THE IRRATIONALITY EXPONENT OF SMALL-HEIGHT BINARY LACUNARY SERIES IS 2

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ABSTRACT. Let $N \geq 2$ be an integer and let λ_j be nonzero rational numbers with $\log H(\lambda_j) = o(2^j)$, where $H(u/v) = \max(|u|, v)$ for u/v in lowest terms with $v > 0$. We prove that $\sum_{j\geq 0} \lambda_j N^{-2^j}$ is irrational and has irrationality exponent 2. The proof uses Padé approximants to the tails $\sum_{r\geq 0} \lambda_{k+r} z^{2^r}$. The two Hankel families they need satisfy parity recursions, so they are nonzero for all nonzero weights. After the finite prefix of the series is cleared, the approximants of order n give integer linear forms with height $N^{(n+1/2)2^k}$ and error $N^{-(n+1/2)2^k}$, up to factors $e^{o(2^k)}$. Only finitely many orders are used for each target exponent. The height condition cannot be weakened to $\log H(\lambda_j) \leq C2^j$ for any constant $C > 0$: for every C some such weights give exponent greater than 2, and for larger C positive integer weights can give a rational number or a Liouville number.

1. INTRODUCTION

For an irrational real number ξ, its irrationality exponent is

$$\mu(\xi) = \sup\left\{\nu \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < q^{-\nu} \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0}\right\}.$$

The pigeonhole principle gives $\mu(\xi) \geq 2$ (see the end of Section 5). Write $L = \log N$. For a nonzero rational number in lowest terms, put

$$H(u/v) = \max(|u|, v), \qquad u \in \mathbb{Z} \setminus \{0\}, \quad v \in \mathbb{Z}_{>0}, \quad \gcd(u,v) = 1. \tag{1.1}$$

We show that equality holds for binary lacunary series with rational weights of sufficiently small height.

Theorem 1. *Let $N \in \mathbb{Z}$, $N \geq 2$, and let $(\lambda_j)_{j\geq 0}$ be a sequence in $\mathbb{Q} \setminus \{0\}$ such that $\log H(\lambda_j) = o(2^j)$. Then the absolutely convergent series*

$$x = \sum_{j=0}^{\infty} \lambda_j N^{-2^j} = \frac{\lambda_0}{N} + \frac{\lambda_1}{N^2} + \frac{\lambda_2}{N^4} + \frac{\lambda_3}{N^8} + \cdots \tag{1.2}$$

is irrational and satisfies $\mu(x) = 2$.

Geometric weights $\lambda_j = \lambda^j$, with $\lambda \in \mathbb{Q} \setminus \{0\}$, satisfy the hypothesis because $\log H(\lambda^j) = O(j)$. Every sequence drawn in arbitrary order from a finite subset of $\mathbb{Q} \setminus \{0\}$ also satisfies the hypothesis, since its heights are bounded.

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Duverney's criterion [3, Corollary 3.4] already gives irrationality under this height condition. For $w_j = N^{2^j}$, its rationality exception would require

$$\lambda_j\lambda_{j+2} = w_j\lambda_{j+1}^2$$

eventually. Here $\log|\lambda_j\lambda_{j+2}/\lambda_{j+1}^2| = o(2^j)$, whereas $\log w_j = 2^jL$, so the exception cannot occur. Duverney [3, Theorem 3.2] also studies finite exponent bounds. For related work, see his later paper [4].

Duverney, Kurosawa and Shiokawa [5, Theorem 1] consider signed series $\sum \epsilon_j/x_j$, where $x_j > 1$ are rational and $x_{j+1} \geq x_j^2$ eventually. Their denominator condition is

$$\delta_1 = \operatorname{den}(x_1), \qquad \delta_{j+1} = \delta_j^2 \operatorname{den}(x_{j+1}/x_j^2), \qquad \log\delta_{j+1} = o(\log x_j), \tag{1.3}$$

where den is the reduced positive denominator. They conclude $\mu(\sum \epsilon_j/x_j) = \limsup_j \log x_{j+1}/\log x_j$. For $x_j = N^{2^j}/(t|\lambda_j|)$ with a fixed rational $t > 0$, the ratio is $t|\lambda_j|^2/|\lambda_{j+1}|$; the accumulated condition in (1.3) can fail even for a finite set of weights.

Amou and Bugeaud [1, Lemma 3] give an earlier criterion for $\sum B_j/A_j$, with strictly increasing positive integers A_j and nonzero integers B_j. Their hypotheses include ratios $S_j = (A_{j+1}/|B_{j+1}|)/(A_j/|B_j|)^2 \geq 2$ eventually and $\prod_{r=0}^{j} d_r^{2^{j-r}} = A_j^{o(1)}$, where $d_j = \operatorname{den}(S_j)$ and $d_0 = |B_1|$; small heights alone do not imply this cumulative arithmetic condition.

For unit weights, Shallit [8, Theorems 8–9] proved bounded partial quotients when $N \geq 3$. For $N = 2$, he states the corresponding result for $2\sum_{j\geq 0} 2^{-2^j}$ [8, Theorem 11 and following example]. He also states a recursion for weights $(-1)^j$ when $N \geq 3$ [8, Theorem 12].

For nonzero integral geometric weights, Bugeaud, Han, Wen and Yao [2, Theorem 2.1] give exponent 2: $g(z) = \sum_{r\geq 0} \lambda^r z^{2^r-1}$ has integer coefficients, satisfies $g(z) = 1 + \lambda z g(z^2)$, and has nonzero Hankel determinants of every order by Lemma 2.1. The multiplier λz is nonzero on the orbit of $1/N$. The same theorem covers periodic rational weights. If the period is $p \geq 1$ and $D_{\mathrm{per}} \in \mathbb{Z}_{>0}$ clears the denominators, then $f(z) = D_{\mathrm{per}} \sum_{j\geq 0} \lambda_j z^{2^j-1} \in \mathbb{Z}[[z]]$ converges in the unit disk and satisfies $f(z) = P(z) + z^{2^p-1} f(z^{2^p})$ with $P \in \mathbb{Z}[z]$. Lemma 2.1 gives every Hankel order, and z^{2^p-1} does not vanish on the orbit of $1/N$. Since $x = f(1/N)/(D_{\mathrm{per}}N)$, [2, Theorem 2.1] gives $\mu(x) = 2$. Su [9, Theorem 7.3] gives a formal continued fraction for $\sum_{j\geq 0} c_j z^{2^j-1}$ with $c_0 = 1$ and nonzero c_j in any field.

Rajchert [7] and Su [9] study related questions on irrationality exponents and continued fractions of Mahler-type series. None of our results depends on these works; apart from standard facts, every step is proved here.

For geometric weights, Mahler's method also gives transcendence [6, Corollary 2]. The series $F(z) = \sum_{r\geq 0} \lambda^r z^{2^r}$ has radius 1 and Hadamard gaps, so it is transcendental over $\mathbb{C}(z)$. It satisfies $F(z) = z + \lambda F(z^2)$; the matrix of the system for $(1, F(z))^{\mathsf{T}}$ and its inverse have polynomial entries. The orbit of $1/N$ therefore contains no singular point.

Our proof uses moving tails and their Hankel recursions. After clearing the finite prefix, adjacent Padé approximants give integer pairs with nonzero determinant. For each target exponent the finite set of orders is fixed first, so all coefficient costs are $o(2^k)$ at tail shift k. Section 6 shows that the height scale is sharp.

Throughout the proof write

$$\lambda_j = \frac{u_j}{v_j}, \qquad u_j \in \mathbb{Z} \setminus \{0\}, \quad v_j \in \mathbb{Z}_{>0}, \quad \gcd(u_j, v_j) = 1, \tag{1.4}$$
$$h_j = \log H(\lambda_j).$$

In particular,

$$e^{-h_j} \le |\lambda_j| \le e^{h_j}. \tag{1.5}$$

The upper bound and $h_j = o(2^j)$ give absolute convergence of (1.2). The moving tails are

$$F_k(z) = \sum_{r \ge 0} \lambda_{k+r} z^{2^r}, \qquad F_k(z) = \lambda_k z + F_{k+1}(z^2). \tag{1.6}$$

For each fixed k, the series converges absolutely for $|z| < 1$.

2. Hankel determinants

Put

$$g_k(z) = \frac{F_k(z)}{z} = \sum_{i \ge 0} a_i^{(k)} z^i. \tag{2.1}$$

By (1.6),

$$a_0^{(k)} = \lambda_k, \qquad a_{2i}^{(k)} = 0 \quad (i \ge 1), \qquad a_{2i+1}^{(k)} = a_i^{(k+1)} \quad (i \ge 0). \tag{2.2}$$

Define

$$\begin{aligned} H_n^{(k)} &= \det(a_{i+j}^{(k)})_{0 \le i,j < n}, & H_0^{(k)} &= 1, \\ K_n^{(k)} &= \det(a_{i+j+1}^{(k)})_{0 \le i,j < n}, & K_0^{(k)} &= 1. \end{aligned} \tag{2.3}$$

Lemma 2.1. *For all shifts $k \ge 0$, the determinants satisfy*

$$H_{2m}^{(k)} = (-1)^m (H_m^{(k+1)})^2 \qquad (m \ge 1), \tag{2.4}$$

$$H_{2m+1}^{(k)} = (-1)^m \lambda_k (K_m^{(k+1)})^2 \qquad (m \ge 0), \tag{2.5}$$

$$K_n^{(k)} = H_{\lceil n/2 \rceil}^{(k+1)} K_{\lfloor n/2 \rfloor}^{(k+1)} \qquad (n \ge 1). \tag{2.6}$$

In particular, $H_n^{(k)} K_n^{(k)} \ne 0$ for every $n, k \ge 0$.

Proof. Ordering rows and columns by parity makes the two permutation signs cancel. For $H_{2m}^{(k)}$, the matrix becomes

$$\begin{pmatrix} \lambda_k E_{00} & A \\ A^{\mathsf{T}} & 0 \end{pmatrix}, \qquad A = (a_{r+s}^{(k+1)})_{0 \le r,s < m},$$

where E_{00} has its sole nonzero entry 1 in the upper left corner. Exchanging the two blocks of m columns contributes $(-1)^{m^2} = (-1)^m$ and leaves diagonal blocks A and A^{T}. This proves (2.4).

For $H_{2m+1}^{(k)}$, separate index 0, the remaining even indices, and the odd indices. The case $m = 0$ gives $H_1^{(k)} = \lambda_k$. For $m \ge 1$, put

$$\boldsymbol{a} = (a_0^{(k+1)}, \dots, a_{m-1}^{(k+1)})^{\mathsf{T}}, \qquad G_m = (a_{r+s+1}^{(k+1)})_{0 \le r,s < m}.$$

The matrix is

$$\begin{pmatrix} \lambda_k & 0 & \boldsymbol{a}^{\mathsf{T}} \\ 0 & 0 & G_m \\ \boldsymbol{a} & G_m^{\mathsf{T}} & 0 \end{pmatrix}. \tag{2.7}$$

Since $\lambda_k \neq 0$, pivoting on the first entry gives the Schur complement

$$\begin{pmatrix} 0 & G_m \\ G_m^{\mathsf{T}} & -\lambda_k^{-1}\boldsymbol{a}\boldsymbol{a}^{\mathsf{T}} \end{pmatrix}.$$

For square blocks of size m, column exchange gives

$$\det \begin{pmatrix} 0 & B \\ C & X \end{pmatrix} = (-1)^m \det B \det C. \tag{2.8}$$

The determinant of (2.7) is $(-1)^m \lambda_k (\det G_m)^2$, proving (2.5).

For $K_n^{(k)}$, the mixed parity blocks vanish by (2.2). The diagonal blocks are

$$(a_{r+s}^{(k+1)})_{0\leq r,s<\lceil n/2\rceil}, \qquad (a_{r+s+1}^{(k+1)})_{0\leq r,s<\lfloor n/2\rfloor}.$$

This proves (2.6). Strong induction on n, simultaneously for all k, starts from

$$H_0^{(k)} = K_0^{(k)} = 1, \qquad H_1^{(k)} = \lambda_k, \qquad K_1^{(k)} = \lambda_{k+1}.$$

Every remaining factor has smaller size, so all determinants are nonzero. □

Only offsets 0 and 1 are needed; other Hankel determinants may vanish, since $a_2^{(k)} = 0$. The weights may have either sign. For geometric weights, $g_k = \lambda^k g_0$; writing $H_n = H_n^{(0)}$ and $K_n = K_n^{(0)}$, the recursions specialize to

$$H_{2m} = (-1)^m \lambda^{2m} H_m^2, \qquad H_{2m+1} = (-1)^m \lambda^{2m} K_m^2, \qquad K_n = \lambda^n H_{\lceil n/2\rceil} K_{\lfloor n/2\rfloor}.$$

3. Padé approximants to the moving tails

Lemma 3.1. *For each $n \geq 1$ and $k \geq 0$, there exist a positive integer $d_{n,k}$ and polynomials $U_{n,k}, V_{n,k} \in \mathbb{Z}[z]$ with*

$$\deg U_{n,k}, \deg V_{n,k} \leq n, \qquad V_{n,k}(0) = d_{n,k} > 0, \tag{3.1}$$

such that

$$E_{n,k}(z) := V_{n,k}(z)F_k(z) - U_{n,k}(z) = c_{n,k} z^{2n+1} + O(z^{2n+2}), \qquad c_{n,k} \in \mathbb{Z}\backslash\{0\}. \tag{3.2}$$

They can be chosen so that, for each fixed finite set of orders, all their coefficients, $d_{n,k}$ and $|c_{n,k}|$ are at most $\exp(o(2^k))$, uniformly over that set as $k \to \infty$. They also satisfy the exact identity

$$U_{n+1,k}V_{n,k} - U_{n,k}V_{n+1,k} = \Omega_{n,k} z^{2n+1}, \qquad \Omega_{n,k} = d_{n+1,k}c_{n,k} \in \mathbb{Z} \setminus \{0\}. \tag{3.3}$$

Proof. Fix n, k. By Lemma 2.1, the system

$$\sum_{i=0}^{n-1} a_{i+j}^{(k)} b_i = -a_{n+j}^{(k)} \qquad (0 \leq j < n) \tag{3.4}$$

has a unique rational solution. For a power series W, the notation $[W]_{<n}$ retains only powers below n. Put

$$p_{n,k}(T) = T^n + \sum_{i=0}^{n-1} b_i T^i, \qquad Q_{n,k}(z) = z^n p_{n,k}(1/z), \tag{3.5}$$

$$P_{n,k}(z) = z[Q_{n,k}(z)g_k(z)]_{<n}. \tag{3.6}$$

Both polynomials have degree at most n, and $Q_{n,k}(0) = 1$. Here $[z^\ell]W$ denotes the coefficient of z^ℓ in W.

Define the linear functional $\mathcal{L}_k(T^j) = a_j^{(k)}$. For $n \le \ell < 2n$, (3.4) gives

$$[z^\ell](Q_{n,k}g_k) = \mathcal{L}_k(T^{\ell-n}p_{n,k}) = 0.$$

At $\ell = 2n$, the bordered determinant formula gives

$$\mathcal{L}_k(T^n p_{n,k}) = a_{2n}^{(k)} - \boldsymbol{h}^{\mathsf{T}}(a_{i+j}^{(k)})_{i,j<n}^{-1}\boldsymbol{h} = \frac{H_{n+1}^{(k)}}{H_n^{(k)}}, \tag{3.7}$$
$$\boldsymbol{h} = (a_n^{(k)}, \dots, a_{2n-1}^{(k)})^{\mathsf{T}}.$$

Hence

$$Q_{n,k}F_k - P_{n,k} = \frac{H_{n+1}^{(k)}}{H_n^{(k)}} z^{2n+1} + O(z^{2n+2}). \tag{3.8}$$

We choose the integerizer explicitly, because its size enters the height. Set

$$\ell_n = \lfloor \log_2(2n+1) \rfloor, \qquad \mathcal{D}_{n,k} = \prod_{r=0}^{\ell_n} v_{k+r}, \qquad \Sigma_{n,k} = \sum_{r=0}^{\ell_n} h_{k+r}. \tag{3.9}$$

Every $\mathcal{D}_{n,k}a_i^{(k)}$ with $0 \le i \le 2n$ is an integer. Let

$$\mathfrak{D}_{n,k} = \det(\mathcal{D}_{n,k}a_{i+j}^{(k)})_{0\le i,j<n}, \qquad d_{n,k} = \mathcal{D}_{n,k}|\mathfrak{D}_{n,k}|. \tag{3.10}$$

The determinant $\mathfrak{D}_{n,k}$ is a nonzero integer. Cramer's rule in (3.4) shows that every coefficient of $d_{n,k}Q_{n,k}$ is an integer multiple of $\mathcal{D}_{n,k}$. Consequently

$$U_{n,k} = d_{n,k}P_{n,k}, \qquad V_{n,k} = d_{n,k}Q_{n,k} \tag{3.11}$$

are integer polynomials. The coefficient of degree $2n+1$ in $V_{n,k}F_k - U_{n,k}$ is also an integer: all coefficients of F_k needed at that degree have denominator dividing $\mathcal{D}_{n,k}$. By (3.8), it is

$$c_{n,k} = d_{n,k}\frac{H_{n+1}^{(k)}}{H_n^{(k)}} \ne 0. \tag{3.12}$$

In particular, $|c_{n,k}| \ge 1$.

Next we bound the sizes in Lemma 3.1. Write $\Sigma = \Sigma_{n,k}$. Then

$$\mathcal{D}_{n,k} \le e^{\Sigma}, \qquad |\mathcal{D}_{n,k}a_i^{(k)}| \le e^{2\Sigma} \quad (0 \le i \le 2n).$$

Expansion of the determinant, and of the integer Cramer numerators, gives

$$\begin{aligned} 1 \le |\mathfrak{D}_{n,k}| &\le n!e^{2n\Sigma}, \\ 1 \le d_{n,k} &\le n!e^{(2n+1)\Sigma}, \\ \max\Big\{\max_i |[z^i]U_{n,k}|, \max_i |[z^i]V_{n,k}|, |c_{n,k}|\Big\} &\le (n+1)!e^{(2n+2)\Sigma}. \end{aligned} \tag{3.13}$$

For the last line, a coefficient of $U_{n,k}$ or the leading error is a convolution of at most $n+1$ terms; a coefficient of F_k contributes at most e^{Σ}. The division by $\mathfrak{D}_{n,k}$ cancels when the Cramer solution is multiplied by $d_{n,k}$. Since n is fixed and $h_j = o(2^j)$, $\Sigma_{n,k} = o(2^k)$, uniformly over finitely many orders. The polynomials, their integerizer and $c_{n,k}$ depend only on the weights in (3.9); the remainder $E_{n,k}$ depends on the whole tail.

Finally, the polynomial on the left of (3.3) has degree at most $2n+1$. Substituting $U_{j,k} = V_{j,k}F_k - E_{j,k}$ shows that it has no terms below degree $2n+1$, and that the coefficient at that degree is $d_{n+1,k}c_{n,k} \neq 0$. Thus it is exactly the monomial in (3.3). □

We evaluate at a point that moves with k. Put

$$\Delta = 2^k, \qquad y = N^{-\Delta}, \qquad \varepsilon_k = \sup_{j \geq k} \frac{h_j}{2^j}. \tag{3.14}$$

The hypothesis gives $\varepsilon_k \to 0$. Moreover,

$$\sum_{j<k} h_j = o(2^k). \tag{3.15}$$

Given $\rho > 0$, choose J such that $h_j \leq \rho 2^j$ for $j \geq J$. The fixed sum before J, divided by 2^k, tends to zero, and the remaining sum divided by 2^k is at most ρ.

Lemma 3.2. *For any fixed finite set of orders, uniformly over that set as $k \to \infty$,*

$$Q_{n,k}(y) = 1 + O\big(\exp(-L\Delta + o(\Delta))\big), \tag{3.16}$$

$$E_{n,k}(y) = c_{n,k}y^{2n+1}\left(1 + O\big(\exp(-L\Delta + o(\Delta))\big)\right). \tag{3.17}$$

In particular, $V_{n,k}(y) > 0$ and $E_{n,k}(y) \neq 0$ for all sufficiently large k, and

$$\log|E_{n,k}(y)| = -(2n+1)L\Delta + o(\Delta). \tag{3.18}$$

Proof. The integer Cramer numerators used above have absolute value at most $n!\exp(2n\Sigma_{n,k})$, and their denominator is a nonzero integer. Thus the nonconstant coefficients of $Q_{n,k}$ have size $\exp(o(\Delta))$. Since $y = \exp(-L\Delta)$, their finite sum proves (3.16). Also $V_{n,k} = d_{n,k}Q_{n,k}$ with $d_{n,k} > 0$, proving eventual positivity.

Fix n, and split F_k after the terms with $r \leq \ell_n$. The first omitted exponent is

$$t_n = 2^{\ell_n+1} > 2n+1.$$

For every $r \geq 0$, the definition of ε_k gives

$$|\lambda_{k+r}|y^{2^r} \leq \exp\big(-(L-\varepsilon_k)\Delta 2^r\big).$$

For all sufficiently large k, $L - \varepsilon_k \geq L/2$ and

$$\begin{aligned}\sum_{r \geq \ell_n+1} |\lambda_{k+r}|y^{2^r} &\leq \sum_{s \geq 0} \exp\big(-(L-\varepsilon_k)\Delta t_n 2^s\big) \\ &\leq 2\exp\big(-(L-\varepsilon_k)\Delta t_n\big) = \exp(-Lt_n\Delta + o(\Delta)).\end{aligned} \tag{3.19}$$

The second inequality uses $2^s \geq s+1$ and, for large k, the geometric ratio $\exp(-(L-\varepsilon_k)\Delta t_n) \leq 1/2$.

Multiplying the finite part by $V_{n,k}$ and subtracting $U_{n,k}$ gives a polynomial whose leading term is $c_{n,k}z^{2n+1}$. All its subsequent terms have degree at least $2n+2$ and coefficients of size $\exp(o(\Delta))$ by (3.13). At y, their sum is therefore at most $\exp(-(2n+2)L\Delta + o(\Delta))$. The omitted part, after multiplication by $V_{n,k}(y)$, is bounded by (3.19) with an additional factor $\exp(o(\Delta))$. Since $t_n \geq 2n+2$ and $|c_{n,k}| \geq 1$, both parts, relative to $c_{n,k}y^{2n+1}$, have absolute value at most $\exp(-L\Delta + o(\Delta))$. This proves (3.17), and the factor in parentheses is eventually nonzero. Finally, $0 \leq \log|c_{n,k}| = o(\Delta)$ proves (3.18); there are finitely many orders, so all bounds are uniform. □

4. Moving-tail iteration and integerization

Fix $k \geq 1$, so that $N^{\Delta/2}$ below is an integer, and retain $\Delta = 2^k$, $y = N^{-\Delta}$. Repeatedly splitting (1.6) gives the exact identity

$$x = S_k + F_k(y), \qquad S_k = \sum_{j=0}^{k-1} \frac{u_j}{v_j N^{2^j}}. \tag{4.1}$$

Define

$$C_k = \operatorname{lcm}(v_0, \dots, v_{k-1}), \qquad T_k = C_k N^{\Delta/2}, \qquad J_k = T_k S_k. \tag{4.2}$$

All three are integers: for $j < k$, v_j divides C_k and $2^j \leq \Delta/2$. Equations (3.15) and (4.2) give

$$0 \leq \log C_k \leq \sum_{j<k} h_j = o(\Delta), \qquad \log T_k = \tfrac{1}{2} L\Delta + o(\Delta). \tag{4.3}$$

Put

$$\begin{aligned} &\widetilde{U}_{n,k} = N^{n\Delta} U_{n,k}(y), && \widetilde{V}_{n,k} = N^{n\Delta} V_{n,k}(y), \\ &B_{n,k} = T_k \widetilde{V}_{n,k}, && A_{n,k} = J_k \widetilde{V}_{n,k} + T_k \widetilde{U}_{n,k}. \end{aligned} \tag{4.4}$$

The degree bounds give integer evaluations and hence integer pairs $A_{n,k}, B_{n,k}$. For all sufficiently large k in a fixed finite order set, $B_{n,k} > 0$ and

$$\frac{A_{n,k}}{B_{n,k}} = S_k + \frac{U_{n,k}(y)}{V_{n,k}(y)}, \tag{4.5}$$

$$B_{n,k} x - A_{n,k} = T_k N^{n\Delta} E_{n,k}(y) \neq 0. \tag{4.6}$$

Put

$$\tau_{n,k} = (n + \tfrac{1}{2}) L\Delta. \tag{4.7}$$

Write the relative error in (3.17) as $r_{n,k} = o(1)$. Then

$$B_{n,k} = C_k d_{n,k} e^{\tau_{n,k}} Q_{n,k}(y), \tag{4.8}$$

$$B_{n,k} x - A_{n,k} = C_k c_{n,k} e^{-\tau_{n,k}} (1 + r_{n,k}). \tag{4.9}$$

Every additional logarithm in these two expressions is $o(\Delta)$, uniformly over a fixed finite set of orders. Hence

$$\log B_{n,k} = \tau_{n,k} + o(\Delta), \tag{4.10}$$

$$\log |B_{n,k} x - A_{n,k}| = -\tau_{n,k} + o(\Delta), \tag{4.11}$$

$$\log \left| x - \frac{A_{n,k}}{B_{n,k}} \right| = -2\tau_{n,k} + o(\Delta). \tag{4.12}$$

The individual factors have logarithmic size

quantity	logarithmic size at fixed n
T_k	$\frac{1}{2} L\Delta + o(\Delta)$
$N^{n\Delta}$	$nL\Delta$
$d_{n,k},\ \lvert c_{n,k}\rvert$	$o(\Delta)$
$V_{n,k}(y)$	$o(\Delta)$
$\lvert E_{n,k}(y)\rvert$	$-(2n+1)L\Delta + o(\Delta)$.

The prefix costs $\frac{1}{2}L\Delta$, and the factor z in $F_k = zg_k$ raises the error order to $2n+1$; the two halves balance, since $2(n+\frac{1}{2}) = 2n+1$. Also $|A_{n,k}| \le |x|B_{n,k} + |B_{n,k}x - A_{n,k}|$, so $A_{n,k}$ obeys the same height bound.

The adjacent identity (3.3) gives

$$A_{n+1,k}B_{n,k} - A_{n,k}B_{n+1,k} = C_k^2 N^{\Delta}\Omega_{n,k} \neq 0. \tag{4.13}$$

The two terms containing J_k cancel, leaving

$$\begin{aligned} &T_k^2 N^{(2n+1)\Delta}\big(U_{n+1,k}(y)V_{n,k}(y) - U_{n,k}(y)V_{n+1,k}(y)\big) \\ &\quad = C_k^2 N^{(2n+2)\Delta}\Omega_{n,k}y^{2n+1} = C_k^2 N^{\Delta}\Omega_{n,k}. \end{aligned}$$

Hence, for every $(p,q) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, the two integers

$$B_{n,k}p - qA_{n,k}, \qquad B_{n+1,k}p - qA_{n+1,k} \tag{4.14}$$

cannot both vanish.

5. Proof of Theorem 1

First fix $n = 1$ and let $k \to \infty$. By (4.6) and (4.11), the forms $B_{1,k}x - A_{1,k}$ are eventually nonzero and tend to zero. If $x = a/b$ were rational, with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$, then

$$b(B_{1,k}x - A_{1,k}) = B_{1,k}a - bA_{1,k}$$

would be a nonzero integer tending to zero, a contradiction. This proves irrationality.

For the upper bound we use a fixed finite set of orders. Fix an integer $M \ge 1$ and use primary orders $M \le n \le 2M$, each with companion $n+1$. The whole order set is therefore

$$\mathcal{B} = \{M, M+1, \dots, 2M+1\}. \tag{5.1}$$

Put

$$\alpha_M = 1 + \frac{1}{M+1/2}. \tag{5.2}$$

Order the primary scales $\tau_{n,k}$ by increasing k, and then increasing n. This is their numerical order, because the largest scale in one block is $(2M+\frac{1}{2})L2^k$, and the first in the next is $(2M+1)L2^k$. Within a block, successive ratios are at most

$$\frac{M+3/2}{M+1/2} = \alpha_M.$$

At a block transition the ratio is

$$\frac{2(M+1/2)}{2M+1/2} < \alpha_M.$$

The companion scale also satisfies $\tau_{n+1,k} \le \alpha_M \tau_{n,k}$.

Fix $0 < \eta < 1$. Since the order set $\mathcal{B}$ is finite, (4.10) and (4.11) hold beyond one common threshold. For each sufficiently large primary scale $\tau = \tau_{n,k}$, its two rows $i = n, n+1$ satisfy

$$0 < B_{i,k} \le \exp((\alpha_M + \eta)\tau), \qquad |B_{i,k}x - A_{i,k}| \le \exp(-(1-\eta)\tau). \tag{5.3}$$

The companion has a larger scale, which only improves its error. Since τ is a fixed multiple of Δ for each order in the order set $\mathcal{B}$, the $o(\Delta)$ terms are absorbed once M and η are fixed.

Let q be any sufficiently large positive integer. Choose the first primary scale for which

$$\tau \geq \frac{\log(4q)}{1-\eta}. \tag{5.4}$$

For sufficiently large q it has a predecessor beyond the common threshold. The scale-ratio bound therefore gives

$$\frac{\log(4q)}{1-\eta} \leq \tau \leq \frac{\alpha_M \log(4q)}{1-\eta}. \tag{5.5}$$

Set

$$\beta = \frac{\alpha_M(\alpha_M+\eta)}{1-\eta}. \tag{5.6}$$

For both selected rows, (5.3)–(5.5) imply

$$B_{i,k} \leq (4q)^{\beta}, \qquad q|B_{i,k}x - A_{i,k}| \leq \frac{1}{4}. \tag{5.7}$$

Suppose $p \in \mathbb{Z}$ and

$$\left|x - \frac{p}{q}\right| < 4^{-(\beta+1)} q^{-(1+\beta)}. \tag{5.8}$$

For $i = n, n+1$, the integer $I_{i,k} = B_{i,k}p - qA_{i,k}$ satisfies

$$\begin{aligned} |I_{i,k}| &\leq B_{i,k} q \left|\frac{p}{q} - x\right| + q|B_{i,k}x - A_{i,k}| \\ &< (4q)^{\beta} q\, 4^{-(\beta+1)} q^{-(1+\beta)} + \frac{1}{4} = \frac{1}{2}. \end{aligned} \tag{5.9}$$

Both integers would vanish, contrary to (4.13). Hence the reverse of (5.8) holds for every integer p and every sufficiently large q, whether or not p/q is reduced or equals a selected approximant.

Let $\nu > 2$. Since $\alpha_M \to 1$, choose M such that $1 + \alpha_M^2 < \nu$, and then choose $\eta > 0$ small enough that $1 + \beta < \nu$. With these choices fixed, the proved lower bound

$$\left|x - \frac{p}{q}\right| \geq 4^{-(\beta+1)} q^{-(1+\beta)}$$

is larger than $q^{-\nu}$ for all sufficiently large q. Only finitely many pairs with the remaining denominators can satisfy $|x - p/q| < q^{-\nu}$. Thus $\mu(x) \leq \nu$, and letting ν decrease to 2 gives $\mu(x) \leq 2$.

The choices have the order

$$\big((\lambda_j)_{j\geq 0}, N\big) \;\longrightarrow\; \nu > 2 \;\longrightarrow\; M, \eta, \mathcal{B} \;\longrightarrow\; k_0, q_0 \;\longrightarrow\; (p,q) \;\longrightarrow\; k, n. \tag{5.10}$$

The thresholds k_0, q_0 depend on the sequence and its little-o modulus.

For the reverse bound, fix an integer $R \geq 1$ and partition $[0,1)$ into R intervals of length $1/R$. Two fractional parts among $\{jx\}$, $0 \leq j \leq R$, lie in one interval. Their difference gives integers p, q with

$$1 \leq q \leq R, \qquad |qx - p| < 1/R, \qquad \left|x - \frac{p}{q}\right| < \frac{1}{qR} \leq \frac{1}{q^2}.$$

Since x is irrational, for each fixed D the finitely many distances $\|qx\|$, $1 \leq q \leq D$, have a positive minimum $m_D > 0$, where $\|t\|$ denotes distance from t to the nearest

integer. For $R > 1/m_D$, the inequality $|qx - p| < 1/R$ forces $q > D$. These denominators are unbounded as $R \to \infty$, and $\mu(x) \geq 2$. □

6. Sharpness of the height scale

The little-o condition cannot be replaced by a bound $\log H(\lambda_j) \leq C_* 2^j$ with any constant $C_* > 0$. Below, small coefficients give exponent greater than 2, and the coefficient $\frac{1}{2}L$ gives both a rational and a Liouville sum.

Fix $0 < \theta < 1/3$, and put

$$m_j = \lfloor \theta 2^j \rfloor, \qquad \lambda_j = N^{(-1)^j m_j}, \qquad e_j = 2^j - (-1)^j m_j. \tag{6.1}$$

The weights are positive rationals, with

$$\log H(\lambda_j) = m_j L = \theta L 2^j + O(1). \tag{6.2}$$

The associated sum has the exact digit expansion

$$x = \sum_{j \geq 0} N^{-e_j}.$$

The exponents are strictly increasing: for odd j,

$$e_{j+1} - e_j = 2^j - m_j - m_{j+1} \geq (1 - 3\theta) 2^j > 0,$$

and for even j their difference is $2^j + m_j + m_{j+1} > 0$. The gaps tend to infinity, so the expansion is not eventually periodic and x is irrational.

Let p_j/q_j be the truncation through index j, in lowest terms. Its denominator is exactly $q_j = N^{e_j}$, because its numerator is 1 modulo N and hence is coprime to every prime dividing N. The increasing integer exponents give

$$0 < x - \frac{p_j}{q_j} \leq \frac{N^{-e_{j+1}}}{1 - N^{-1}} \leq 2N^{-e_{j+1}}. \tag{6.3}$$

Along even indices,

$$\frac{e_{j+1}}{e_j} \longrightarrow \frac{2(1+\theta)}{1-\theta} > 2.$$

For every $\nu < 2(1+\theta)/(1-\theta)$, (6.3) is less than $q_j^{-\nu}$ at all sufficiently large even indices. Consequently

$$\mu(x) \geq \frac{2(1+\theta)}{1-\theta} > 2. \tag{6.4}$$

Equivalently, for each $0 < \kappa < L/3$ this gives

$$\log H(\lambda_j) \sim \kappa 2^j, \qquad \mu(x) \geq \frac{2(L+\kappa)}{L-\kappa}.$$

Given any $C_* > 0$, choose $0 < \kappa < \min(C_*, L/3)$. Then the same weights satisfy $\log H(\lambda_j) \leq C_* 2^j$ for all j, but their sum has exponent greater than 2.

A rational sum is possible with positive integer weights. Set

$$\lambda_0 = N - 1, \qquad \lambda_j = N^{2^{j-1}} - 1 \quad (j \geq 1). \tag{6.5}$$

For $j \geq 1$,

$$\lambda_j N^{-2^j} = N^{-2^{j-1}} - N^{-2^j},$$

so the partial sums telescope:

$$\sum_{j=0}^{m} \lambda_j N^{-2^j} = 1 - N^{-2^m}.$$

The infinite sum is therefore 1, whereas

$$\log H(\lambda_j) \sim \frac{1}{2} L\, 2^j. \tag{6.6}$$

The same coefficient also allows a Liouville number. Keep $\lambda_0 = N - 1$, and replace the later weights by

$$\widetilde{\lambda}_j = N^{2^{j-1}} - 1 + \mathbf{1}_{\{j=r^2 \text{ for some } r\geq 1\}} \quad (j \geq 1).$$

These are again positive integers and satisfy (6.6). Their sum is

$$\widetilde{x} = 1 + \xi_N, \qquad \xi_N = \sum_{r\geq 1} N^{-2^{r^2}}.$$

Its digit expansion is not eventually periodic, so ξ_N is irrational. The truncation through $r = R$ has reduced denominator $q_R = N^{2^{R^2}}$, since its numerator is 1 modulo N, and

$$0 < \xi_N - \sum_{r=1}^{R} N^{-2^{r^2}} \leq 2N^{-2^{(R+1)^2}} = 2q_R^{-2^{2R+1}}.$$

The same bound holds after adding 1. Since $2^{2R+1} \to \infty$, this proves $\mu(\widetilde{x}) = \infty$.

The height condition is not necessary for an individual series. For example, given an integer $b \geq 2$, the weights $\lambda_j = b^{-2^j}$ have $\log H(\lambda_j) = (\log b)2^j$, but their sum equals $\sum_{j\geq 0}(Nb)^{-2^j}$. Theorem 1, applied with base Nb and constant weights 1, gives exponent 2 for that sum.

Use of AI tools

The author devised the main argument. Codex (OpenAI, GPT-6 Astra) carried out exact checks of the Hankel identities and searched the literature. Claude Code (Anthropic, Claude Opus 5.5) reviewed the arguments and references. The author takes responsibility for the proof and references.

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