# EQUAL RADICALS OF CONSECUTIVE PRODUCTS

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ABSTRACT. For every fixed $k \geq 2$ and $\varepsilon > 0$, $F_{k,k}(X) \ll_{k,\varepsilon} X^{2/3+\varepsilon}$. For $k \geq 3$ this improves the $X^{1+o(1)}$ bound of Lebowitz–Lockard. For $k = 2$, Kane's localized estimates give the large-quotient endpoint bound. Our independent proof combines a Pell bound for partner multiplicities, a quotient-tail estimate, and Farey spacing for small quotients. Reuss's determinant estimate also gives $N(H, Y) \ll_\varepsilon H^\varepsilon(Y/H + H^{7/12})$, uniformly for $H \geq 2$ and $1 \leq Y < H^2$, where $N(H, Y)$ counts integers $t \in (H, 2H]$ with $\operatorname{rad}(t(t+1)) \leq Y$. A cubic family gives $F(X) \geq (4^{1/3}/3)X^{1/3} - O(1)$.

## 1. INTRODUCTION

For a positive integer $a$, let $\operatorname{rad}(a) = \prod_{p|a} p$. We study the ordered count

$$F(X) = \#\{1 \leq m < n \leq X : \operatorname{rad}(m(m+1)) = \operatorname{rad}(n(n+1))\}.$$

More generally, put

$$P_k(t) = \prod_{j=0}^{k-1}(t+j), \qquad F_{k,k}(X) = \#\{1 \leq m < n \leq X : \operatorname{rad}(P_k(m)) = \operatorname{rad}(P_k(n))\}.$$

Thus $F(X) = F_{2,2}(X)$. We impose no disjointness condition on the two blocks, and all implied constants may depend on the fixed integer $k$.

**Theorem 1.** *For every fixed integer $k \geq 2$, every $\varepsilon > 0$, and every real $X \geq 2$,*

$$F_{k,k}(X) \ll_{k,\varepsilon} X^{2/3+\varepsilon}.$$

Erdős asked for the size of $F(X)$ [3]. Balasubramanian, Shorey and Waldschmidt [1, §5] studied non-overlapping blocks of consecutive integers with the same prime divisors. Lebowitz–Lockard [7, Theorems 5 and 10] proved $F_{k,k}(X) \leq X^{1+o(1)}$. For $k \geq 3$, Theorem 1 replaces this exponent by $2/3 + \varepsilon$. Pell equations of this kind go back to Lehmer's work on Størmer's problem [8]; see [1, §2].

For $k = 2$, Kane's localized estimates [6, Propositions 7 and 10] (Propositions 6 and 9 of [5]) give the large-quotient endpoint bound after specialization to $(1, t, -(t+1))$ and dyadic summation. In the proof below, Lemma 2.4 handles the small-quotient range; together with the endpoint-fibre and quotient-tail bounds, it gives an independent proof of the $k = 2$ case.

Put $q_k(n) = P_k(n)/\operatorname{rad}(P_k(n))$. Splitting at a cutoff $K$ and applying the endpoint-fibre, quotient-tail and small-quotient estimates gives, for $\max\{k, 2\} \leq K \leq X$,

$$F_{k,k}(X) \ll_{k,\varepsilon} X^\varepsilon\big(K^2 + X/K + X^{1/2}\big). \tag{1.1}$$

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For $X \geq k^3$, taking $K = X^{1/3}$ in (1.1) proves Theorem 1. The bounded range $2 \leq X < k^3$ is absorbed into the implied constant, since $k$ is fixed. Theorem 2 gives an explicit lower-bound family for Erdős's count. Theorem 3 follows from known estimates; its deduction appears in Appendix A. Section 6 gives two consequences of the pair bound. The tools used for numerical checks and literature searches are identified at the end of the paper.

## 2. Proof of the fixed-block theorem

Fix an integer $k \geq 2$. Put

$$P_k(n) = \prod_{j=0}^{k-1} (n+j), \qquad q_k(n) = \frac{P_k(n)}{\operatorname{rad}(P_k(n))}.$$

The function $P_k$ is positive and strictly increasing on the positive integers. Write $\omega(D)$ for the number of distinct prime divisors of $D$. All implied constants in this section may depend on $k$.

Lemma 2.1 is a Pell-equation count; compare Lebowitz–Lockard [7, Lemma 8] and Tao [11, Lemma 2.10].

**Lemma 2.1.** *For every squarefree positive integer $Q$ and every real $X \geq 2$,*

$$\#\{1 \leq z \leq X : \operatorname{rad}(z(z+1)) \mid Q\} \ll 2^{\omega(Q)} \log(2X),$$

*with an absolute implied constant. Consequently, for every $\delta > 0$, every value of $\operatorname{rad}(P_k(n))$ has at most*

$$C_{k,\delta} X^{k\delta} \log(2X) \tag{2.1}$$

*preimages among the positive integers $n \leq X$.*

*Proof.* Write $z(z+1) = dy^2$, where $d$ is squarefree. Then $d \mid Q$ and

$$(2z+1)^2 - d(2y)^2 = 1.$$

The case $d = 1$ has no solution with $z \geq 1$, since $(2z+1-2y)(2z+1+2y) = 1$ is impossible. For a fixed $d \geq 2$, put $A = 2z+1$, $B = 2y$ and $\alpha = A + B\sqrt{d}$. Thus

$$1 < \alpha < 2A \leq 4X + 2, \qquad \alpha^{-1} = A - B\sqrt{d}.$$

If $\alpha_2 > \alpha_1$ arise from two solutions, then

$$\frac{\alpha_2}{\alpha_1} = U_0 + V_0\sqrt{d} > 1, \qquad U_0, V_0 \in \mathbb{Z}, \qquad U_0^2 - dV_0^2 = 1.$$

Its conjugate is its positive reciprocal. Hence

$$U_0 = \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_1}{\alpha_2}\right) \geq 2, \qquad \frac{\alpha_2}{\alpha_1} \geq 2 + \sqrt{3}.$$

There are $O(\log(2X))$ such solutions for each $d \mid Q$. Summing over the squarefree divisors of $Q$ proves the first assertion.

For every $\delta > 0$,

$$2^{\omega(Q)} \ll_\delta Q^\delta.$$

Indeed, $2 \leq p^\delta$ for all sufficiently large primes $p$, and the remaining primes contribute a constant. If $Q = \operatorname{rad}(P_k(n))$ with $n \leq X$, then $Q \leq (X+k-1)^k \ll_k X^k$. Every preimage $m$ of $Q$ satisfies $\operatorname{rad}(m(m+1)) \mid Q$, since $k \geq 2$. This gives (2.1). □

Erdős and Straus [4, p. 63] also use a Pell equation for products of consecutive integers.

For a positive integer $D$, let

$$\rho_k(D) = \#\{x \bmod D : P_k(x) \equiv 0 \pmod{D}\}.$$

**Lemma 2.2.** *For every positive integer $D$,*

$$\rho_k(D) \le (k-1)!\, k^{\omega(D)}.$$

*Proof.* If $p > k-1$, the $k$ factors of $P_k$ have distinct roots modulo $p$. At most one factor is divisible by $p$, so $\rho_k(p^j) = k$ for every $j \ge 1$.

For $p \le k-1$, put $C_p = v_p((k-1)!)$. Given an integer $x$, choose $i \in \{0, \dots, k-1\}$ for which $v_p(x+i)$ is maximal, taking $v_p(0) = \infty$. For $j \ne i$, the ultrametric inequality and maximality give

$$v_p(x+j) \le v_p(j-i).$$

Thus

$$\sum_{j \ne i} v_p(x+j) \le v_p\big(i!(k-1-i)!\big) \le C_p.$$

If $p^j \mid P_k(x)$, then $p^{\max(j-C_p,0)} \mid x+i$. For each $i$ there are at most $p^{C_p}$ such residue classes modulo $p^j$. Hence $\rho_k(p^j) \le kp^{C_p}$. The Chinese remainder theorem now gives

$$\rho_k(D) \le k^{\omega(D)} \prod_{p \le k-1} p^{C_p} = (k-1)!\, k^{\omega(D)}. \qquad \square$$

Sanna [10, Lemma 3.3] proves the related bound $O_r(x/y^{1/2})$ for the number of $n \le x$ such that one of $n, \dots, n+r-1$ has powerful part exceeding $y$.

**Lemma 2.3.** *For all real $X \ge 2$, $K \ge \max(k, 2)$ and $0 < \eta < 1$,*

$$\#\{1 \le n \le X : q_k(n) > K\} \ll_{k,\eta} XK^{-1+\eta} + K^2(1+\log K)^{k-1} + X^{1/2}. \tag{2.2}$$

*Proof.* Let $1 \le n \le X$ satisfy $q_k(n) > K$. Suppose first that a prime $p > K$ divides $q_k(n)$. Then $p^2 \mid P_k(n)$. Since $p > K \ge k$, exactly one factor is divisible by $p$, and $p^2 \mid n+i$ for some $0 \le i < k$. The number of these $n$ is at most

$$k \sum_{K < p \le \sqrt{X+k-1}} \left( \frac{X}{p^2} + 1 \right) \ll_k \frac{X}{K} + X^{1/2}.$$

The last bound follows by summing over all integers in place of primes.

Otherwise every prime factor of $q_k(n)$ is at most $K$. Multiply its prime factors, with multiplicity, until the product first exceeds $K$. This gives an integer $g$ such that

$$g \mid q_k(n), \qquad K < g \le K^2, \qquad g \operatorname{rad}(g) \mid P_k(n).$$

For the last divisibility, if $p^a \mid g$, then $a \le v_p(P_k(n)) - 1$. Lemma 2.2 bounds the number of these $n$ by

$$(k-1)! \sum_{K < g \le K^2} k^{\omega(g)} \left( \frac{X}{g \operatorname{rad}(g)} + 1 \right). \tag{2.3}$$

For $0 < \eta < 1$, the Euler product

$$\sum_{g \ge 1} \frac{k^{\omega(g)} g^{1-\eta}}{g \operatorname{rad}(g)} = \prod_p \left( 1 + \frac{k}{p(p^\eta - 1)} \right)$$

converges, since its local factor is $1+O_{k,\eta}(p^{-1-\eta})$. The first term of (2.3) is therefore $O_{k,\eta}(XK^{-1+\eta})$.

Let $\tau_k(g)$ count the ordered factorizations of $g$ into $k$ positive integers. Then $k^{\omega(g)} \le \tau_k(g)$ and, for $Y \ge 1$,

$$\sum_{g\le Y} \tau_k(g) \le Y \left(\sum_{r\le Y} \frac{1}{r}\right)^{k-1} \le Y(1+\log Y)^{k-1}.$$

The second term of (2.3) is $O_k(K^2(1+\log K)^{k-1})$. This proves (2.2). □

**Lemma 2.4.** *For every real $X \ge 2$ and $K \ge 1$,*

$$\#\left\{(m,n) : \begin{array}{c} 1 \le m < n \le X, \quad q_k(m), q_k(n) \le K, \\ \operatorname{rad}(P_k(m)) = \operatorname{rad}(P_k(n)) \end{array}\right\} \ll_k K^2 \log(2X).$$

*Proof.* Put $\mu = q_k(m)$ and $\nu = q_k(n)$. Strict monotonicity of $P_k$ gives

$$1 \le \mu < \nu \le K, \qquad \nu P_k(m) = \mu P_k(n). \tag{2.4}$$

Set

$$\xi = 2m+k-1, \qquad \zeta = 2n+k-1, \qquad R_k(U) = \prod_{j=0}^{k-1}(U+2j-k+1).$$

Then $R_k(\zeta) = 2^k P_k(n)$. With

$$\mathcal{H}_k = \{h : 1 \le h \le k-1,\ h \equiv k-1 \pmod 2\}, \qquad A_k = \sum_{h\in\mathcal{H}_k} h^2,$$

we have

$$R_k(U) = U^k G_k(U), \qquad G_k(U) = \prod_{h\in\mathcal{H}_k}\left(1-\frac{h^2}{U^2}\right).$$

For $U \ge k+1$, the function $G_k$ is strictly increasing and

$$0 < G_k(U) < 1, \qquad 1 - G_k(U) \le \frac{A_k}{U^2}. \tag{2.5}$$

Indeed, $\mathcal{H}_k$ is nonempty and each factor is positive and strictly increasing. The last inequality follows from $1-\prod_j(1-x_j) \le \sum_j x_j$ for $0 \le x_j \le 1$.

Fix $\mu, \nu$ in (2.4), and put $\gamma = (\nu/\mu)^{1/k}$ and $\lambda = \zeta/\xi$. Thus $\zeta > \xi \ge k+1$ and

$$\gamma^k = \lambda^k \frac{G_k(\zeta)}{G_k(\xi)}, \qquad 1 < \lambda < \gamma.$$

Using (2.5), we obtain

$$\begin{aligned} 0 < \gamma - \lambda &= \gamma\left(1 - \left(\frac{G_k(\xi)}{G_k(\zeta)}\right)^{1/k}\right) \\ &\le \gamma\big(1 - G_k(\xi)^{1/k}\big) \\ &\le \gamma\big(1 - G_k(\xi)\big) \\ &\le \frac{A_k\gamma}{\xi^2}. \end{aligned}$$

Write $\lambda = h/q$ in lowest terms, with $h, q > 0$. Since $q \mid \xi$,

$$0 < \gamma - \frac{h}{q} \le \frac{A_k \gamma}{q^2}, \qquad q \le 2X + k - 1.$$

In a block $2^j \le q < 2^{j+1}$, these fractions lie in an interval of length $A_k\gamma/2^{2j}$. Distinct reduced fractions in the block are more than $2^{-2j-2}$ apart. There are at most $1 + 4A_k\gamma$ fractions in each block, and hence $O_k(\gamma \log(2X))$ possible values of $\lambda$.

For each such $\lambda$, every possible $\xi$ is a root of

$$\nu R_k(U) - \mu R_k(\lambda U) = 0.$$

This polynomial has degree $k$, since its leading coefficient is $\nu - \mu\lambda^k = \mu(\gamma^k - \lambda^k) > 0$. It has at most $k$ real roots. Each root determines at most one pair $(m, n)$. Therefore, for fixed $\mu, \nu$, the number of pairs in (2.4) is

$$\ll_k (\nu/\mu)^{1/k} \log(2X).$$

Finally,

$$\sum_{1 \le \mu < \nu \le K} (\nu/\mu)^{1/k} \le \sum_{\nu \le K} \nu^{1/k} \frac{k}{k-1} \nu^{1-1/k} \ll_k K^2. \qquad \square$$

*Proof of Theorem 1.* Let $\varepsilon > 0$. A counted pair either has both quotients at most $K$, or has an endpoint in the exceptional set of Lemma 2.3. Multiplying the latter count by (2.1) and applying Lemma 2.4 gives, for $K \ge \max(k, 2)$,

$$\begin{aligned} F_{k,k}(X) &\ll_{k,\delta,\eta} K^2 \log(2X) \\ &\quad + X^{k\delta} \log(2X) \left( XK^{-1+\eta} + K^2 (1 + \log K)^{k-1} + X^{1/2} \right). \end{aligned} \tag{2.6}$$

The fibre bound counts all possible partners $1 \le m \le X$, so this also covers pairs whose smaller endpoint is exceptional.

Ignoring the factors $X^{k\delta}$, $K^\eta$ and the logarithms, $K = X^\tau$ gives the exponent $\max(2\tau, 1 - \tau, 1/2)$, minimized at $\tau = 1/3$. Set $K = \max(k, 2, X^{1/3})$. If $X \ge k^3$, then $K = X^{1/3}$. With this choice fixed, take

$$\eta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right), \qquad \delta = \frac{\varepsilon}{4k}.$$

Every power of $X$ in (2.6) is at most

$$\frac{2}{3} + k\delta + \frac{\eta}{3} \le \frac{2}{3} + \frac{5\varepsilon}{12}.$$

The logarithmic factors together are $O_{k,\varepsilon}(X^{\varepsilon/2})$. Hence

$$F_{k,k}(X) \ll_{k,\varepsilon} X^{2/3+\varepsilon}.$$

The bounded range $2 \le X < k^3$ is absorbed into the implied constant. $\square$

## 3. A cubic lower-bound family

**Theorem 2.** *There is an absolute constant $C$ such that, for every real $X \ge 2$,*

$$F(X) \ge \frac{4^{1/3}}{3} X^{1/3} - C.$$

*Proof.* Erdős posed the product-support counting problem in [3]. We know of no published source for this particular polynomial family. Take integers $t \geq 7$ satisfying

$$t \not\equiv 0 \pmod 4, \qquad t \equiv 1, 2, 7, 8 \pmod 9,$$

and put

$$m = \frac{t^2 - 4}{3}, \qquad n = \frac{t^3 - 3t - 2}{4}.$$

The resulting integers are positive, and

$$12(n - m) = 3t^3 - 4t^2 - 9t + 10 > 0$$

for $t \geq 7$, so $m < n$. Direct multiplication gives

$$m(m + 1) = \frac{(t^2 - 4)(t^2 - 1)}{9}, \qquad n(n + 1) = \frac{(t^2 - 4)(t^2 - 1)^2}{16}.$$

For every prime other than 2 and 3, the two products have the same prime support. If $t \equiv \pm 1 \pmod 9$, then $9 \mid t^2 - 1$ and $3 \mid t^2 - 4$; if $t \equiv \pm 2 \pmod 9$, then $9 \mid t^2 - 4$ and $3 \mid t^2 - 1$. Thus division by 9 leaves a factor 3 in the first product, and the second product is also divisible by 3.

If $t$ is odd, then $8 \mid t^2 - 1$, so both products remain even after their displayed divisions. If $t \equiv 2 \pmod 4$, then $32 \mid t^2 - 4$, which again leaves both products even after division by 16. Hence the two products have equal radicals.

The conditions specify twelve residue classes modulo 36, of density $1/3$. Since $n = t^3/4 + O(t)$, the number of such $t$ with $n \leq X$ is $(4^{1/3}/3)X^{1/3} + O(1)$. The map $t \mapsto m$ is strictly increasing for $t \geq 7$, so these give distinct pairs counted by $F(X)$. □

## 4. A radical-threshold estimate from known results

For real $H \geq 2$ and $1 \leq Y < H^2$, define

$$N(H, Y) = \#\{t \in \mathbb{Z} : H < t \leq 2H, \operatorname{rad}(t(t + 1)) \leq Y\}.$$

**Theorem 3.** *For every $\varepsilon > 0$,*

$$N(H, Y) \ll_\varepsilon H^\varepsilon \left( \frac{Y}{H} + H^{7/12} \right)$$

*uniformly for real $H \geq 2$ and $1 \leq Y < H^2$.*

Theorem 3 is a consequence of known estimates, stated here in this notation. In the proof of Kane's Proposition 7 [6], the dyadic lattice count for triples of height $N_0$ is bounded by

$$N_0^\varepsilon \left[ \frac{\operatorname{rad}(A) \operatorname{rad}(B) \operatorname{rad}(C)}{|C|} + v(A)v(B)v(C) \right].$$

For $H < t \leq 2H$, take $N_0 = \lceil 2H + 1 \rceil$ and $(A, B, C) = (1, t, -(t + 1))$. Put

$$S_2(n) = \prod_{p^j \| n} p^{\lfloor j/2 \rfloor}, \qquad T_2(n) = \frac{n}{S_2(n)^2}, \qquad \Pi = S_2(t)S_2(t + 1).$$

Here $v(n) = \prod_{p^2 \mid n} p = \operatorname{rad}(S_2(n))$ in Kane's notation. When $\Pi \leq H^{2/3}$, the displayed lattice inequality gives $H^\varepsilon(Y/H + \Pi)$. For $\Pi > H^{2/3}$, Kane's Proposition 10

[6, Proposition 10] gives the square-part estimate

$$H^{\varepsilon}\left(T_2(t)T_2(t+1)+\sqrt{\Pi T_2(t)T_2(t+1)}\right)\ll H^{\varepsilon}\left(\frac{H^2}{\Pi^2}+\frac{H}{\sqrt{\Pi}}\right),$$

since $t(t+1)\ll H^2$. Both ranges are $O_{\varepsilon}(H^{\varepsilon}(Y/H+H^{2/3}))$, with the split point $\Pi=H^{2/3}$ balancing the terms in Kane's proof. Diao's Lemma 2.1 [2] gives the $7/12$ estimate in the large-product range. Appendix A gives a detailed deduction via Reuss's theorem. Theorem 3 records this known consequence in the notation used here.

## 5. Lebowitz–Lockard's Conjecture 11

Lebowitz–Lockard state the following conjecture [7, Conjecture 11]:

> If $k,\ell>1$ and $(k,\ell)\neq(2,2)$, then the equation
>
> $$\operatorname{rad}(m(m+1)\cdots(m+k-1))=\operatorname{rad}(n(n+1)\cdots(n+\ell-1))$$
>
> only has finitely many solutions.

Their preceding definition of $F_{k,\ell}$ imposes $m<n$. The conjecture also fails under this order condition, for $(k,\ell)=(3,2)$ and $(4,2)$. Equivalent forms of the identities below appear in Erdős–Straus [4, p. 64]. For every positive integer $t$,

$$P_2(t(t+2))=t(t+2)(t+1)^2,$$
$$P_2\left(\frac{t(t+3)}{2}\right)=\frac{1}{4}P_4(t).$$

Since $t+1\mid P_3(t)$, the first identity shows that $(m,n)=(t,t(t+2))$ gives equal radicals for $(k,\ell)=(3,2)$. For the second identity, four consecutive integers have product divisible by 24: one is divisible by 4, a distinct one is even, and one is divisible by 3. Thus $P_4(t)/4$ remains divisible by 6, so division by 4 leaves its prime support unchanged. The integer $t(t+3)/2$ is integral, and the second identity gives equal radicals for $(m,n)=(t,t(t+3)/2)$ and $(k,\ell)=(4,2)$. Both families have $m<n$ for every positive integer $t$. Swapping the variables gives infinite families for $(2,3)$ and $(2,4)$ if no order condition is imposed. Conjecture 11 is false as literally stated; its intended form is unclear. Erdős and Straus [4, p. 64] conjectured that $A(n,k)=\rho A(m,\ell)$ has only finitely many solutions with $m\geq n+k$ for each rational $\rho$ when $\min(k,\ell)>1$ and $\max(k,\ell)>2$, where $A(n,k)=(n+k)!/n!$. The second identity reads $A(t-1,4)=4A(t(t+3)/2-1,2)$, and $t(t+3)/2-1>(t-1)+4$ for every integer $t\geq 3$. Their conjecture is therefore also false as literally stated. They print the underlying four-factor identity on the same page in their divisibility argument.

## 6. Further consequences

### 6.1. **Radical fibres and moments.**

*Remark* 6.1. Lemma 2.1 is a growth form of the Pell argument; compare Lehmer [8, Theorem 1] and Lebowitz–Lockard [7, Lemma 8]. In particular, if

$$\nu_R(X)=\#\{1\leq n\leq X:\operatorname{rad}(n(n+1))=R\},$$

then Lemma 2.1 gives

$$\nu_R(X)\ll 2^{\omega(R)}\log(2X)\ll_{\varepsilon}X^{\varepsilon}\qquad(R\ll X^2),$$

so $\max_R \nu_R(X) \ll_\varepsilon X^\varepsilon$. Since

$$\sum_R \nu_R(X) = \lfloor X \rfloor, \qquad \sum_R \nu_R(X)(\nu_R(X)-1) = 2F(X),$$

the main theorem gives

$$\#\{\mathrm{rad}(n(n+1)) : 1 \le n \le X\} = \lfloor X \rfloor - O_\varepsilon(X^{2/3+\varepsilon}),$$
$$\sum_R \nu_R(X)^r = \lfloor X \rfloor + O_{r,\varepsilon}(X^{2/3+\varepsilon})$$

for each fixed integer $r \ge 2$. For $r \ge 3$, summing the following pointwise bound over $R$ gives

$$0 \le \nu_R(X)^r - \nu_R(X) \le (r-1)\big(\max_S \nu_S(X)\big)^{r-2} \nu_R(X)(\nu_R(X)-1),$$

and hence the stated moment estimate. The case $r = 2$ follows from the pair identity above. That identity also gives

$$0 \le \lfloor X \rfloor - \#\{R : \nu_R(X) > 0\} = \sum_{\nu_R(X)>0} (\nu_R(X)-1) \le F(X).$$

6.2. **Short gaps.**

*Remark* 6.2. For $1 \le \Delta \le X$, let $G(X,\Delta)$ count the pairs in $F(X)$ with $n-m \le \Delta$. For a fixed gap $d = n - m \ge 2$, every prime dividing the common radical divides one factor from each of $m(m+1)$ and $n(n+1)$. Subtracting these factors shows that it divides one of $d-1, d, d+1$. There are at most $2^{\omega(d(d-1)(d+1))}$ possible common radicals. Since $d(d-1)(d+1) \le X^3$, take $\delta = \varepsilon/6$ in $2^{\omega(q)} \ll_\delta q^\delta$. This contributes at most $O_\varepsilon(X^{\varepsilon/2})$ radicals. In the fibre estimate, take its exponent parameter to be $\varepsilon/8$; then $X^{\varepsilon/4}\log(2X) \ll_\varepsilon X^{\varepsilon/2}$. Thus each gap contributes $O_\varepsilon(X^\varepsilon)$ pairs. Summing over $d \le \Delta$ gives

$$G(X,\Delta) \ll_\varepsilon \Delta X^\varepsilon.$$

For $d = 1$, equality of radicals reduces to $\mathrm{rad}(m) = \mathrm{rad}(m+2)$, which forces $m = 2$ and gives the single pair $(m,n) = (2,3)$: their common radical divides $\mathrm{rad}(\gcd(m, m+2)) \mid 2$. It cannot be 1, so both terms are powers of 2, which leaves $m = 2$.

## Appendix A. Proof of Theorem 3

*Proof of Theorem 3.* Write uniquely

$$t = d^2 r, \qquad t+1 = f^2 s,$$

where $r, s$ are squarefree positive integers. Then

$$f^2 s - d^2 r = 1, \qquad (d,f) = (r,s) = 1. \tag{A.1}$$

Put

$$\alpha_0 = \gcd(r, \mathrm{rad}(d)), \qquad \beta_0 = \gcd(s, \mathrm{rad}(f)).$$

The radical identity is

$$\mathrm{rad}(t(t+1)) = \frac{rs\,\mathrm{rad}(d)\,\mathrm{rad}(f)}{\alpha_0\beta_0}. \tag{A.2}$$

Since $rs > H^2/(d^2 f^2)$, every point counted by $N(H,Y)$ satisfies

$$d^2 f^2 \alpha_0 \beta_0 \ge \frac{H^2\,\mathrm{rad}(d)\,\mathrm{rad}(f)}{Y}. \tag{A.3}$$

Set $H_0 = H^{7/12}$ and first count the points with $df \le H_0$. For fixed $d, f, \alpha_0, \beta_0$, the divisibilities $\alpha_0 \mid r$ and $\beta_0 \mid s$ give

$$t \equiv 0 \pmod{d^2\alpha_0}, \qquad t \equiv -1 \pmod{f^2\beta_0}.$$

The moduli are coprime, since every prime dividing the first divides $t$ and every prime dividing the second divides $t+1$. Thus the Chinese remainder theorem gives at most one class modulo $d^2f^2\alpha_0\beta_0$, containing at most $H/(d^2f^2\alpha_0\beta_0)+1$ integers in the interval. By (A.3),

$$N_{\mathrm{lo}} \ll \frac{Y}{H} \sum_{df \le H_0} \frac{\tau(\mathrm{rad}(d))\tau(\mathrm{rad}(f))}{\mathrm{rad}(d)\,\mathrm{rad}(f)} + \sum_{df \le H_0} \tau(d)\tau(f). \tag{A.4}$$

For every $\eta > 0$, Rankin's trick and the convergent product

$$\sum_{n \ge 1} \frac{\tau(\mathrm{rad}(n))}{\mathrm{rad}(n) n^{\eta}} = \prod_p \left(1 + \frac{2}{p(p^{\eta}-1)}\right) < \infty$$

give

$$\sum_{df \le H_0} \frac{\tau(\mathrm{rad}(d))\tau(\mathrm{rad}(f))}{\mathrm{rad}(d)\,\mathrm{rad}(f)} \ll_{\eta} H_0^{\eta}.$$

Also $\sum_{df \le H_0} \tau(d)\tau(f) \ll H_0(\log(2H_0))^3$, by summing the four-fold divisor function. Taking $\eta$ sufficiently small in terms of $\varepsilon$ and absorbing logarithms gives

$$N_{\mathrm{lo}} \ll_{\varepsilon} H^{\varepsilon}\left(\frac{Y}{H} + H^{7/12}\right). \tag{A.5}$$

It remains to count solutions of (A.1) with $df > H_0$; here we discard the radical restriction. Since $r, s \ge 1$, $t \le 2H$, and $t+1 \le 3H$, we have

$$d \le \sqrt{2H}, \qquad f \le \sqrt{3H}. \tag{A.6}$$

Partition the $d, f$ ranges into dyadic boxes and refine each into multiplicative cells of ratio $\lambda = 2^{1/8}$. On a cell write

$$S < t \le \lambda S, \qquad \mathcal{D} < d \le \lambda \mathcal{D}, \qquad \mathcal{E} < f \le \lambda \mathcal{E},$$

and put $x = S/\lambda^2$, $U = x/\mathcal{D}^2$, and $V = x/\mathcal{E}^2$. Then $r = t/d^2 \in (U, \lambda^3 U)$ and $s = (t+1)/f^2 \in (V, \lambda^3 V + \mathcal{E}^{-2})$. Since $\lambda^3 + \lambda^2/S < 2$ for $S \ge 2$, we have

$$U < r < 2U, \qquad V < s < 2V,$$

and the variables lie in dyadic boxes with the exact scale identities

$$U\mathcal{D}^2 = V\mathcal{E}^2 = x \asymp H.$$

Only a bounded number of such cells is used in each dyadic box. In every cell,

$$UV \asymp \frac{H^2}{(\mathcal{D}\mathcal{E})^2}. \tag{A.7}$$

Suppose first that $UV \le H^{1/2}$. Fix squarefree $r, s$. If $rs = 1$, (A.1) would give $f^2 - d^2 = 1$, which is impossible for positive $d, f$. Otherwise

$$(sf)^2 - rsd^2 = s.$$

The element

$$\xi = sf + d\sqrt{rs} = \sqrt{s}\left(\sqrt{t+1} + \sqrt{t}\right)$$

is totally positive and has norm $s$. The relevant quadratic order is maximal unless $rs \equiv 1 \pmod 4$; in that case it has conductor 2, and $s$ is odd and coprime to the

conductor. Thus the ideals of norm $s$ in the order are invertible and correspond to ideals of norm $s$ in the maximal order. Every prime dividing $s$ ramifies, so at most $\tau(s)$ such ideals arise. Generators of the same principal ideal differ by a unit of the order; for two totally positive generators their ratio is a totally positive norm-one unit. If $\epsilon > 1$ is such a unit, its integer trace is at least 3, whence $\epsilon \geq (3+\sqrt{5})/2 > 2$. In contrast, the displayed formula for $\xi$ shows that its ratio across $H < t \leq 2H$ is less than $\sqrt{2}$. Thus each ideal contributes at most one solution. Since $s \leq 3H$, $\tau(s) \ll_\varepsilon H^\varepsilon$. There are $O(UV)$ choices of $r, s$ in their boxes, and therefore

$$N_{UV \leq H^{1/2}} \ll_\varepsilon H^\varepsilon UV \ll_\varepsilon H^{1/2+\varepsilon}. \tag{A.8}$$

Now suppose $UV > H^{1/2}$. Equations (A.7) and $df > H_0$ give

$$H^{7/12} \ll \mathcal{D}\mathcal{E} \ll H^{3/4}, \qquad \log(\mathcal{D}\mathcal{E}) \asymp \log(UV) \asymp \log H.$$

Apply Reuss's determinant estimate [9, Theorem 1] to $sf^2 - rd^2 = 1$ with $(k, \ell, h) = (2, 1, 1)$ and scales $U = x/\mathcal{D}^2$, $V = x/\mathcal{E}^2$. They satisfy $U\mathcal{D}^2 = V\mathcal{E}^2 = x$. Since $df > H_0$ and (A.6) gives $d \leq \sqrt{2H}$ and $f \leq \sqrt{3H}$, we have

$$d > H_0/\sqrt{3H} \gg H^{1/12}, \qquad f > H_0/\sqrt{2H} \gg H^{1/12}.$$

As $d \leq \lambda\mathcal{D}$, $f \leq \lambda\mathcal{E}$, and $x \asymp H$, this implies $\mathcal{D}, \mathcal{E}, x > 1$ for large $H$. Define

$$M = \exp\left(\frac{9\log(\mathcal{D}\mathcal{E})\log(UV)}{8\log x}\right).$$

In this range $\mathcal{D}\mathcal{E} > 1$, $UV > H^{1/2} > 1$, and $x > 1$, so $M > 1$. If the sides are interchanged, the equation has $h = -1$. Reuss's hypotheses are $1 \leq \ell < k$, $h \neq 0$, $\log(\mathcal{D}\mathcal{E}) \asymp \log(UV) \asymp \log x$, and, when $\ell = 1$, $\mathcal{D}\mathcal{E} \gg x^{1/k}$. We have $\mathcal{D}\mathcal{E} \gg H^{7/12} \gg x^{1/2}$, and the logarithmic comparabilities were established above. The estimate applies for large $x$; bounded values of $H$ are absorbed in the implied constant. The resulting bound for the number $N$ in a cell is

$$\begin{aligned} N &\ll_\varepsilon H^\varepsilon \min\left\{(\mathcal{D}\mathcal{E}M)^{1/2} + \mathcal{D} + \mathcal{E}, (UVM)^{1/2} + U + V\right\}, \\ \log M &= \frac{9\log(\mathcal{D}\mathcal{E})\log(UV)}{8\log x}. \end{aligned} \tag{A.9}$$

Write $\mathcal{D}\mathcal{E} = H^\theta$ up to constant factors. Since $x \asymp H$ and $UV \asymp H^2/(\mathcal{D}\mathcal{E})^2$, the determinant term in (A.9) has exponent at most

$$\phi(\theta) + O\left(\frac{1}{\log H}\right), \qquad \phi(\theta) = \frac{1}{2}\min(\theta, 2-2\theta) + \frac{9}{8}\theta(1-\theta), \quad \frac{7}{12} \leq \theta \leq \frac{3}{4}.$$

The function increases up to $\theta = 2/3$ and decreases afterwards, with

$$\phi(2/3) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

It remains to bound the additive terms. For $\theta \leq 2/3$, use the first branch of (A.9); by (A.6), $\mathcal{D} + \mathcal{E} \ll H^{1/2}$. For $\theta \geq 2/3$, orient the cell so $\mathcal{D} \leq \mathcal{E}$, hence $U \geq V$, and use the second branch. If $U \leq H^{7/12}$, its additive term is already small enough. Otherwise $U = x/\mathcal{D}^2 > H^{7/12}$ gives $\mathcal{D} \ll H^{5/24}$; together with $\mathcal{D}\mathcal{E} \gg H^{2/3}$ this gives $\mathcal{E} \gg H^{11/24}$ and $V \ll H^{1/12}$. Fixing $d, s$, the equation gives

$$f^2 s \equiv 1 \pmod{d^2},$$

or $f^2 s \equiv -1 \pmod{d^2}$ in the reversed orientation. Since $(d, s) = 1$, each sign has at most $4^{\omega(d)} \ll_\varepsilon H^\varepsilon$ roots modulo $d^2$. Counting $f$ in its interval for each root and summing over $d \ll \mathcal{D}$ and $s \ll V$ gives

$$N \ll_\varepsilon H^\varepsilon \left( \frac{\mathcal{E}V}{\mathcal{D}} + \mathcal{D}V \right) \ll_\varepsilon H^\varepsilon \left( \frac{H}{\mathcal{D}\mathcal{E}} + \mathcal{D}V \right) \ll_\varepsilon H^{1/3+\varepsilon},$$

since $\mathcal{D}V \ll H^{7/24}$. Thus all additive terms are below $H^{7/12+\varepsilon}$. There are $O((\log H)^2)$ dyadic boxes and a bounded number of cells in each. Summing (A.8) and the bounds above over all cells, the logarithms are absorbed into $H^\varepsilon$ and

$$N_{\mathrm{hi}} \ll_\varepsilon H^{7/12+\varepsilon}. \tag{A.10}$$

Combining (A.5) and (A.10) proves the theorem. □

*Remark* A.1. Diao's Lemma 2.1 [2] gives the 7/12 exponent for $\delta_1 x_1^2 - \delta_2 x_2^2 = \vartheta$ in dyadic boxes with $\Delta_1 \le \Delta_2$, $\Delta_1 \Delta_2 \le B$, and $\Delta_1 X_1^2 = \Delta_2 X_2^2 = B$, where $\vartheta \in \{\pm 1, \pm 2\}$. In our cell the scales are $(\Delta_1, X_1, \Delta_2, X_2, B) = (V, \mathcal{E}, U, \mathcal{D}, x)$, and the equation is $sf^2 - rd^2 = 1$. If the scales need to be ordered, interchanging the two sides changes $\vartheta$ to $-1$. Since $\mathcal{D}\mathcal{E} \gg H^{7/12}$ and $x \asymp H$, $UV = x^2/(\mathcal{D}\mathcal{E})^2 \ll H^{5/6} \ll x$ for sufficiently large $H$. Thus Diao's condition $\Delta_1 \Delta_2 \le B$ holds. Diao proves Lemma 2.1 by combining the determinant estimate (2.1) with the Fouvry–Jouve estimate (2.2). The determinant input is the $(k, \ell) = (2, 1)$ case of Reuss's Theorem 1 [9], applied above with $h = \pm 1$. Diao cites this estimate as Theorem 5 of Reuss's 2015 thesis; his lemma therefore gives the high-product bound directly.

## Use of AI tools

The author devised the main argument. Codex (OpenAI, GPT-6 Astra) carried out the numerical checks of the identities and of the cubic family and searched the literature. Claude Code (Anthropic, Claude Opus 5.5) checked the arguments and the references. The author checked every step of the proof and every reference, and takes full responsibility for the content of the paper.

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