i, after applying (9.1). Define

$$T_0 = \max\{3, T_{\mathrm{geom}}, \log Y_1(\beta_0), \ldots, \log Y_{13}(\beta_0)\}, \qquad Y_{\mathrm{final}} = \lceil e^{T_0} \rceil + 1. \tag{9.4}$$

Since $T_0 \geq 3$, $Y_{\mathrm{final}} \geq \lceil e^3 \rceil + 1 = 22 > 4$. Thus the elementary conditions $Y \geq 4$ of the 5W accounting hold at the common cutoff. The maximum runs over the thirteen fixed row estimates and their fixed finite degree families. It does not run over moving source labels. It includes the separate post-determinant thresholds of the $3t$ row.

Proposition 9.1 (Common-loss assembly)**.** *For $Y \geq Y_{\mathrm{final}}$, each row $i \neq 5W$ is*

$$\#\mathrm{row}_i \ll_{\beta_0} Y^{2/3 - r_i + \beta_0}, \qquad r_i \geq S_0,$$

and the last row satisfies

$$\#5W \ll_{\beta_0} Y^{2/3 - S_0 + \beta_0 + g_0/4}.$$

Consequently $\#\Omega_Y \ll_\varepsilon Y^{109/165+\varepsilon}$.

Proof. The first twelve row estimates are complete counts. They include the moving labels, shifts, coefficient factorizations and inverse multiplicities. By Section 8, their minimum saving is S_0. By (7.46), the three limiting values $L_{5W}, s_{\mathrm{int}}, s_{\mathrm{HD}}$ in the definition of σ_{5W} are each at least S_0. Hence the corrected minimum satisfies

$$\sigma_{5W} \geq S_0 - \max\left(6\rho_Y,\ c_N,\ \frac{E_q + c_N}{2} + 12d_2\right).$$

The three corresponding source-box contributions are added in (7.45), so this minimum controls their sum exponent. The corrections have separate sources. First, $6\rho_Y$ is the finite source-to-lower-corner allowance in (7.40)–(7.43). Next, c_N is the ceiling correction from $N = \lceil Y^{a_{\mathrm{grid}}} \rceil$ in

(7.6). Last, $(E_q + c_N)/2 + 12d_2$ bounds the finite square-root term $\sqrt{4MN}$. This uses the source-product and grid-ceiling bounds just before (7.45). The margins add no losses beyond these three terms. By (9.3) and the correction bounds above, each of the three corrections is at most $g_0/4$ for $Y \geq Y_{\text{final}}$. Hence $\sigma_{5W} \geq S_0 - g_0/4$. Apply (7.45) with $\eta = \beta_0$, using the remaining $\beta_0/2$ for the box sum. Thus the total count is

$$\ll_{\beta_0} Y^{2/3 - S_0 + \beta_0 + g_0/4} = Y^{109/165 - 11g_0/16} \ll Y^{109/165}.$$

There are thirteen rows. Their finite sum changes only the implied constant. □

10. Dyadic shells

For $Y = 2^i$, the shells $2^{i-1} < n \leq 2^i$ are disjoint. By Proposition 9.1, for every i with $2^i \geq Y_{\text{final}}$,

$$\#\Omega_{2^i} \ll_\varepsilon 2^{i(109/165+\varepsilon)}. \tag{10.1}$$

Let I be the least integer with $X \leq 2^I$. Then $2^I < 2X$ and

$$\sum_{i \leq I} 2^{i(109/165+\varepsilon)} \ll_\varepsilon (2X)^{109/165+\varepsilon}. \tag{10.2}$$

The finitely many shells below Y_{final} are absorbed by

$$\#\Omega_Y \leq \frac{Y(Y-1)}{2} \leq \frac{1}{2}\max(1, Y_{\text{final}}^{2-\alpha})Y^\alpha, \qquad \alpha = 109/165 + \varepsilon.$$

Each pair counted by $F(X)$ lies in exactly one shell. This proves Theorem 1.1.

Appendix A. Boundary rules for the first-match partition

All cuts in the partition apply to the original pair. The table gives the first-match row for each equality, after all earlier rows are tested. The notation is that of Section 3.

(A.1)

boundary	first-match row
$R = Y^{5/3-\epsilon_0}$	O_R
$M = Y^{2/3-\epsilon_0}$	O_M
$\operatorname{rad}(M) = Y^{2/3-k_0}$	$O_{\operatorname{rad} M}$
$K_{\text{res}} = Y^{\Delta_{\exp}}$	fails $Q0$; continue testing
$L = Y^{1/3-l_0}$	fails $Q1$; continue testing
$\min(J, K) = Y^{1/3-j_0}$	fails strict $Q2$; test later rows
$u = t$	u-chart
$s = Y^{a_0}$	$3u$ or $3t$, as selected
$s = Y^{b_0}$	$5W$
$\min(v, w) = Y^{p_0}$	$4E$
$\Pi = Y^{p_0}, \quad v, w > Y^{p_0}$	$4H$
$\Pi > Y^{p_0}, \quad v, w > Y^{p_0}$	$4P$

The rows for $4E, 4H, 4P$ require that all earlier predicates fail and that $Y^{a_0} < s < Y^{b_0}$. Equality at the lower-corner boundary fails the strict $Q2$ cut. It then goes to the first later row whose predicate holds. The weak lower-corner inequalities describe the complements of the strict $Q2$ cuts.

The row predicates are Boolean functions of the original pair. A counting argument may round an interval endpoint outward to an integer. This only enlarges the counted set. No real rounding enters the definition of a source row. Take the explicit inverse in (2.2). Then test positivity, integrality, the gcd, the shell and the mask. Each accepted reconstructed tuple then gives at most one original pair.

Fix the ordered residual tuple that determines L. Then the identity $ut = L$ determines the opposite shift. Hence a sum over u or over t is a sum over one source coordinate. It is not a product of independent shift ranges. In the $4H$ and $4P$ rows this identity forces the opposite shift. In $5W$ both shifts stay in the original-cell source. Its geometric envelope adds no new shift sum.

The four $Q2$ count contributions fall into two populations:

population	contributions	selected chart
$Q2u$	$Q2u_{\mathrm{int}} + Q2u_{\mathrm{harm}}$	$u \le t$
$Q2t$	$Q2t_{\mathrm{int}} + Q2t_{\mathrm{harm}}$	$t < u$

Section 8 computes their savings. The two terms on a line are added to bound one chart population. They are not separate rows. Their exponents and savings are

(A.2)

term	chart contribution	exponent	saving
$Q2t_{\mathrm{int}}$	interval term in $Q2t$	$1/2 + 6S_0$	$1/6 - 6S_0 = 972179/7500000$
$Q2t_{\mathrm{harm}}$	harmonic term in $Q2t$	$2/3 - S_0$	S_0
$Q2u_{\mathrm{int}}$	interval term in $Q2u$	$1/2 + 3S_0$	$1/6 - 3S_0 = 2222179/15000000$
$Q2u_{\mathrm{harm}}$	harmonic term in $Q2u$	$2/3 - 4S_0$	$4S_0 = 30869/1250000$

In each chart the larger exponent controls the sum. This is the smaller saving. No term is an additional source population.

The partition is disjoint and exhaustive. Hence

$$\begin{aligned}\#\Omega_Y = {} & \#O_R + \#O_M + \#O_{\mathrm{rad}\,M} + \#Q0 + \#Q1 + \#Q2u + \#Q2t \\ & + \#3u + \#3t + \#4E + \#4H + \#4P + \#5W. \end{aligned} \tag{A.3}$$

The first-match convention is essential. A relaxed set that bounds one row may overlap another relaxed counting set. Still, every original pair goes to exactly one row. So the populations satisfy the exact identity (A.3).

Appendix B. A uniform square-fibre bound

Tao's Lemma 2.10 [8, Lemma 2.10] gives a related square-fibre bound. Appendix E states its hypotheses. The self-contained theorem below gives the fixed-ε, fixed-polynomial-height version used here. It assumes no uniform Pell-orbit or regulator bound.

Theorem B.1 (Uniform square fibres)**.** *Fix $A \ge 1$. For every $\varepsilon > 0$ there is a constant $C_{A,\varepsilon}$ such that, for $x \ge 2$, positive integers $a, b \le x^A$, and a nonzero integer h with $|h| \le x^A$,*

$$\#\{(n,m) \in \mathbb{Z}_{>0}^2 : an^2 + h = bm^2,\ n \le x\} \le C_{A,\varepsilon} x^{\varepsilon}. \tag{B.1}$$

The constant is independent of a, b, h. The exponent A is fixed.

Lemma B.2 (Divisor bound)**.** *For every $\delta > 0$, $\tau_{\mathrm{div}}(N) \ll_\delta N^\delta$ for $N \ge 1$.*

Proof. For primes $p \ge 2^{1/\delta}$, $j + 1 \le 2^j \le p^{\delta j}$. There are finitely many smaller primes. For each of them $\sup_{j \ge 0}(j+1)p^{-\delta j} < \infty$. Multiply over the prime factorization. □

Lemma B.3 (Roots of a quadratic congruence)**.** *If $D \ge 2$ is squarefree and $M \ge 1$, then*

$$\#\{r \bmod M : r^2 \equiv D \pmod M\} \le 4 \cdot 2^{\omega(M)}. \tag{B.2}$$

For $M = 1$ there is one residue class.

Proof. It suffices to count roots modulo prime powers. For an odd prime $p \mid D$, there is one root modulo p and none modulo p^e, $e \ge 2$. Here a root would have valuation at least one. Its square would then have valuation at least two. If $p \nmid D$, a root r_0 is a unit. For any other root

r, $p^e \mid (r-r_0)(r+r_0)$. The two factors are not both divisible by p. So $r \equiv r_0$ or $-r_0 \pmod{p^e}$. There are at most two roots.

Let $p = 2$. If D is even, squarefreeness gives $D \equiv 2 \pmod 4$. Then there are no roots modulo 2^e for $e \geq 2$, and at most one modulo 2. Let D be odd and let r_0 be one root. Multiplication by r_0^{-1} injects the roots into solutions of $z^2 \equiv 1 \pmod{2^e}$. For $e = 1, 2$ there are at most two. Let $e \geq 3$. Then z is odd, and exactly one of $z-1, z+1$ is divisible by 2 but not 4. Their product is divisible by 2^e. So the other is divisible by 2^{e-1}. Hence $z \equiv 1$ or $-1 \pmod{2^{e-1}}$. This gives at most four classes modulo 2^e. The Chinese remainder theorem multiplies these local bounds, and (B.2) follows. □

Lemma B.4 (Spacing in one norm class)**.** *Let $D \geq 2$ be squarefree, $N \in \mathbb{Z} \setminus \{0\}$, and $H \geq 1$. The number of positive integral solutions of*

$$X^2 - DY^2 = N, \qquad X + Y\sqrt{D} \leq H$$

is at most

$$4\tau_{\mathrm{div}}(|N|)^2 \left(1 + \frac{\log H}{\log(2+\sqrt{3})}\right). \tag{B.3}$$

Proof. Put $g = \gcd(X, Y)$. Then $g^2 \mid N$. For each of at most $\tau_{\mathrm{div}}(|N|)$ choices of g, set $U = X/g$, $V = Y/g$, and $N_0 = N/g^2$. The pair (U, V) is primitive. Also $\gcd(V, |N_0|) = 1$. Otherwise, by the norm equation, a common prime would divide U^2, hence U. This contradicts primitivity. Thus $r = UV^{-1} \pmod{|N_0|}$ is defined and satisfies $r^2 \equiv D$. For modulus one use its unique class. By (B.2), there are at most $4\tau_{\mathrm{div}}(|N|)$ such classes.

Take two solutions in one class. Write $\alpha_i = U_i + V_i\sqrt{D} > 1$. Direct multiplication gives

$$\frac{\alpha_1}{\alpha_2} = s + t\sqrt{D}, \quad s = \frac{U_1U_2 - DV_1V_2}{N_0}, \quad t = \frac{V_1U_2 - U_1V_2}{N_0}. \tag{B.4}$$

Here s, t are the integer coefficients of this norm-one quotient. This local t is unrelated to either source shift. Both numerators are divisible by N_0. To see this, work modulo $|N_0|$ and use $U_i \equiv rV_i$ and $r^2 \equiv D$. Hence s, t are integers even when $N_0 < 0$. The quotient has norm one. Let $\alpha_1/\alpha_2 > 1$. Its conjugate is the positive reciprocal. Thus $s = (\alpha_1/\alpha_2 + \alpha_2/\alpha_1)/2 > 1$. Since s is an integer, $s \geq 2$, and

$$\frac{\alpha_1}{\alpha_2} = s + \sqrt{s^2 - 1} \geq 2 + \sqrt{3}.$$

Distinct pairs have distinct α_i, as $\sqrt{D}$ is irrational. Hence each class contains at most $1 + \log H/\log(2+\sqrt{3})$ solutions. Multiply by the choices of g and of the residue class to get (B.3). □

Proof of Theorem B.1. Write $ab = Dc^2$ with D squarefree and put

$$X = bm, \qquad Y_0 = cn, \qquad N = bh.$$

Then

$$X^2 - DY_0^2 = b^2m^2 - abn^2 = bh = N \neq 0. \tag{B.5}$$

This map is injective. From the original equation and $n \leq x$,

$$bm \leq \sqrt{abx^2 + b|h|} \leq \sqrt{2}\, x^{A+1}, \qquad cn\sqrt{D} = \sqrt{ab}\, n \leq x^{A+1}.$$

Thus $X + Y_0\sqrt{D} \leq 3x^{A+1}$ and $|N| \leq x^{2A}$.

If $D \geq 2$, apply Lemma B.4. Then Lemma B.2 absorbs the two divisor factors and the logarithm into x^ε. Its exponent is chosen in terms of A, ε. If $D = 1$, (B.5) factors as $(X - Y_0)(X + Y_0) = N$. Each positive divisor $X + Y_0$ of $|N|$ fixes the signed factor $X - Y_0 = N/(X + Y_0)$. So it gives at most one pair. The divisor bound then proves (B.1) in this case too. □

Remark B.5. Every use has positive coefficients and a nonzero shift. The coefficients and the shift have fixed polynomial height. One variable is bounded, and the equation then bounds the other. No discriminant-dependent constant or fundamental-unit estimate occurs.

In the conic uses of Sections 6 and 7, put $(a, b, h, n, m) = (f, g, s, J, K)$. The source bounds give $1 \le f, g, s \le (Y+1)^2$ and $J, K \le Y+1$. Take $x = 2Y$ and the fixed height exponent $A_{\mathrm{ht}} = 2$, with theorem parameter $A = A_{\mathrm{ht}}$. All height hypotheses hold, since $(Y+1)^2 \le (2Y)^2$ and $Y+1 \le 2Y$. The coefficients are positive and the selected shift satisfies $s > 0$. Suppose the factorization does not fix the shift. Then we first fix the ordered residual factors, which determine L. Next we enumerate $s \mid L$. As $L \le (Y+1)^2$, this costs at most $\tau_{\mathrm{div}}(L) \ll_\delta Y^\delta$. Thus the theorem is applied once per fixed nonzero shift, with its uniform constant. The shift divisor count is part of the allocated loss.

Appendix C. Application of the low-radical estimate

Theorem 3 of [1], stated in Appendix E, holds uniformly for real $H \ge 2$ and $1 \le V < H^2$. Its determinant-method input is Reuss's Theorem 1 [7]; we use the consequence stated in Paper 1.

For the outer row O_R, set

$$H = Y/2, \qquad V = Y^{5/3-S_0}, \qquad S_0 = 30869/5000000.$$

For $Y \ge 64$, $H \ge 2$, $V \ge 1$, and

$$\frac{V}{H^2} = 4Y^{-1/3-S_0} < 1.$$

Hence that theorem applies. Its two terms satisfy

$$\frac{V}{H} = 2Y^{2/3-S_0}, \qquad H^{7/12} = 2^{-7/12}Y^{7/12} \le Y^{2/3-S_0},$$

because

$$\frac{2}{3} - S_0 - \frac{7}{12} = \frac{1157393}{15000000} > 0.$$

Keep the common factor $H^\eta = (Y/2)^\eta$. Then

$$(Y/2)^\eta \left(2Y^{2/3-S_0} + 2^{-7/12}Y^{7/12}\right) \ll_\eta Y^{2/3-S_0+\eta}.$$

Consequently

$$\#\{Y/2 < t \le Y : \mathrm{rad}(t(t+1)) \le Y^{5/3-S_0}\} \ll_\eta Y^{2/3-S_0+\eta},$$

with one constant for every $Y \ge 64$. This is the low-radical input of Section 4. The partner count there separately uses Lemma 2.1 of Paper 1 [1].

Appendix D. Exact exponents and finite optimization

All numbers in this appendix are exact rationals. Set

$$\begin{gathered} S_0 = \epsilon_0 = k_0 = \frac{30869}{5000000}, \qquad l_0 = 2S_0 = \frac{30869}{2500000}, \qquad j_0 = p_0 = 3S_0 = \frac{92607}{5000000}, \\ a_0 = \frac{6}{125}, \qquad b_0 = \frac{4166537}{30000000}, \qquad \Delta_{\mathrm{exp}} = 3S_0 = \frac{92607}{5000000}, \\ \mu_{\mathrm{HP}} = \frac{16}{25}, \qquad \nu = \frac{1}{64}, \qquad \xi = \frac{1}{10000}. \end{gathered} \tag{D.1}$$

The target saving and reserve are

$$S_{\mathrm{target}} = \frac{1}{165}, \qquad g_0 = S_0 - S_{\mathrm{target}} = \frac{18677}{165000000}. \tag{D.2}$$

The table gives every source-count contribution. The margin is the displayed saving minus S_0. A zero margin marks a binding contribution. Two chart terms of one population are both listed,

since their counts add. The smaller saving controls their sum. The three displayed $5W$ savings are additive contributions to the complete count. Their minimum controls the exponent of the sum. The chart and assigned-corner routes are selected in each source box as in Section 7.

Contribution	Saving	Margin above S_0
O_R	30869/5000000	0
O_M	30869/5000000	0
$O_{\mathrm{rad}\,M}$	30869/5000000	0
$Q0$	2/3	9907393/15000000
$Q1$	30869/5000000	0
$Q2t_{\mathrm{int}}$	972179/7500000	1851751/15000000
$Q2t_{\mathrm{harm}}$	30869/5000000	0
$Q2u_{\mathrm{int}}$	2222179/15000000	532393/3750000
$Q2u_{\mathrm{harm}}$	30869/1250000	92607/5000000
$3u_{\mathrm{sqrt}}$	5102312049347/600000000000000	1398032049347/600000000000000
$3u_{\mathrm{int}}$	23206592049347/300000000000000	21354452049347/300000000000000
$3u_{\mathrm{core}}$	2473/12000	2998643/15000000
$3u_{\mathrm{small},1}$	1619393/6000000	7911751/30000000
$3u_{\mathrm{small},2}$	8003/500000	49161/5000000
$3t_{\mathrm{sqrt}}$	388751/60000000	18323/60000000
$3t_{\mathrm{int}}$	2199179/30000000	402793/6000000
$3t_{\mathrm{axes}}$	2726537/15000000	263393/1500000
$4E$	30869/5000000	0
$4H$	140441/5000000	27393/1250000
$4P_{\mathrm{graph}}$	30869/5000000	0
$4P_{\mathrm{HD}}$	107393/15000000	7393/7500000
$4P_{\mathrm{cover}}$	2/75	307393/15000000
$4P_{\mathrm{axes}}$	4166537/30000000	3981323/30000000
$5W_{\mathrm{geom}}$	$\frac{1203113}{90000000}$ (0.013368)	$\frac{647471}{90000000}$ (0.007194)
$5W_{\mathrm{int}}$	$\frac{235120131355662893}{9520800917171070000}$ (0.024695)	$\frac{88170305326616070517}{4760400458585535000000}$ (0.018522)
$5W_{\mathrm{HD}}$	$\frac{29390272487700347267}{4760400458585535000000}$ (0.006174)	$\frac{128034121242821}{1190100114646383750000}$ (1.076×10^{-7})

Decimals in parentheses are rounded annotations. All comparisons and margins use the exact rational entries.

Thus the seven binding source contributions are

$$O_R,\ O_M,\ O_{\mathrm{rad}\,M},\ Q1,\ Q2t_{\mathrm{harm}},\ 4E,\ 4P \text{ graph},$$

each with saving exactly S_0. The high-degree term limits the $3u$ aggregate. The high-degree term also limits the $3t$ aggregate. The graph term limits the $4P$ aggregate. For $5W$, the minimum saving among the geometric, interval and high-degree contributions controls the combined count in (7.45).

The limiting linear program. Let $x_0, \dots, x_7$ be the base-Y logarithms of the eight assigned corner factors, in the order

$$(a_A, a_B, b_C, b_D, c_A, c_C, d_B, d_D).$$

Put

$$\begin{gathered}\alpha = x_0 + x_1 + x_2 + x_3, \quad \gamma = x_4 + x_5 + x_6 + x_7, \quad q = \alpha + \gamma,\\ c_1 = x_0 + x_1 + x_4 + x_5, \quad c_2 = x_0 + x_1 + x_6 + x_7,\\ P = x_0 + x_4, \quad Q = x_3 + x_7, \quad R_{\mathrm{LP}} = x_1 + x_6, \quad T = x_2 + x_5.\end{gathered} \tag{D.3}$$

The other variables are $W, z_u, z_t, \mu \geq 0$. Here W is a candidate saving, $z_u = \log_Y u$, $z_t = \log_Y t$, and $\mu = \log_Y m$. Every branch uses the limiting source constraints below. They are not finite-source identities. Section 7 records the finite relations and their collars.

$$\begin{gathered}\frac{2407393}{7500000} \leq \alpha \leq \frac{5092607}{15000000}, \qquad \frac{9907393}{30000000} \leq \gamma \leq \frac{2092607}{6000000},\\ \frac{9907393}{15000000} \leq q \leq \frac{5092607}{7500000}, \qquad \frac{2407393}{7500000} \leq z_u + z_t \leq \gamma,\\ \mu + \frac{\gamma - \alpha}{2} = 1, \qquad \mu \leq 1, \qquad \mu \geq \frac{9907393}{15000000} + \frac{q}{2},\\ z_u, z_t \geq b_0, \qquad z_u + z_t \leq \frac{5092607}{7500000}.\end{gathered} \tag{D.4}$$

The four assigned-corner capacity inequalities are

$$\frac{3}{2}x_0 + \frac{1}{2}x_1 + \frac{3}{2}x_4 + \frac{1}{2}x_5 - \frac{1}{2}z_u - \frac{1}{2}\mu \leq \frac{92607}{5000000}, \tag{D.5}$$

$$\frac{1}{2}x_2 + \frac{3}{2}x_3 + \frac{1}{2}x_6 + \frac{3}{2}x_7 - \frac{1}{2}z_u - \frac{1}{2}\mu \leq \frac{92607}{5000000}, \tag{D.6}$$

$$\frac{1}{2}x_0 + \frac{3}{2}x_1 - \frac{1}{2}x_4 - \frac{1}{2}x_5 + x_6 + \frac{1}{2}z_u - \frac{1}{2}\mu \leq \frac{92607}{5000000}, \tag{D.7}$$

$$-\frac{1}{2}x_0 - \frac{1}{2}x_1 + x_2 + \frac{1}{2}x_4 + \frac{3}{2}x_5 + \frac{1}{2}z_u + \frac{1}{2}\mu \leq \frac{5092607}{5000000}. \tag{D.8}$$

The selected-chart condition is $z_u \leq z_t$ in chart 1 and $z_t \leq z_u$ in chart 2. In chart 1 also impose

$$\mu + z_u - c_1 \geq \frac{4722179}{7500000}, \qquad \mu + z_u - (x_2 + x_3 + x_6 + x_7) \geq \frac{4722179}{7500000}; \tag{D.9}$$

in chart 2 impose

$$x_0 + x_1 - x_4 - x_5 + z_u - \mu \leq -\frac{4722179}{7500000}, \qquad -x_0 - x_1 + x_4 + x_5 + z_u + \mu \leq \frac{10277821}{7500000}. \tag{D.10}$$

The four assigned-corner route scores are the G_A, G_D, G_B, G_C in (7.38). A branch also imposes $W \geq 0$ and $W \geq G_A, G_D, G_B, G_C$. For $i, j \in \{0,1,2,3,4,5\}$ impose $W \geq C_{1,i}$ and $W \geq C_{2,j}$. The six values in each chart are defined by

i	$C_{1,i}$	j	$C_{2,j}$
0	$2/3 - c_1$	0	$2/3 - c_2$
1	$2/3 - q + c_1$	1	$2/3 - q + c_2$
2	$9722179/15000000 - x_1 - x_2 - x_5 - x_6$	2	$9722179/15000000 - x_0 - x_3 - x_4 - x_7$
3	$9722179/15000000 - x_1 - x_3 - x_5 - x_7$	3	$9722179/15000000 - x_0 - x_2 - x_5 - x_7$
4	$9722179/15000000 - x_0 - x_2 - x_4 - x_6$	4	$9722179/15000000 - x_1 - x_3 - x_4 - x_6$
5	$9722179/15000000 - x_0 - x_3 - x_4 - x_7$	5	$9722179/15000000 - x_1 - x_2 - x_5 - x_6$

(D.11)

Derivation of the 22 limiting source rows. The source rows are R1–R6 and R9–R24 in the certificate order below. The LP linear forms P, Q are those of (D.3). For the source products write $P_{\mathrm{src}} = ab$ and $Q_{\mathrm{src}} = cd$, and write $L = Q_{\mathrm{src}} - P_{\mathrm{src}}$. Set $\ell_{\mathrm{src}} = \log_Y L$, $\alpha_{\mathrm{src}} = \log_Y P_{\mathrm{src}}$, $\gamma_{\mathrm{src}} = \log_Y Q_{\mathrm{src}}$, and $q_{\mathrm{src}} = \alpha_{\mathrm{src}} + \gamma_{\mathrm{src}} = \log_Y M$. Write $\ell = z_u + z_t$ for the limiting LP coordinate. The finite logarithmic collars come from $\kappa_Y = 1 + Y^{-1}$, from factors of two and from the endpoint terms in $Z^2 M = m(m+1)n(n+1)$. They are included in (7.42)–(7.43). The LP rows are their limiting affine inequalities. The table writes them in limiting LP endpoint coordinates q, ℓ, z_u, z_t, distinct from the finite source logarithms with subscript src. The limiting coordinates are denoted

by q and ℓ; z_u, z_t are the LP variables corresponding to the finite logarithms $\log_Y u, \log_Y t$. The subscript src marks source quantities before taking limits. Labels R1–R28 refer throughout to the certificate row numbering below.

Limiting LP row	Source derivation
R1	If $P_{\mathrm{src}} \le L$, then $P_{\mathrm{src}} \le L < \kappa_Y Y^{1/3+S_0}$. If $P_{\mathrm{src}} > L$, then $P_{\mathrm{src}} < \sqrt{M} < \kappa_Y Y^{q_{\max}/2}$. Since $q_{\max}/2 = 1/3 + S_0$, these give the displayed limiting upper bound.
R2	If $P_{\mathrm{src}} \le L$, then $Q_{\mathrm{src}} \le 2L$, so $P_{\mathrm{src}} = M/Q_{\mathrm{src}} > Y^{q_{\min}}/(2\kappa_Y Y^{1/3+S_0})$, whose limiting exponent is $1/3 - 2S_0$. If $P_{\mathrm{src}} > L$, then $Q_{\mathrm{src}} < 2P_{\mathrm{src}}$ and $M < 2P_{\mathrm{src}}^2$, giving the stronger limiting lower exponent $q_{\min}/2 = 1/3 - S_0/2$.
R3	If $P_{\mathrm{src}} \le L$, then $Q_{\mathrm{src}} \le 2L$; otherwise $Q_{\mathrm{src}} < 2P_{\mathrm{src}} < 2\sqrt{M}$. Thus the source gives $\gamma_{\mathrm{src}} \le 1/3 + S_0 + \log_Y(2\kappa_Y)$; for sufficiently large Y this implies the weaker $\gamma_{\mathrm{src}} \le 1/3 + 5S_0/2$.
R4	Since $P_{\mathrm{src}} < Q_{\mathrm{src}}$, we have $\alpha_{\mathrm{src}} < \gamma_{\mathrm{src}}$; with $q_{\mathrm{src}} = \alpha_{\mathrm{src}} + \gamma_{\mathrm{src}} \ge q_{\min}$, this gives $\gamma_{\mathrm{src}} \ge q_{\min}/2$.
R5	The failed O_R cut gives $M < \kappa_Y^2 Y^{q_{\max}}$, hence $q_{\mathrm{src}} \le q_{\max} + 2\lambda$, where $\lambda = \log_Y \kappa_Y$; the LP uses the limiting bound $q \le q_{\max}$.
R6	The failed O_M cut gives $M > Y^{q_{\min}}$, hence $q_{\mathrm{src}} \ge q_{\min}$.
R9	Failure of $Q1$ gives $L \ge Y^{1/3-l_0}$, so $\ell_{\mathrm{src}} \ge 1/3 - l_0$.
R10	Since $L = Q_{\mathrm{src}} - P_{\mathrm{src}} < Q_{\mathrm{src}}$, taking logarithms gives $\ell_{\mathrm{src}} \le \gamma_{\mathrm{src}}$.
R11	The mask gives $a_A c_A \le AK_{\mathrm{res}}$. Taking logarithms and replacing the finite $\log_Y A$ by its lower-corner reference $\mathsf{A} = (\mu + z_u - c_1)/2$ from (7.41)–(7.42) gives $x_0 + x_4 - \mathsf{A} \le \Delta_{\exp}$, which is (D.5).
R12	The mask gives $b_D d_D \le DK_{\mathrm{res}}$. Replacing the finite $\log_Y D$ by its lower-corner reference $\mathsf{D} = (\mu + z_u - q + c_1)/2$ from (7.41)–(7.42) gives $x_3 + x_7 - \mathsf{D} \le \Delta_{\exp}$, which is (D.6).
R13	The source B-corner capacity is $a_B d_B \le BK_{\mathrm{res}}$. Taking logarithms and using the limiting corner form $\mathsf{B} = 1/2 - z_u/2 + q/4 - c_2/2$, where $R_{\mathrm{LP}} = x_1 + x_6$ and $c_2 = x_0 + x_1 + x_6 + x_7$, gives the limiting inequality below. Here q denotes the LP coordinate corresponding to the source exponent q_{src} defined above; its finite endpoint error is included in the source collar: $$R_{\mathrm{LP}} + z_u/2 - q/4 + c_2/2 \le 1/2 + \Delta_{\exp}.$$ Use $1/2 = \mu/2 + (\gamma - \alpha)/4$, from the limiting source equality, and expand. This gives row R13, namely (D.7).
R14	The source C-corner capacity is $b_C c_C \le CK_{\mathrm{res}}$, so $T + z_u/2 + q/4 - c_2/2 \le 1/2 + \Delta_{\exp}$, where $T = x_2 + x_5$. Using the limiting equality first rewrites this as $$T + z_u/2 + \alpha/2 - c_2/2 - \mu/2 \le \Delta_{\exp}.$$ Now eliminate $\alpha = \gamma + 2\mu - 2$ and expand. This gives row R14, namely (D.8) with right side $1 + \Delta_{\exp} = 5092607/5000000$; its finite endpoint error is included in the source collar. Using (7.41), after the $h_m - h_n$ terms cancel, the finite right sides in rows R13 and R14 are respectively $$\Delta_{\exp} + \frac{H - \nu_n}{2}, \qquad 1 + \Delta_{\exp} + \frac{H + \nu_n - 2}{2}.$$ After removing the limiting row constants, each endpoint residual is at most $d_2/2$ before lower-corner binning; that binning is included in ρ_Y.
R15–R16	Put $\nu_n = \log_Y n$ and $h_r = \log_Y(1 + 1/r)$. The exact source identity is $$\mu + (\gamma_{\mathrm{src}} - \alpha_{\mathrm{src}})/2 = \nu_n + (h_n - h_m)/2,$$ obtained by taking logarithms of $Q_{\mathrm{src}} m(m+1) = P_{\mathrm{src}} n(n+1)$. Since $Y/2 < n \le Y$, $-d_2 < \nu_n - 1 \le 0$, and $\lvert h_n - h_m \rvert \le \ell_{\mathrm{end}}$. The finite discrepancy is absorbed in the source collar. In the limiting LP, row R15 is $\mu + (\gamma - \alpha)/2 \le 1$, and row R16 is the reverse inequality $\mu + (\gamma - \alpha)/2 \ge 1$.
R17	The source has $m < n \le Y$, hence $\mu \le 1$.

Limiting LP row	Source derivation
R18	Since $m+1 \le 2m$, $n+1 \le 2Y$, and $Z > Y^{5/3-S_0}$, the identity $Z^2M = m(m+1)n(n+1)$ gives $Y^{10/3-2S_0+q_{\mathrm{src}}} < 4m^2Y^2$, hence the finite bound $\mu > q_{\min} + q_{\mathrm{src}}/2 - d_2$, where $d_2 = \log_Y 2$. The LP uses the limiting weak inequality $\mu \ge q_{\min} + q/2$; the finite d_2 loss is included in the source collar ρ_Y.
R19–R20	The 5W predicate has $u, t \ge Y^{b_0}$, hence $z_u, z_t \ge b_0$.
R21	The source has $L < \kappa_Y Y^{1/3+S_0}$, so its limiting bound is $\ell_{\mathrm{src}} \le 1/3 + S_0$. The displayed LP bound $\ell \le q_{\max}$ is a redundant, weaker relaxation; it only enlarges the feasible set and can lower the LP minimum.
R22	The selected-chart rule (2.3) gives $z_u \le z_t$ in chart 1 and $z_t \le z_u$ in chart 2.
R23–R24	The strict $Q2$ cuts have failed. In chart 1, the complement is the lower-corner inequality (D.9); in chart 2, it is (D.10).

Rows R11–R14 are the four limiting logarithmic capacity inequalities. They come from the two-sided assignment of each residual prime to its unique eligible corner. Rows R13–R14 use equality rows R15–R16. Rows R3 and R21 weaken stronger source bounds. So the relaxed feasible set only grows. The six counting rows R7, R8 and R25–R28 come from the chart costs (7.37), the route scores (7.38), the endpoint patterns (6.27), (7.34), and the route-score and endpoint bounds (7.41)–(7.43). They are not extra source premises.

Fix an actual source. Keep its selected chart. Then choose a minimizing form for each of the two counting orientations. Its endpoint data then place it in one of the $2 \cdot 6 \cdot 6 = 72$ cases. Dropping the inequalities that enforce the minimizing choices enlarges that case. Hence a lower bound on every displayed relaxation applies to the original source.

The certificates number the inequalities as follows.

- R1–R6: the upper and lower bounds for α, then for γ, then for q.
- R7–R8: $W \ge C_{1,i}$ and $W \ge C_{2,j}$.
- R9–R10: the lower bound for $z_u + z_t$ and its upper bound by γ.
- R11–R14: the four capacity inequalities as displayed.
- R15–R16: the two directions of the equality in (D.4).
- R17–R18: the upper and lower bounds for μ.
- R19–R20: the lower bounds for z_u, z_t.
- R21: the upper bound for $z_u + z_t$.
- R22: the selected-chart inequality.
- R23–R24: the two core inequalities for that chart.
- R25–R28: $W \ge G_A, G_D, G_B, G_C$, in this order.

The six counting rows are R7, R8, and R25–R28.

Each branch has 28 inequalities. These are the displayed source bounds, the two selected chart score rows from (D.11) and the four assigned-corner rows. The table below gives their exact dual certificates. Write the rows as $a_r \cdot x \le b_r$. The listed nonnegative multipliers satisfy $\sum_r \lambda_r a_r \ge -e_W$ coordinatewise, $\sum_{r \in \mathcal{C}} \lambda_r = 1$ over the six counting rows, and $-\sum_r \lambda_r b_r = W_{\sigma ij}$. As $x \ge 0$, these relations give $-W \le \sum_r \lambda_r a_r \cdot x \le \sum_r \lambda_r b_r$. Thus every branch satisfies

$$W \ge L_{5W}, \qquad L_{5W} = \frac{1203113}{90000000}. \tag{D.12}$$

The value is attained in the relaxed branch $(1,2,5)$ by

$$\begin{aligned}&(x_0,\dots,x_7,W,z_u,z_t,\mu)=\\&\left(0,\frac{24907393}{90000000},\frac{5648249}{90000000},0,0,\frac{2092607}{9000000},\right.\\&\left.\frac{5648249}{90000000},\frac{3981323}{90000000},\frac{1203113}{90000000},\frac{4166537}{30000000},\frac{6018677}{30000000},1\right).\end{aligned}\tag{D.13}$$

Exchange the two corner pairs. This gives a feasible point of the chart-swapped branch $(2,5,2)$ with the same value. In the coordinate order of (D.3), the permutation is

$$(x_0,\dots,x_7,z_u,z_t)\longmapsto(x_1,x_0,x_3,x_2,x_6,x_7,x_4,x_5,z_t,z_u),$$

with W and μ fixed. Consequently the exact optimization margin is

$$\frac{1203113}{90000000}-S_0=\frac{647471}{90000000}>0.\tag{D.14}$$

The common-loss reserve is

$$S_0-\frac{g_0}{16}-\frac{g_0}{4}=\frac{1}{165}+\frac{11g_0}{16}>\frac{1}{165}.$$

The threshold in Section 9 bounds each of the $5W$ geometric, interval, and high-degree corrections by $g_0/4$. The assembly charges their maximum, not their sum. The exact fractions and the certificate table give the finite optimization explicitly.

Table of branch certificates. In the table, $W_{\sigma,i,j}$ is the lower bound for branch $(\sigma;i,j)$ in the lower-bound column. An entry $r:c$ means $\lambda_r=c$ for the inequality row numbered above. Omitted multipliers are zero. A multiplier list may wrap onto continuation lines with blank branch and lower-bound columns; all such lines belong to the same certificate.

Table 2: Exact dual certificates for the 72 chart and score branches.

Branch $(\text{chart};i,j)$	lower bound	Nonzero entries of λ
$1;0,0$	4722179/30000000	1:1/2; 5:1/2; 7:1/2; 8:1/2
$1;0,1$	4722179/30000000	5:1/2; 7:1/2; 8:1/2; 15:1/2; 18:1/2
$1;0,2$	21943969/150000000	7:3/5; 8:2/5; 10:1/10; 11:2/5; 15:1/10; 17:1; 18:9/10; 22:1/10
$1;0,3$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 23:1
$1;0,4$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 23:1
$1;0,5$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 23:1
$1;1,0$	4722179/30000000	5:1/2; 7:1/2; 8:1/2; 15:1/2; 18:1/2
$1;1,1$	4722179/30000000	1:1/2; 5:1/2; 7:1/2; 8:1/2
$1;1,2$	21943969/150000000	1:1/10; 5:2/5; 7:3/5; 8:2/5; 10:1/10; 12:2/5; 15:1/5; 22:1/10
$1;1,3$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 24:1
$1;1,4$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 24:1
$1;1,5$	3796109/30000000	5:1/4; 7:1; 10:1/2; 15:1/2; 17:1/2; 22:1/2; 24:1
$1;2,0$	21017899/150000000	7:2/5; 8:3/5; 13:2/5; 15:1/5; 17:1; 18:1; 19:1/5
$1;2,1$	21017899/150000000	1:1/5; 7:2/5; 8:3/5; 14:2/5; 17:3/5; 18:4/5; 19:1/5
$1;2,2$	1157393/3750000	7:1/2; 8:1/2; 17:1; 18:1
$1;2,3$	18424903/180000000	1:1/6; 7:1/2; 8:1/2; 14:1/3; 17:5/6; 18:1; 19:1/6
$1;2,4$	18424903/180000000	7:1/2; 8:1/2; 13:1/3; 15:1/6; 17:7/6; 18:7/6; 19:1/6
$1;2,5$	1203113/90000000	8:1/3; 10:2/3; 15:2/3; 17:2/3; 18:4/3; 19:1; 27:1/3; 28:1/3
$1;3,0$	3796109/90000000	7:2/9; 8:1/3; 10:5/18; 13:2/9; 15:7/18; 17:7/9; 18:5/6; 22:5/18; 26:4/9
$1;3,1$	3796109/90000000	7:2/9; 8:1/3; 10:5/18; 14:2/9; 15:1/6; 17:7/9; 18:5/6; 22:5/18; 26:4/9
$1;3,2$	4413489/40000000	7:1/4; 8:1/4; 10:3/8; 15:3/8; 17:3/4; 18:7/8; 22:3/8; 26:1/2
$1;3,3$	4166537/30000000	7:1/2; 8:1/2; 15:1/2; 17:1; 18:3/2
$1;3,4$	4166537/30000000	7:1/2; 8:1/2; 16:1/2; 17:2; 18:3/2
$1;3,5$	5462257/135000000	7:1/3; 8:4/9; 10:2/9; 15:2/9; 17:11/9; 18:10/9; 23:1/3; 27:1/9; 28:1/9
$1;4,0$	3796109/90000000	7:2/9; 8:1/3; 10:5/18; 13:2/9; 15:7/18; 17:7/9; 18:5/6; 22:5/18; 25:4/9

Branch (chart; i, j)	lower bound	Nonzero entries of λ
$1; 4, 1$	3796109/90000000	7:2/9; 8:1/3; 10:5/18; 14:2/9; 15:1/6; 17:7/9; 18:5/6; 22:5/18; 25:4/9
$1; 4, 2$	4413489/40000000	7:1/4; 8:1/4; 10:3/8; 15:3/8; 17:3/4; 18:7/8; 22:3/8; 25:1/2
$1; 4, 3$	4166537/30000000	7:1/2; 8:1/2; 16:1/2; 17:2; 18:3/2
$1; 4, 4$	4166537/30000000	7:1/2; 8:1/2; 15:1/2; 17:1; 18:3/2
$1; 4, 5$	5462257/135000000	7:1/3; 8:4/9; 10:2/9; 15:2/9; 17:11/9; 18:10/9; 24:1/3; 27:1/9; 28:1/9
$1; 5, 0$	3981323/90000000	7:1/3; 10:1/2; 15:1/2; 17:2/3; 18:5/6; 22:1/2; 25:1/3; 26:1/3
$1; 5, 1$	3981323/90000000	7:1/3; 10:1/2; 15:1/2; 17:2/3; 18:5/6; 22:1/2; 25:1/3; 26:1/3
$1; 5, 2$	3981323/90000000	8:1/3; 10:1/2; 15:1/2; 17:2/3; 18:5/6; 22:1/2; 25:1/3; 26:1/3
$1; 5, 3$	3981323/90000000	7:1/3; 10:1/2; 15:1/2; 17:2/3; 18:5/6; 22:1/2; 25:1/3; 26:1/3
$1; 5, 4$	3981323/90000000	7:1/3; 10:1/2; 15:1/2; 17:2/3; 18:5/6; 22:1/2; 25:1/3; 26:1/3
$1; 5, 5$	1157393/3750000	7:1/2; 8:1/2; 17:1; 18:1
$2; 0, 0$	4722179/30000000	7:1/2; 8:1/2; 16:1/2; 17:2; 18:3/2
$2; 0, 1$	4722179/30000000	7:1/2; 8:1/2; 15:1/2; 17:1; 18:3/2
$2; 0, 2$	21017899/150000000	7:3/5; 8:2/5; 10:1/5; 11:2/5; 15:1/5; 17:1; 18:1; 20:1/5
$2; 0, 3$	3796109/90000000	7:1/3; 8:2/9; 10:1/6; 11:2/9; 15:1/6; 17:7/9; 18:5/6; 22:5/18; 28:4/9
$2; 0, 4$	3796109/90000000	5:7/18; 7:1/3; 8:2/9; 10:1/6; 11:2/9; 15:1/6; 18:1/18; 22:5/18; 27:4/9
$2; 0, 5$	3981323/90000000	8:1/3; 10:1/6; 15:1/6; 17:2/3; 18:5/6; 22:1/2; 27:1/3; 28:1/3
$2; 1, 0$	4722179/30000000	7:1/2; 8:1/2; 15:1/2; 17:1; 18:3/2
$2; 1, 1$	4722179/30000000	1:1/2; 5:1/2; 7:1/2; 8:1/2
$2; 1, 2$	21017899/150000000	5:1/2; 7:3/5; 8:2/5; 10:1/5; 12:2/5; 15:1/5; 20:1/5
$2; 1, 3$	3796109/90000000	5:7/18; 7:1/3; 8:2/9; 10:1/6; 12:2/9; 15:1/6; 18:1/18; 22:5/18; 28:4/9
$2; 1, 4$	3796109/90000000	5:7/18; 7:1/3; 8:2/9; 10:1/6; 12:2/9; 15:1/6; 18:1/18; 22:5/18; 27:4/9
$2; 1, 5$	3981323/90000000	8:1/3; 10:1/6; 15:1/6; 17:2/3; 18:5/6; 22:1/2; 27:1/3; 28:1/3
$2; 2, 0$	5416537/37500000	7:2/5; 8:3/5; 9:1/10; 13:2/5; 15:1/5; 17:1; 18:1; 22:1/10
$2; 2, 1$	5416537/37500000	7:2/5; 8:3/5; 9:1/10; 14:2/5; 16:1/5; 17:1; 18:1; 22:1/10
$2; 2, 2$	1157393/3750000	7:1/2; 8:1/2; 17:1; 18:1
$2; 2, 3$	4413489/40000000	7:1/4; 8:1/4; 10:1/8; 15:1/8; 17:3/4; 18:7/8; 22:3/8; 28:1/2
$2; 2, 4$	4413489/40000000	7:1/4; 8:1/4; 10:1/8; 15:1/8; 17:3/4; 18:7/8; 22:3/8; 27:1/2
$2; 2, 5$	3981323/90000000	8:1/3; 10:1/6; 15:1/6; 17:2/3; 18:5/6; 22:1/2; 27:1/3; 28:1/3
$2; 3, 0$	219893/1875000	5:1/2; 8:1; 9:1/2; 15:1; 22:1/2; 23:1
$2; 3, 1$	219893/1875000	5:1/2; 8:1; 9:1/2; 16:1; 22:1/2; 24:1
$2; 3, 2$	18424903/180000000	7:1/2; 8:1/2; 10:1/6; 12:1/3; 15:1/6; 17:7/6; 18:7/6; 20:1/6
$2; 3, 3$	4166537/30000000	7:1/2; 8:1/2; 15:1/2; 17:1; 18:3/2
$2; 3, 4$	4166537/30000000	1:1/2; 5:1/2; 7:1/2; 8:1/2
$2; 3, 5$	3981323/90000000	8:1/3; 10:1/6; 15:1/6; 17:2/3; 18:5/6; 22:1/2; 27:1/3; 28:1/3
$2; 4, 0$	219893/1875000	5:1/2; 8:1; 9:1/2; 15:1; 22:1/2; 23:1
$2; 4, 1$	219893/1875000	5:1/2; 8:1; 9:1/2; 16:1; 22:1/2; 24:1
$2; 4, 2$	18424903/180000000	7:1/2; 8:1/2; 10:1/6; 11:1/3; 15:1/6; 17:7/6; 18:7/6; 20:1/6
$2; 4, 3$	4166537/30000000	1:1/2; 5:1/2; 7:1/2; 8:1/2
$2; 4, 4$	4166537/30000000	5:1/2; 7:1/2; 8:1/2; 15:1/2; 18:1/2
$2; 4, 5$	3981323/90000000	8:1/3; 10:1/6; 15:1/6; 17:2/3; 18:5/6; 22:1/2; 27:1/3; 28:1/3
$2; 5, 0$	219893/1875000	8:1; 9:1/2; 15:1; 17:1; 18:1; 22:1/2; 23:1
$2; 5, 1$	219893/1875000	1:1/2; 8:1; 9:1/2; 16:1/2; 18:1/2; 22:1/2; 24:1
$2; 5, 2$	1203113/90000000	5:1/3; 8:1/3; 10:1; 15:1; 18:2/3; 20:1; 25:1/3; 26:1/3
$2; 5, 3$	5462257/135000000	1:1/9; 5:1/2; 7:4/9; 8:1/3; 15:4/9; 23:1/3; 25:1/9; 26:1/9
$2; 5, 4$	5462257/135000000	1:1/3; 5:5/18; 7:4/9; 8:1/3; 18:2/9; 24:1/3; 25:1/9; 26:1/9
$2; 5, 5$	1157393/3750000	7:1/2; 8:1/2; 17:1; 18:1

Appendix E. Cited statements and their applications

This appendix lists the inputs from earlier work and the parameter substitutions at their applications. Results proved here are not attributed to these sources.

For integers $k, n \geq 1$, define

$$P_k(n) = \prod_{j=0}^{k-1}(n+j), \qquad q_k(n) = \frac{P_k(n)}{\operatorname{rad}(P_k(n))}.$$

For real $X \geq 2$, define

$$F_{k,k}(X) = \#\{1 \leq m < n \leq X : \operatorname{rad}(P_k(m)) = \operatorname{rad}(P_k(n))\}.$$

The author's earlier papers. Theorem 1 of [1] states that for every fixed integer $k \geq 2$, every $\varepsilon > 0$, and $X \geq 2$,

$$F_{k,k}(X) \ll_{k,\varepsilon} X^{2/3+\varepsilon}.$$

At $k = 2$, this is the earlier $2/3$ baseline for $F(X)$. It is given for comparison only. No row below uses it.

Lemma 2.1 of [1] states that for squarefree Q and $X \geq 2$,

$$\#\{1 \leq z \leq X : \operatorname{rad}(z(z+1)) \mid Q\} \ll 2^{\omega(Q)} \log(2X),$$

with an absolute implied constant. For the O_R row take $Q = \operatorname{rad}(n(n+1))$ and $X = n$. The source equality of radicals gives the divisibility condition for every partner m. Also Q is squarefree and $Q \leq Y(Y+1)$. Thus $2^{\omega(Q)} \ll_\eta Y^{2\eta}$. Choose the exponents of the logarithm and divisor loss below the requested row loss. Then both are absorbed. No other uniformity in Q is needed.

Lemma 2.3 of [1] gives, for fixed $k \geq 2$, $X \geq 2$, $K \geq \max(k, 2)$, and $0 < \eta < 1$,

$$\#\{n \leq X : q_k(n) > K\} \ll_{k,\eta} XK^{-1+\eta} + K^2(1+\log K)^{k-1} + X^{1/2}.$$

Lemma 2.4 of [1] gives

$$\#\{1 \leq m < n \leq X : q_k(m), q_k(n) \leq K, \operatorname{rad}(P_k(m)) = \operatorname{rad}(P_k(n))\} \ll_k K^2 \log(2X).$$

Here $X \geq 2$ and $K \geq 1$, and $k \geq 2$ is fixed. The stronger lower restriction $K \geq \max(k, 2)$ belongs only to Lemma 2.3 above. These quotient-tail and Farey-spacing estimates are stated for context only. Neither is a premise of the proof.

Theorem 3 of [1] gives, for every $\eta > 0$, uniformly for real $H \geq 2$ and $1 \leq V < H^2$,

$$\#\{H < t \leq 2H : \operatorname{rad}(t(t+1)) \leq V\} \ll_\eta H^\eta \left(\frac{V}{H} + H^{7/12}\right).$$

For the outer low-radical estimate put $H = Y/2$ and $V = Y^{5/3-S_0}$. For all sufficiently large Y, $H \geq 2$, $V \geq 1$, and $V/H^2 = 4Y^{-1/3-S_0} < 1$. The right side becomes

$$(Y/2)^\eta \left(2Y^{2/3-S_0} + (Y/2)^{7/12}\right).$$

The common factor $(Y/2)^\eta$ is kept. Fix a requested row loss $\varepsilon > 0$. Choose the theorem parameter below both $\varepsilon/2$ and half the fixed gap $2/3 - S_0 - 7/12 > 0$. Then the display is $O_\varepsilon(Y^{2/3-S_0+\varepsilon})$. These are the hypotheses and bound used in (4.1) and Appendix C.

Theorems 1.1 and 1 of the author's earlier papers [2, 3] state, respectively,

$$N_1(X) \ll_\varepsilon X^{7/12+\varepsilon}, \qquad N_1(X) \ll_\varepsilon X^{20/37+\varepsilon},$$

where $N_1(X)$ counts primitive positive triples $x + y = z$ with $\operatorname{rad}(xyz) < z$ and $\max(x, y, z) \leq X$. These theorems are cited for context only. The source rows below count equal-radical pairs of consecutive-product arguments. They do not count primitive *abc* triples. So neither theorem enters a row estimate.

The four-corner parametrization and the identities of Section 2 follow from the four gcd definitions. They give the coordinates used below. Their derivation uses no further cited result.

External results. We use Heath–Brown's published [4, Lemma 2, p. 8]. Let G be an absolutely irreducible bihomogeneous integer form of bidegree (a, b), with $a, b \geq 1$, and let $X \geq 1$. Then the number of primitive pairs (x_1, x_2), (y_1, y_2) with $G = 0$ and $\max_i |x_i| \leq X$ is

$$O_{a,b,\varepsilon}\big(X^{2/b+\varepsilon}\|G\|^{\varepsilon}\big).$$

There is no size condition on the y-pair. Section 6 treats a factor with corner degree $a_c \geq 2$ and coefficient degree $b_q \geq 1$. There the coefficient-lattice pair is first and the primitive corner pair is second. The reductions in Section 6 make both pairs primitive. The first has radius $O(T_1 + 1)$. The transformed factor height is at most Y^C for a fixed C. The source also gives $T_1 + 1 \leq Y^{C'}$ for a fixed C'. The lemma with bidegree (b_q, a_c) gives

$$O((T_1 + 1)^{2/a_c+\varepsilon} Y^{C\varepsilon}).$$

Fix a requested total loss $\eta > 0$. Choose a fixed exponent $A_{\mathrm{HB}} \geq \max(C, C', 1)$. Apply Heath–Brown with $0 < \varepsilon < \eta/(4A_{\mathrm{HB}})$. Then its per-cell loss is at most $Y^{\eta/2}(T_1 + 1)$. Use the lattice census with its own loss parameter $\eta/2$. As $2/a_c \leq 1$,

$$(T_1 + 1)^{2/a_c+\varepsilon} Y^{C\varepsilon} \leq Y^{2A_{\mathrm{HB}}\varepsilon}(T_1 + 1) \leq Y^{\eta/2}(T_1 + 1).$$

Summation by (6.34) then gives the requested total factor Y^{η}. Section 5 uses the same pair order and degree conditions. Its lattice census gets the separate $\eta/2$ loss. Thus the hypotheses $X \geq 1$, $a, b \geq 1$, primitivity and one-sided size hold in Sections 5 and 6.

Section 7 uses the same theorem for each absolutely irreducible factor there with $a_c \geq 2$ and $b_q \geq 1$. The pair order and the primitivity are the same. Let H_7 be the fixed exponent in the factor-height bound $\|F\| \leq Y^{H_7}$. The source bounds give $U_0, V_*, N \leq Y$ for sufficiently large Y, since $U_0 \leq f \leq M$, $V_* \leq 4g \leq 4M$, $M \ll Y^{q_{\max}}$, $q_{\max} < 1$, and $N = \lceil Y^{a_{\mathrm{grid}}} \rceil$ with $a_{\mathrm{grid}} < 1$. Set the temporary census loss parameter $\eta_{\mathrm{census}} = 1$ in (7.30). This choice serves only the coarse height bound below. It differs from the later allocated loss η. It yields $\sum_i (T_{1,i}+1) \ll Y^{5/2}$. After raising the fixed lower threshold, every $T_{1,i} + 1 \leq Y^3$. This coarse bound serves only to give this pointwise height cap. Set $A_{\mathrm{HB}} = H_7 + 3$. For a requested total loss $\eta > 0$, take $0 < \varepsilon < \eta/(2A_{\mathrm{HB}})$ in the Heath–Brown estimate. Then

$$(T_{1,i} + 1)^{2/a_c+\varepsilon}\|F\|^{\varepsilon} \leq Y^{(H_7+3)\varepsilon}(T_{1,i} + 1) \leq Y^{\eta/2}(T_{1,i} + 1),$$

and (7.30) with loss parameter $\eta/2$ pays for the cell sum. The combined loss is Y^{η}. Section 7 treats the cases $a_c = 0$, $a_c = 1$, and $b_q = 0$ separately. The same result applies there.

The unnumbered theorem of Bombieri–Schmidt [5, pp. 69–70] reads as follows. Let $F \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $r \geq 3$ and let $h \geq 1$. Then the number of primitive integer solutions of $|F(x, y)| = h$ is at most $C_{\mathrm{BS}} r^{1+\omega(h)}$, with absolute C_{BS}. The 1989 correction to their proof of Lemma 7 [6] changes no hypothesis of this theorem. In Section 6 the form is the irreducible binary cubic $c_1x^3 - c_2z^3$. Its content and the common divisor of x, z are removed first. The right side is nonzero. The theorem gives $O(3^{1+\omega(|h|)})$ primitive solutions. The reducible case occurs when both coprime coefficients are cubes. Section 6 counts it separately by factoring the difference of cubes.

Tao's Lemma 2.10 [8, Lemma 2.10] reads as follows. Let x be large. Let a, b be positive integers and h a nonzero integer, all of polynomial size in x. Then

$$\#\{(n, m) \in \mathbb{Z}^2_{>0} : an^2 + h = bm^2,\ n \leq x\} \ll x^{o(1)}.$$

Sections 4–7 use the fixed-height Theorem B.1, proved in Appendix B. Tao's statement is cited as the precedent. The applications are as follows. In Section 4, the outer fibre (4.4) uses

$$(a, b, h, n, m) = (P, Q_{\mathrm{prod}}, Q_{\mathrm{prod}} - P, 2n + 1, 2m + 1), \qquad x = 2Y + 1, \qquad A_{\mathrm{ht}} = 4.$$

Here $P, Q_{\mathrm{prod}}, |Q_{\mathrm{prod}} - P| \le 4Y^4$, $2n+1 \le 2Y+1$, and the shift is nonzero. In Section 5, the two applications use, respectively,

$$(a,b,h,n,m) = (\alpha,\beta,u,A,D), \qquad (a,b,h,n,m) = (\gamma,\delta,t,B,C), \qquad x = 2Y, \quad A_{\mathrm{ht}} = 2.$$

The equations are $\alpha A^2 + u = \beta D^2$ and $\gamma B^2 + t = \delta C^2$. Section 5 gives $\alpha, \beta, u, \gamma, \delta, t \le (Y+1)^{21/20} \le (2Y)^2$ and $A, B \le Y+1 \le 2Y$ for $Y \ge 2$. Thus the fixed height exponent is $A_{\mathrm{ht}} = 2$. The bounded variables are A and B. In Sections 6–7, every corner-degree-zero use takes

$$(a,b,h,n,m) = (f,g,s,J,K), \qquad x = 2Y, \qquad A_{\mathrm{ht}} = 2.$$

The source gives positive f, g, s, nonzero s, $f, g, s \le (Y+1)^2$, and $J, K \le Y+1$. In the corner-degree-zero branches the ordered residual factors first fix L. The selected shift is then counted among $s \mid L$. As L has polynomial height in Y, for every $\eta > 0$ the divisor bound gives the uniform estimate $\tau_{\mathrm{div}}(L) \ll_\eta Y^\eta$. The proof here uses the self-contained Theorem B.1.

Reuss's Theorem 1 is the determinant-method input behind this low-radical estimate; we use the consequence stated in Theorem 3 of [1], as applied in Appendix C.

Use of AI tools

The author devised the main argument. Codex (OpenAI, GPT-6 Astra) carried out numerical checks of identities and of the cubic family, and searched the literature. Claude Code (Anthropic, Claude Opus 5.5) reviewed the arguments and references. The author takes responsibility for the proof and references.

Statements and declarations

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