AN AGGREGATE CM-PAIR ESTIMATE AND A 5/3 BOUND FOR VALUATION PRODUCTS OF abc TRIPLES

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ABSTRACT. Let $a+b=c$ be pairwise coprime positive integers. Then, we introduce $R=\mathrm{rad}(abc)$ and $P(abc)=\prod_{p\mid abc} v_p(abc)$. We present here the proof of

$$P(abc),\ \mathbf{d}(abc) \ll_\varepsilon R^{5/3+\varepsilon},$$

where $\mathbf{d}$ is taken to be the divisor function. The first step in the deduction gives an aggregate fixed-radius estimate for CM points on an indefinite Shimura curve. In the case where the CM discriminants lie in $[\Delta, 2\Delta]$, the number of nearby ordered pairs is

$$\#\{(P,Q)\in \mathscr{P}_\Delta^2 : d(P,Q)\le r_0\} \ll |\mathscr{P}_\Delta| + \Delta\left(1+\frac{\Delta^2}{N}\right)N^\varepsilon,$$

which holds uniformly in all polynomial ranges $\Delta \le N^A$. This is established by an elementary local orbit count in the Bruhat–Tits tree, using the characteristic-matrix framework of Bruinier, Li and Möller. The spatial Cauchy–Schwarz then yields a bound for the first absolute moment of the CM-packet. The ordinary AM–GM now applies directly to the logarithmic average in Pasten's Arakelov estimate to complete the argument, and neither a fourth moment nor a pointwise subconvexity estimate is required.

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1. Introduction

Let a, b, c be pairwise coprime positive integers satisfying $a + b = c$. Here

$$R = \operatorname{rad}(abc), \qquad P(abc) = \prod_{p \mid abc} v_p(abc), \qquad \mathbf{d}(n) = \prod_{p \mid n} (v_p(n) + 1).$$

Using a comparison between classical and quaternionic modular degrees, Pasten was able to derive the unconditional bounds $P(abc), \mathbf{d}(abc) \ll_\varepsilon R^{8/3+\varepsilon}$ by evaluating an integral differential at CM points [12, Theorem 1.11 and Theorem 16.8]. Our main result is a reduction of the exponent.

Rickards has established precise formulas regarding the intersections of closed geodesics associated with pairs of real quadratic embeddings. These formulas include x-linking and local Eichler-order information [15, Theorems 10 and 63]. More recently, Bruinier–Li–Möller considered all signatures of a pair of quadratic vectors. More specifically, their formula gives the number of pairs of Heegner points at a given distance, and this number is a product of local factors [3, Theorem 1.1(iv) and Corollary 5.9]. We use Bruinier–Li–Möller's characteristic-matrix and distance formalism, but prove the coarse orbit bound needed here directly from the local structure of quaternion orders and the Bruhat–Tits tree [16, Equation (23.2.4), Theorem 13.3.11, and Sections 23.5.11–23.5.16]. The aggregate estimate provides us with the divisor-sum bound we need for Pasten's argument.

In the context of the theory of special cycles, Heegner points, and quaternionic Shimura varieties, the natural objects in that framework are CM points [8, 20, 19, 18]. Complete or structured packets satisfy the equidistribution, variance, and small-scale bounds [10, 5, 7, 9, 1, 6, 17]. None of these works states our fixed-radius bound for arbitrary selected submultisets. Our approach is to substitute the classical sup-norm techniques [2, 4] with a local L^1 inequality. Later abc work treats different functionals or additional size hypotheses [13].

Theorem 1.1. *For every $\varepsilon > 0$ and for every pairwise coprime triple of positive integers $a + b = c$, we have*

$$P(abc),\ \mathbf{d}(abc) \ll_\varepsilon \operatorname{rad}(abc)^{5/3+\varepsilon}.$$

Geometrically speaking, let R_N stand for an Eichler order of reduced discriminant N in an indefinite quaternion division algebra, putting $X_N = R_N^1 \backslash \mathfrak{H}$. We assume that the odd part of N is square-free, and that $v_2(N) \leq 8$. For odd fundamental discriminants, $-d$ being coprime to N, and $d \in [\Delta, 2\Delta]$, choose $\mathscr{P}_\Delta$ as any multiset of the related CM centers where the multiplicity of each one does not exceed a fixed $B_0 \geq 1$.

Theorem 1.2 (Aggregate CM-pair estimate)**.** *Fix $A, r_0 > 0$ and $B_0 \geq 1$. If $\Delta \leq N^A$ and every multiplicity in $\mathscr{P}_\Delta$ is at most B_0, then*

$$\#\{(P, Q) \in \mathscr{P}_\Delta^2 : d(P, Q) \leq r_0\} \ll_{A, r_0, B_0, \varepsilon} |\mathscr{P}_\Delta| + \Delta \left(1 + \frac{\Delta^2}{N}\right) N^\varepsilon.$$

The estimate is uniform both in the chosen discriminant submultiset and in the admissible factorization $N = DM$.

For pairs lying on the off-diagonal, we will set $-d, -e$ to be the discriminants and b to be the integral pairing parameter. The local conditions state that the odd part of N has to divide $b^2 - de$. A direct local orbit count gives an upper bound of N^{ε} for each global orbit count. After we have fixed the radius we will be down to $O_{r_0}(\Delta(1 + \Delta^2/N))$ choices for b and the quotient. These choices fix de, and de can have at most N^{ε} admissible factorizations. Further, the thinning of the packets considered here will only eliminate pairs.

In the case of $\mathscr{P}$, that is, the corresponding full Galois CM multiset from Pasten's construction, we use $x = N^{2/3+\delta}$ and $\vartheta = \delta/5$. Then, using theorem 1.2, the local holomorphic L^1 inequality, and the spatial form of the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{H}\sum_{d}\frac{1}{2h_d}\sum_{P\in\mathscr{C}_d}|F_N(P)| \ll N^{\tau+\varepsilon}, \qquad \tau = \left(\frac{2}{3}+\delta\right)\frac{6\delta}{5}.$$

Here $\|F_N\|_{L^2(\mu_{\mathrm{prob}})} = N^{o(1)}$. Applying zero-counting and Markov's inequality at this point, we obtain a full packet that is free of zeros, on which the first absolute moment is $\ll N^{\tau+\varepsilon}$. We write $u_\sigma = |f(\tau_\sigma)|\,\mathrm{Im}\,\tau_\sigma > 0$, and for these we need only AM–GM:

$$\frac{1}{2h_d}\sum_{\sigma}\log u_\sigma \le \log\left(\frac{1}{2h_d}\sum_{\sigma}u_\sigma\right).$$

Using these first moment and AM–GM bounds along with Pasten's Proposition 14.2, height formula, and modular-degree comparison gives the cell exponent $q_\delta = 5/3 + 13\delta/5 + 12\delta^2/5$. In place of Pasten's norm exponent value of $5/6$, we can put the value $1/3 + o(1)$ that our aggregate first-moment estimate gives us. With that replacement, $1 + 2(5/6) = 8/3$ changes to $1 + 2(1/3) = 5/3$. Finally, the cyclic pairs and triples will distribute the level factors that are left, and we let $\delta \to 0$ to obtain the proof of theorem 1.1.

2. Quaternionic CM packets

Let B represent an indefinite quaternion division algebra over $\mathbb{Q}$, and let R_N denote an Eichler order of reduced discriminant $N = DM$, where D is the discriminant of B and M the Eichler level. As in Pasten, a factorization $N = DM$ is called admissible when $(D, M) = 1$ and D is the product of an even number of distinct primes; in the compact case considered below, $D > 1$. We assume that the odd part of N is square-free and that $v_2(N) \le 8$. Write

$$\Gamma_N = R_N^1/\{\pm 1\}, \qquad X_N = \Gamma_N\backslash\mathfrak{H}.$$

In this notation we write $d\mu_{\mathrm{phys}} = dx\,dy/y^2$ for the hyperbolic measure in the physical sense, we let μ_{prob} be the probability measure, and we set

$V_N = \mathrm{vol}(X_N)$. Using the exact volume formula from [3, Equation (2.7)], we get, for every $\varepsilon > 0$,

$$N^{1-\varepsilon} \ll_\varepsilon V_N \ll_\varepsilon N^{1+\varepsilon}.$$

As for the Gross lattice, we can write it out in the following form

$$L_N = \{2a - \mathrm{trd}(a) : a \in R_N\} \subset B^0.$$

In what follows we adopt the source convention $Q = -\,\mathrm{nrd}$ on the trace-zero space, with its full polarization

$$B_Q(x, y) = Q(x + y) - Q(x) - Q(y)$$

on the same space, so that $B_Q(x, y) = \mathrm{trd}(xy)$. We note that the primitive vectors representing imaginary quadratic discriminant $-d$ satisfy $Q(x) = -d < 0$. Taking lifts and signs for the associated negative lines, we read off the relation

$$\frac{|B_Q(x, y)|}{2\sqrt{Q(x)Q(y)}} = \cosh d(x, y).$$

In fact, if we divide x and y by the square roots of $-Q(x)$ and $-Q(y)$, we recognize the standard distance equation for the negative sheet of the hyperboloid model, the 2 being the contribution of the full polarization.

Lemma 2.1 (Lattice and characteristic-matrix normalization)**.** *The lattice L_N is exactly the index-four lattice L that we find in* [3, Equation (2.9)]. *An optimal embedding of the imaginary quadratic maximal order of odd fundamental discriminant $-d$ determines, up to sign, a primitive $x \in L_N$ with $Q(x) = -d$, and the converse holds as well. For a pair x, y, the characteristic matrix in the notation of Bruinier–Li–Möller is*

$$T = \frac{1}{2}\big(B_Q(x_i, x_j)\big)_{i,j} = \begin{pmatrix} -d & b \\ b & -e \end{pmatrix}, \qquad b = \frac{1}{2}B_Q(x, y).$$

Proof. If $a \in R_N$, we can check that either $2a - \mathrm{trd}(a)$ lies in $2(R_N \cap B^0)$ or $(1 + 2a - \mathrm{trd}(a))/2 \in R_N$, according to the parity of $\mathrm{trd}(a)$. In both cases the converse holds as well, and we obtain the equality with the lattice in [3, Equation (2.9)]. Let us assume now that $x \in L_N$ and $Q(x) = -d$, with $-d$ being an odd fundamental discriminant. Then x cannot belong to $2(R_N \cap B^0)$, therefore $\alpha = (1 + x)/2 \in R_N$. Since

$$\alpha^2 - \alpha + \frac{d + 1}{4} = 0,$$

α defines an embedding of the maximal order $\mathbb{Z}[(1 + \sqrt{-d})/2]$ into R_N, and this embedding is optimal since the source order is maximal. Conversely, an optimal embedding φ sends $\sqrt{-d}$ to the vector $x = \varphi(\sqrt{-d}) \in L_N$ with $Q(x) = x^2 = -d$. This vector is primitive. Indeed, $Q(L_N) \subset \mathbb{Z}$ because $Q(2a - \mathrm{trd}(a)) = \mathrm{trd}(a)^2 - 4\,\mathrm{nrd}(a)$ for $a \in R_N$, and hence a factorization $x = kx_0$ in L_N, with $k > 1$, would force $k^2 \mid d$, contrary to the square-freeness of d. This proves the stated correspondence, and it also proves the normalization in the characteristic matrix $\frac{1}{2}(B_Q(x_i, x_j))$. In addition, let us

state the values $x = 2\alpha - 1$ and $y = 2\beta - 1$, where $\alpha, \beta \in R_N$ both have the reduced trace value of one. Then

$$B_Q(x, y) = 4\,\mathrm{trd}(\alpha\beta) - 2, \qquad b = 2\,\mathrm{trd}(\alpha\beta) - 1 \in \mathbb{Z},$$

so b is odd. This direct calculation fixes the normalization that we will use below, and we can check that its off-diagonal entry agrees with [3, Equation (2.14)]. □

In what follows, we will take CM discriminants to be coprime to N and to lie in $[\Delta, 2\Delta]$.

Remark 2.2. In the following, we will write h_d for the class number of $\mathbb{Q}(\sqrt{-d})$, while $\mathscr{C}_d$ will denote the complete $2h_d$-element Galois multiset of Pasten, and not a selected orientation packet. Moreover, we can use the fact that the CM point of Pasten has a residue field equal to the Hilbert class field [12, Section 13.1], which ensures that these $2h_d$ conjugates are distinct quotient points. Each of the elements is a CM center of discriminant $-d$, and hence we can see that it has the primitive-vector realization in lemma 2.1. Since the pair estimate is uniform for submultisets, we do not need to identify this Galois multiset with one prescribed system of local orientations. As for standard optimal-embedding counts with a fixed local class, we can find such counts stated in [11, Theorem 2.5].

Remark 2.3. For nonsingular T, we can recall that Bruinier–Li–Möller identify R_T with the diagonal norm-one orbit set of ordered pairs in L_N^2 that have characteristic matrix T. See their definition of R_T together with [3, Equation (2.9)]. The paragraph preceding their Theorem 5.7 explicitly states that the lattice in use for their subsequent local computations is reset to the localization of this same index-four lattice. Their count allows for the inclusion of imprimitive pairs while the CM centers are derived from primitive vectors, so we can see that the latter classes form a subset. The bound for the entire orbit set below is obtained from a direct local coset count and does not use an exact local-density formula. Indeed, a CM center determines its negative rational line, on which a primitive vector of norm $-d$ is unique up to sign, since any two such vectors differ by a rational scalar with square equal to one. In moving to the quotient, at most an elliptic stabilizer can be added, and by the integral-trace argument in the proof of lemma 4.1, the projective order of that stabilizer is bounded by 3. Thus the fibers of the map from primitive-vector classes to CM centers are uniformly bounded, and the bounded multiplicities in $\mathscr{P}_\Delta$ will be absorbed in the implied constant.

3. The aggregate CM-pair estimate

We prove theorem 1.2.

Lemma 3.1 (Primitive local-factor bound)**.** *Let*

$$T = \begin{pmatrix} -d & b \\ b & -e \end{pmatrix}, \qquad N^\flat = N/2^{v_2(N)},$$

where $-d, -e$ are odd fundamental discriminants, $(de, N) = 1$, b is odd, and T is nonsingular. Assume that the odd part of N is square-free and $v_2(N) \le 8$. We let $R_T(D, M)$ denote the complete set of all norm-one equivalence classes with characteristic matrix T, for the order of discriminant D and level M. When the order is clear we write R_T for that set. If R_T is nonempty, then

$$N^\flat \mid b^2 - de. \tag{1}$$

Moreover, uniformly for $d, e \asymp \Delta$, $|b| \ll_{r_0} \Delta$, and $\Delta \le N^A$,

$$|R_T| \ll_{A, r_0, \varepsilon} N^\varepsilon. \tag{2}$$

Proof. Write $M = 2^k M_0$ with M_0 odd. For $k > 0$, let R_{DM_0} be the Eichler overorder of R_N of level M_0; admissibility gives $2 \nmid D$. Then

$$L_N \subset L_{DM_0}, \qquad \Gamma_N \subset \Gamma_{DM_0}.$$

Thus every Γ_{DM_0}-orbit splits into at most $[\Gamma_{DM_0} : \Gamma_N]$ many orbits of Γ_N. The local Eichler index, equivalently the quotient of the exact volume formulas in [3, Equation (2.7)], is

$$|R_T(D, M)| \le [\Gamma_{DM_0} : \Gamma_N] |R_T(D, M_0)|, \qquad [\Gamma_{DM_0} : \Gamma_N] = 3 \cdot 2^{k-1} \le 384. \tag{3}$$

It remains to treat the odd-level order. Put $N_0 = DM_0$, $R = R_{N_0}$, $L = L(R)$, and $\Gamma = R^1/\{\pm 1\}$ for the rest of this proof. Notice that the odd part of N_0 is $N^\flat$.

Put $G = \mathrm{SO}(B^0) \simeq B^\times/\mathbb{Q}^\times$. Fix a rational pair $X_0 = (x_0, y_0)$ with Gram matrix T, and, for each finite prime p, write

$$\begin{aligned} \mathcal{K}_p &= \mathrm{im}(R_p^\times \longrightarrow G(\mathbb{Q}_p)), \\ S_p(T) &= \{g\mathcal{K}_p : g^{-1} X_0 \in L_p^2\}, \\ c_p(T) &= \# S_p(T). \end{aligned}$$

The action of $G(F)$ on ordered pairs of Gram matrix T is simply transitive for $F = \mathbb{Q}$ and $F = \mathbb{Q}_p$. Indeed, an isometry between the two nondegenerate planes extends by Witt's theorem, its determinant can be corrected on the orthogonal line, and an element of $G(F)$ fixing the pair also fixes that line because it has determinant one. The orbit map is a homeomorphism: its inverse is obtained by completing (x, y) with their cross-product vector, and its only denominator is a fixed nonzero multiple of $\det T$. Hence the inverse image of the closed integral Gram fiber in the compact set L_p^2 is compact. Since $\mathcal{K}_p$ is open, every $S_p(T)$ is finite.

We next record the global-to-local injection. If $\mathcal{K} = \prod_p \mathcal{K}_p$, then

$$G(\mathbb{Q}) \cap \mathcal{K} = \mathrm{im}(R^\times). \tag{4}$$

To see this, represent a class in the intersection by $q \in B^\times(\mathbb{Q})$ and write $q = \lambda_p u_p$ locally, with $\lambda_p \in \mathbb{Q}_p^\times$ and $u_p \in R_p^\times$. Since $v_p(\operatorname{nrd} q) = 2v_p(\lambda_p)$, multiplication by the rational number having valuation $-v_p(\lambda_p)$ at every p gives an element, and its inverse, in every R_p. The identity $R = B(\mathbb{Q}) \cap \prod_p R_p$ proves (4). Simple transitivity now maps full-unit global orbits injectively into the restricted product $\prod_p S_p(T)$. Passing from full units to norm-one units costs an index at most two, and hence

$$|R_T(D, M_0)| \le 2 \prod_p c_p(T). \tag{5}$$

We count the local cosets directly. At an odd prime, $L_p = R_p^0$. If B_p is split and R_p is maximal, then $G(\mathbb{Q}_p)/\mathcal{K}_p$ is the vertex set of the Bruhat–Tits tree. An integral change in the two pair coordinates puts the first elementary divisor first. If it is a unit, choose the first vector x with $Q(x) = \epsilon \in \mathbb{Z}_p^\times$. When ϵ is nonsquare, $\mathbb{Z}_p[x]$ is the maximal unramified quadratic order and its stable lattices give at most one homothety class. When $\epsilon = s^2$, the integral idempotents put

$$x = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & r \\ u & 0 \end{pmatrix}.$$

The x-stable lattices are $\Lambda_n = \mathbb{Z}_p e_1 \oplus p^n \mathbb{Z}_p e_2$, and y stabilizes Λ_n exactly for

$$-v_p(r) \le n \le v_p(u).$$

As $v_p(ru) = t := v_p(\det T)$, this gives at most $t + 1$ vertices. If the first elementary divisor is p, square-freeness of d, e allows us to choose $Q(x) = p\epsilon$. The order $\mathbb{Z}_p[x]$ is then the maximal order of a ramified quadratic field. Its fractional ideals, modulo central homothety, have two parity classes, so $c_p(T) \le 2$; here $t \ge 2$.

If p is an odd division prime, the unique maximal order and its Gross lattice are invariant under conjugation, while

$$B_p^\times / (\mathbb{Q}_p^\times R_p^\times) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Thus a nonempty support has $c_p(T) = 2$. Reduction of R_p^0 modulo the radical is one-dimensional, so such a represented pair has $p \mid \det T$. Finally, when $p \mid M_0$, the group $\mathcal{K}_p$ fixes an oriented edge. Since $(de, N) = 1$, the maximal-order support is either one vertex or an interval of $t + 1$ vertices as above. The former gives no edge, and the latter gives $2t$ oriented edges. Hence

$$c_p(T) \le \begin{cases} t+1, & p \nmid DM_0, \text{ unit first divisor}, \\ 2, & p \nmid DM_0, \text{ first divisor } p, \\ 2, & p \mid D, \\ 2t, & p \mid M_0. \end{cases} \tag{6}$$

These tree and order identifications are standard; see [16, Equation (23.2.4), Theorem 13.3.11, Lemma 23.5.15, and Sections 23.5.11–23.5.16].

It remains to count at 2. Since $-d, -e$ are odd fundamental discriminants and b is odd, $d \equiv e \equiv 3 \pmod 4$ and $t = v_2(\det T) \geq 2$. An integral shear gives

$$T \sim \operatorname{diag}(\epsilon_1, 2^t\epsilon_2), \qquad \epsilon_1 \equiv 1 \text{ or } 5 \pmod 8.$$

In the split algebra the Gross lattice of the standard maximal order is

$$L_2 = \left\{ \begin{pmatrix} z & 2v \\ 2u & -z \end{pmatrix} : z, u, v \in \mathbb{Z}_2 \right\}, \qquad Q = z^2 + 4uv.$$

For a unit-norm first vector x, membership in the Gross lattice of a maximal order O' is equivalent to $(1+x)/2 \in O'$. Indeed, writing $x = 2a - \operatorname{trd}(a)$, the oddness of $Q(x) = \operatorname{trd}(a)^2 - 4\operatorname{nrd}(a)$ forces $\operatorname{trd}(a)$ to be odd; the converse follows from trace one. If $\epsilon_1 \equiv 5 \pmod 8$, this element generates the unramified maximal quadratic order, whose stable lattices form one homothety class. If $\epsilon_1 \equiv 1 \pmod 8$, scale by a unit and take $x = \operatorname{diag}(1, -1)$. The integral idempotent $(1+x)/2$ forces every stable lattice to split along the two eigenlines. Writing $y = \left(\begin{smallmatrix} 0 & r \\ u & 0 \end{smallmatrix}\right)$, the initial Gross integrality gives $r, u \in 2\mathbb{Z}_2$, while $Q(y) = ru = 2^t\epsilon_2$. Gross integrality at Λ_n is equivalent to

$$1 - v_2(r) \leq n \leq v_2(u) - 1,$$

which gives at most $t-1$ vertices. In the dyadic division algebra the unique maximal order is again invariant under conjugation and the projective unit quotient has order two. Consequently

$$\begin{aligned} &c_2(T) \leq 1 && (\epsilon_1 \equiv 5 \bmod 8), \\ &c_2(T) \leq t-1 && (\epsilon_1 \equiv 1 \bmod 8), \\ &c_2(T) = 2 && (2 \mid D). \end{aligned} \tag{7}$$

Every nontrivial local factor is therefore attached to a prime divisor of $\det T$ and, in every branch of (6)–(7),

$$c_p(T) \leq \left(v_p(\det T) + 1\right)^2.$$

At an odd prime dividing DM_0, the same rank-one reductions used above also give $p \mid \det T$, proving $N^\flat \mid b^2 - de$. Thus

$$|R_T(D, M_0)| \leq 2 \prod_{p \mid \det T} \left(v_p(\det T) + 1\right)^2 = 2\mathbf{d}(|\det T|)^2.$$

Together with (3), the bounds $|\det T| \ll_{r_0} \Delta^2$ and $\Delta \leq N^A$, and the standard divisor bound, this proves (2), uniformly in the admissible factorization. □

Proof of theorem 1.2. For each ordered pair of quotient centers, we will need to choose lifts that realize the respective quotient distance after shifting one of the two by a member of Γ_N. In turn, we see that the corresponding simultaneous Γ_N-class of primitive vectors is counted by the appropriate R_T. By lemma 2.1, we see that its characteristic matrix is

$$T = \begin{pmatrix} -d & b \\ b & -e \end{pmatrix}.$$

We need to select signs for the two primitive vectors independently in such a way that $b > 0$. Then, from the hyperboloid calculation preceding lemma 2.1, we can read off $\cosh d(P,Q) = b/\sqrt{de}$. Also, we can switch either sign, replacing b by $-b$, but the CM center remains unchanged. Then we will find that the sign-invariant form is

$$\cosh d(P,Q) = \frac{|b|}{\sqrt{de}}. \tag{8}$$

We let R_T denote the complete set of norm-one equivalence classes with characteristic matrix T, and we write $N^\flat = N/2^{v_2(N)}$. We will treat the singular and the nonsingular cases separately.

If T is singular, then we can see that (8) gives distance zero. Since d, e are both fundamental, we see that $b^2 = de$ also gives $d = e$. Hence we can conclude that the corresponding quotient centers agree. With $m(P)$ the multiplicity of a center, we can read off from the bounded multiplicity that $\sum_P m(P)^2 \leq B_0 \sum_P m(P) = B_0 |\mathscr{P}_\Delta|$. Together with the bounded fibres in remark 2.3, we see that this bound provides a contribution given by $O_{B_0}(|\mathscr{P}_\Delta|)$. For a nonsingular pair we will need lemma 3.1. If the pair is at distance at most r_0, then we see that (8) gives $O_{r_0}(\Delta)$ possibilities for b. We can now put

$$m = \frac{b^2 - de}{N^\flat}.$$

The form $Q = -\operatorname{nrd}$ has signature $(2,1)$. The reverse Cauchy–Schwarz inequality on its negative cone gives

$$B_Q(x,y)^2 \geq 4Q(x)Q(y) = 4de,$$

with equality only when x and y are proportional. Since $b = B_Q(x,y)/2$ and T is nonsingular, the inequality is strict here; therefore $b^2 - de > 0$. Together with (1), this shows that m is a positive integer. Since the standing hypothesis gives $N/N^\flat \leq 2^8$, Equations (8) and (1) give

$$1 \leq m \ll_{r_0} 1 + \frac{\Delta^2}{N}.$$

For fixed b and m, we see that the product $de = b^2 - N^\flat m$ is fixed as well. Then we can bound the number of ordered factor pairs $d, e \in [\Delta, 2\Delta]$ by its divisor function value. That divisor function value is N^ε throughout the polynomial range. The conditions on fundamental discriminants and coprimality thin this set of pairs, but as we can see they add nothing. The summation of (2) over b and m therefore leads us to

$$\ll_{A,r_0,\varepsilon} \Delta \left(1 + \frac{\Delta^2}{N}\right) N^\varepsilon$$

off the diagonal. Restricting to submultisets removes pairs and does nothing else, apart from the bounded multiplicities that were already controlled. This proves the theorem. □

4. A first absolute CM-packet moment

We now place theorem 1.2 in Pasten's family. For an abc triple, let N be the conductor of the Frey curve attached to that triple, and choose an admissible factorization $N = DM$. Let $0 < \delta < 1/10$ be a parameter to be fixed at the end. For this Frey curve, Pasten records

$$N = 2^r \prod_{\substack{p \mid abc \\ p>2}} p, \qquad 0 \le r \le 8;$$

at every odd prime $p \mid abc$ one has $v_p(N) = 1$ [12, Section 3 and proof of Theorem 16.8]. Thus the odd part of N is square-free and the hypotheses of section 2 hold for every admissible factorization used below. Pasten's sieve [12, Theorem 11.1 and Corollary 11.2], along with

$$x = N^{2/3+\delta}, \qquad \vartheta = \delta/5, \tag{9}$$

produces H odd fundamental discriminants $-d$, with $x < d \le 2x$, in which all the primes dividing D are inert while all the primes dividing M are split in $\mathbb{Q}(\sqrt{-d})$, and

$$H \gg x^{1-\delta}, \qquad d^{1/2-\vartheta} < h_d \ll_\varepsilon x^{1/2+\varepsilon}, \qquad \frac{L'}{L}(1,\chi_d) \ll \log\log d. \tag{10}$$

In particular, all these discriminants are coprime to N. In this situation we can apply the corollary that we cited, because $x = N^{2/3+\delta} > N^{1/2+\delta}$, and we can note that its constants are uniform in the admissible factorization $N = DM$. Let $\mathscr{C}_d$ be the full $2h_d$-element Galois multiset in remark 2.2, and set

$$\mathscr{P} = \bigsqcup_d \mathscr{C}_d, \qquad P = |\mathscr{P}|.$$

Then $P \ll Hx^{1/2+\varepsilon}$. Put $C(\mathscr{P}) := \#\{(P,Q) \in \mathscr{P}^2 : d(P,Q) \le 2r_1\}$, where r_1 is the fixed radius in lemma 4.1. Theorem 1.2 gives

$$C(\mathscr{P}) \ll N^\varepsilon \left(P + x + \frac{x^3}{N} \right). \tag{11}$$

Lemma 4.1 (Uniform local L^1 evaluation)**.** *Let $f(z)\,dz$ be a holomorphic differential on X_N, and we put*

$$F_N(z) = \frac{\sqrt{N}}{\|f\|_{\mathrm{Pas}}}\, y f(z), \qquad \|f\|_{\mathrm{Pas}}^2 = \int_{X_N} |y f(z)|^2 \, d\mu_{\mathrm{phys}}(z).$$

We regard F_N as a unitary automorphic section. Its pointwise norm $|F_N|$ is a well defined function on X_N, and we note that every evaluation and every integral below involving F_N refers either to this norm or to its powers. For a hyperbolic radius r_1 that is fixed and sufficiently small, we have

$$|F_N(P)| \ll V_N \int_{B(P,r_1)} |F_N(z)| \, d\mu_{\mathrm{prob}}(z),$$

uniformly for the levels in section 2. Moreover we have $\|F_N\|^2_{L^2(\mu_{\mathrm{prob}})} = N/V_N$.

Proof. Fix a lift $\widetilde{P} = x_0 + iy_0$. Using the holomorphic submean inequality on a Euclidean disk of radius ρy_0, we read off, on a comparable hyperbolic disk,

$$|y_0 f(\widetilde{P})| \ll \int_{B(\widetilde{P},r_1)} |yf(z)|\, d\mu_{\mathrm{phys}}(z),$$

since $y \asymp y_0$ and $d\mu_{\mathrm{phys}} = dx\, dy/y^2$ hold there.

Next we need to make the descent to the orbifold uniform. The groups Γ_N are cocompact, so we conclude that they contain no parabolic elements. Also, a nonelliptic element has an integral reduced trace whose absolute value is at least 3, and so we conclude that its translation length is at least $\ell_0 = 2\operatorname{arcosh}(3/2)$. We let μ_2 denote a Margulis constant for the hyperbolic plane, and choose $r_1 < \min\{\mu_2/2, \ell_0/4\}$. Applying the two-dimensional Margulis lemma [14] and then the classification of elementary Fuchsian groups, we conclude that the subgroup generated by

$$\{\gamma \in \Gamma_N : d(\widetilde{P}, \gamma\widetilde{P}) < 2r_1\}$$

is finite. In fact, we note that an infinite elementary subgroup would either contain a parabolic element or preserve a geodesic. In the latter case, for the subgroup generated by the displayed set to have infinitely many elements, we need either a hyperbolic generator or two distinct half-turn generators. As for the hyperbolic generator, it would contradict the length gap, whereas we can also observe here that the product of the two distinct half-turn generators has a translation length shorter than $4r_1 < \ell_0$. Thus the small-displacement subgroup is finite. A finite subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is cyclic. Also, any nontrivial torsion lift in this situation has its integral reduced trace lying in $\{-1, 0, 1\}$, so its projective order is no greater than 3. As a consequence, we can say that the projection of $B(\widetilde{P}, r_1)$ to X_N has a uniformly bounded multiplicity, because an overlap of $B(\widetilde{P}, r_1)$ with its γ-translate forces the inequality $d(\widetilde{P}, \gamma\widetilde{P}) < 2r_1$. Thus, when we descend the submean inequality that precedes this one, we can see that the effect is that of changing just an absolute constant. Finally, we can take $d\mu_{\mathrm{phys}} = V_N d\mu_{\mathrm{prob}}$ together with the definition of F_N, and in this way we read off the estimate that was asserted. The L^2 identity is immediate from the normalizations. □

Define

$$T_d(F_N) = \frac{1}{2h_d} \sum_{P \in \mathscr{C}_d} |F_N(P)|$$

and the sampling multiplicity $\mu(z) = \sum_{P \in \mathscr{P}} \mathbf{1}_{B(P,r_1)}(z)$. Since any two intersecting balls have their centers at a bounded distance from one another, we can see that

$$\int \mu^2\, d\mu_{\mathrm{prob}} \ll \frac{C(\mathscr{P})}{V_N}.$$

Indeed, after we have expanded the square, we will find that the pairs of centers that contribute are just those whose distance apart is at most $2r_1$, and we can also see that each ball intersection has a probability measure that is equal to $O(1/V_N)$. If we now sum lemma 4.1, and then make use of the lower bound for h_d, and after that apply Cauchy–Schwarz in space, what we read off is the exact volume cancellation, and it comes out in the form

$$\begin{aligned}\frac{1}{H}\sum_d T_d(F_N) &\ll \frac{V_N}{Hx^{1/2-\vartheta}}\int \mu|F_N|\,d\mu_{\mathrm{prob}}\\ &\ll \frac{\sqrt{NC(\mathscr{P})}}{Hx^{1/2-\vartheta}}.\end{aligned} \tag{12}$$

If we now insert (11) together with the bound $P \ll Hx^{1/2+\varepsilon}$, what we are left with is three terms:

$$E_{\mathrm{diag}} = N^{1/2}H^{-1/2}x^{-1/4+\vartheta+\varepsilon}, \quad E_x = N^{1/2}H^{-1}x^{\vartheta+\varepsilon}, \quad E_{\mathrm{off}} = H^{-1}x^{1+\vartheta+\varepsilon}.$$

Under (9)–(10), we can compute that the N-exponents of these three are

$$-\frac{17}{60}\delta + \frac{7}{10}\delta^2, \qquad -\frac{1}{6} - \frac{1}{5}\delta + \frac{6}{5}\delta^2, \qquad \tau = \left(\frac{2}{3}+\delta\right)\frac{6\delta}{5}.$$

It is the last of these three terms that is the limiting one here, and in this way we obtain the following proposition.

Proposition 4.2. *In the notation that we have set up in the preceding discussion, uniformly in the admissible factorization $N = DM$,*

$$\frac{1}{H}\sum_d T_d(F_N) \ll N^{\tau+\varepsilon}, \qquad \tau = \left(\frac{2}{3}+\delta\right)\frac{6\delta}{5}.$$

5. The arithmetic conclusion

First we will need to choose a suitable packet on which logarithms can be taken. Using [12, Lemma 14.3], we can see that if the corresponding rational differential vanishes at one of the CM conjugate points, then it vanishes on the entire Galois orbit as well. Also, we can note that, by the same lemma, the associated modular form has at most $N^{1+\eta}$ zeros on $\mathfrak{H}$, under Γ_N-equivalence, for each fixed $\eta > 0$. Thus, for $\eta = \delta/5$, we can estimate the number B of bad discriminants as

$$B \ll N^{1+\delta/5} x^{-1/2+\delta/5}, \qquad \frac{B}{H} \ll N^{-\delta/2+6\delta^2/5} = o(1).$$

Since the negative line of a CM center determines the stabilizing torus of that CM center, which also determines the imaginary quadratic field, we see that all of the CM orbits corresponding to different quadratic fields are pairwise disjoint. Using Markov's inequality together with the estimate from proposition 4.2, we can get a positive proportion of quadratic fields with $T_d(F_N) \ll N^{\tau+\varepsilon}$. Hence we can be sure that one of them is also zero-free. We now fix such a field K. In fact, this field belongs to the original sieve, and we can also note that it keeps the logarithmic-derivative bound that was recorded in (10).

In what follows we will let $f(z)\,dz = f_{D,M}(z)\,dz$ be the integral quaternionic differential in Pasten's construction, that is to say, the differential which comes from the modular parametrization in the form in which he records it in his Equation (16.3). For all of the conjugates P_σ, and with a lift $\tau_\sigma \in \mathfrak{H}$ chosen for each one of them, we will write $u_\sigma = |f(\tau_\sigma)| \operatorname{Im} \tau_\sigma > 0$. Then, from the normalization that was recorded in lemma 4.1, we can read off $u_\sigma = \|f\|_{\mathrm{Pas}} |F_N(P_\sigma)|/\sqrt{N}$. Now, using ordinary AM–GM together with the first-moment estimate that was selected above, we can then go on to obtain

$$\frac{1}{2h_K} \sum_\sigma \log u_\sigma \leq \log \|f\|_{\mathrm{Pas}} - \frac{1}{2} \log N + (\tau + \varepsilon) \log N + O(1). \tag{13}$$

Also, Proposition 14.2 in Pasten is given with respect to exactly this hyperbolic metric, and therefore we can see that it implies

$$h_{\mathrm{Ar}}(P_K) \geq -\log(2M) - \frac{1}{2h_K} \sum_\sigma \log u_\sigma.$$

Now, using his height formula [12, Theorem 13.2] in the form in which it is recorded there, after the functional equation has been applied, we can go on to obtain

$$h_{\mathrm{Ar}}(P_K) = \frac{L'}{L}(1, \chi_K) + \frac{1}{2} \log(dD) + O(1).$$

Then, combining these formulas, and using $N = DM$ together with $d \leq 2N^{2/3+\delta}$, and together with the logarithmic-derivative estimate that was

retained above, we can now go on to deduce that

$$-\log\|f\|_{\mathrm{Pas}} \le \frac{1}{2}\log M + \left(\frac{1}{3} + \frac{\delta}{2} + \tau + \varepsilon\right)\log N. \tag{14}$$

In preparation for the final step of the argument, we will now single out the one consequence of Pasten's modular-degree comparison that is used there.

Proposition 5.1 (Pasten interface)**.** *For the Frey branch and for an admissible factorization $N = DM$ which has the property that M is divisible by at least two odd primes, we have the following estimate*

$$\prod_{p\mid D} v_p(\Delta_E) \ll_\varepsilon N^{1+\varepsilon}\|f\|_{\mathrm{Pas}}^{-2},$$

uniformly in the admissible factorization.

Proof. Now, from Pasten's Theorem 6.1(b.2), what we can read off is

$$\log\prod_{p\mid D} v_p(\Delta_E) \le \log\delta_{1,N} - \log\delta_{D,M} + O(\omega(D)).$$

Using the degree relations (16.1)–(16.3), which are combined in [12, Theorem 16.1, pp. 305–307] and invoked in his Theorem 16.5(ii), we can then go on to read off the estimate

$$\log\prod_{p\mid D} v_p(\Delta_E) \le 2\log|c_f| + 2\log\|f_{1,N}\|_2 - 2\log\|f_{D,M}\|_2 + O(\log\log N + \omega(D)).$$

Now the minimal isogeny has degree at most 163 by an application of the Mazur–Kenku theorem, in the form in which it is recorded in [12, proof of Theorem 16.1]. Thus the difference between the two terms which are associated with the Faltings height is $O(1)$. In the same normalization, the classical estimate of $\|f_{1,N}\|_2^2 \ll N\log N$, the Manin constant estimate that is due to Pasten [12, Theorem 1.1 and proof of Theorem 16.1], and the bound $\omega(D) \ll \log N/\log\log N$ together imply that for sufficiently large N the right-hand side is at most

$$(1+\varepsilon)\log N - 2\log\|f_{D,M}\|_2.$$

Finally, the non-normalized Petersson norms of Pasten take into account the same physical hyperbolic measure [12, Section 4.6], and therefore we can conclude that $\|f_{D,M}\|_2 = \|f\|_{\mathrm{Pas}}$. All constants just used are absolute or depend only on ε (the Manin-constant theorem is applied with the fixed exceptional set $\{2\}$), so the estimate is uniform in (D, M). This proves the statement while additionally showing explicitly that the Frey branch contributes no further factor of D. □

For every factorization satisfying the hypothesis of proposition 5.1, applying it to (14) gives us the cell estimate

$$\prod_{p|D} v_p(\Delta_E) \ll_\varepsilon N^{q_\delta+\varepsilon} M, \qquad q_\delta = 1+\left(\frac{2}{3}+\delta\right)+2\tau = \frac{5}{3}+\frac{13}{5}\delta+\frac{12}{5}\delta^2. \quad (15)$$

The implied constant in (15) is uniform in the admissible factorization.

It remains to distribute the factor M. In what follows we write the odd part of N as $p_1 \cdots p_m$, but the bounded factor 2^r remains. If m exceeds a fixed cutoff, then we can be sure that all subsequent levels contain at least two odd primes, as required in proposition 5.1. In the even case $m-2$ is even, and in the odd case $m-3$ is even. As a result, each D_j described below is an admissible indefinite quaternion discriminant and after the fixed cutoff has been increased, will exceed 1. Moreover $(D_j, M_j) = 1$, and all constants are uniform in j. We then take each entry (D_j, M_j) and apply to it the sieve, the pair estimates, the first-moment selection, and the Pasten interface. Note that the selected field can depend on j and that the earlier established uniformity makes the cell estimate independent of j. If m is even, apply (15) cyclically with

$$M_j = 2^r p_j p_{j+1}, \qquad D_j = \frac{p_1 \cdots p_m}{p_j p_{j+1}}.$$

We can note that all valuations occur on $m-2$ left-hand sides, and that all odd primes are included in two M_j. If $Q_{\text{odd}} = \prod_{i=1}^m (2v_{p_i}(abc)) = \prod_{i=1}^m v_{p_i}(\Delta_E)$, let $C_{\delta,\varepsilon} \geq 1$ be the uniform implied constant in (15). Multiplication gives

$$Q_{\text{odd}}^{m-2} \leq C_{\delta,\varepsilon}^m N^{m(q_\delta+\varepsilon)} \prod_{j=1}^m M_j, \qquad \prod_{j=1}^m M_j = 2^{r(m-2)} N^2.$$

Since $m/(m-2) \leq 2$ for even $m \geq 4$ and $r \leq 8$, taking the $(m-2)$-nd root leaves at most the bounded constant $C_{\delta,\varepsilon}^2 2^8$. Thus

$$Q_{\text{odd}} \ll N^{q_\delta+(2q_\delta+2)/(m-2)+O(\varepsilon)}. \quad (16)$$

If m is odd, use cyclic triples

$$M_j = 2^r p_j p_{j+1} p_{j+2}, \qquad D_j = \frac{p_1 \cdots p_m}{p_j p_{j+1} p_{j+2}},$$

for which

$$Q_{\text{odd}}^{m-3} \leq C_{\delta,\varepsilon}^m N^{m(q_\delta+\varepsilon)} \prod_{j=1}^m M_j, \qquad \prod_{j=1}^m M_j = 2^{r(m-3)} N^3.$$

Here $m/(m-3) \leq 5/2$ for odd $m \geq 5$, so the rooted constant is at most $C_{\delta,\varepsilon}^{5/2} 2^8$. Consequently

$$Q_{\text{odd}} \ll N^{q_\delta+(3q_\delta+3)/(m-3)+O(\varepsilon)}. \quad (17)$$

If $m \leq m_0$ for the chosen fixed cutoff, then we can see that $\omega(abc) = m+1 \leq m_0+1$. We can first employ Pasten's Proposition 15.1 with a smaller internal

epsilon and then select the maximum over these finitely many values, which gives $\mathbf{d}(abc) \ll_{\varepsilon,m_0} R^{1+\varepsilon}$ [12, Proposition 15.1]. From his Lemma 15.2 we can read off $v_2(abc) \ll_\varepsilon R^\varepsilon$ [12, Lemma 15.2 and Equation (16.4)], and the Frey conductor satisfies $N \asymp R$. Finally,

$$P(abc) \le \mathbf{d}(abc) \le (v_2(abc)+1)Q_{\mathrm{odd}}.$$

In this last step we needed to use the inequality $v+1 \le 2v$, and we needed it for each positive integer value v at the odd primes in question.

Given $\varepsilon_0 > 0$, we first need to choose a fixed $0 < \delta < 1/10$ for which $13\delta/5 + 12\delta^2/5 < \varepsilon_0/4$ holds. In the next step we need to ensure that the combined internal epsilon loss is below $\varepsilon_0/4$ and we need to select the prime-count cutoff such that both the overheads in (16)–(17) fall below $\varepsilon_0/4$ above it. Here we can rely on Proposition 15.1 of Pasten, which takes care of those cases where m is equal to or lower than our cutoff. The finitely many conductors below the previously established thresholds end up in the implied constant, and we do not track the effective dependence of that constant on ε_0. With that we have proved theorem 1.1.

Remark 5.2. We attain the exponent 5/3 in the proof by means of the discriminant scale $\Delta = N^{2/3+o(1)}$. We see that the above reasoning does not prove the abc conjecture. In fact, we note that it handles only the product of valuation exponents and the divisor function, not *abc* itself.

Notation

$R = \mathrm{rad}(abc)$	radical of the abc triple
$P(abc)$, $\mathbf{d}(n)$	valuation product and divisor function
B	indefinite quaternion division algebra over $\mathbb{Q}$
B (in §5)	number of bad discriminants
B^0, $Q = -\,\mathrm{nrd}$	trace-zero space and its quadratic form
B_Q	full polarization of Q, so $B_Q(x,y) = \mathrm{trd}(xy)$
D	discriminant of B
M	Eichler level
$N = DM$	reduced discriminant of R_N, and the Frey conductor
R_N	Eichler order of reduced discriminant N
$\Gamma_N = R_N^1/\{\pm 1\}$	norm-one units modulo ± 1
$\mathfrak{H}$	upper half plane
$X_N = \Gamma_N \backslash \mathfrak{H}$	the Shimura curve
$V_N = \mathrm{vol}(X_N)$	its hyperbolic volume
$L_N \subset B^0$	Gross lattice of [3, Equation (2.9)]
$G = \mathrm{SO}(B^0)$	$\simeq B^\times/\mathbb{Q}^\times$
$\mathcal{K}_p$, $\mathcal{K} = \prod_p \mathcal{K}_p$	local projective unit image and its product
T, $R_T(D,M)$	characteristic matrix, and its norm-one class set
$S_p(T)$, $c_p(T)$	local integral-coset set and its cardinality
$-d$, $-e$	odd fundamental CM discriminants
h_d	class number of $\mathbb{Q}(\sqrt{-d})$
$\mathscr{C}_d$	the complete $2h_d$-element Galois multiset
Δ	the CM discriminant scale
$\mathscr{P}_\Delta$	a multiset of CM centers with $d \in [\Delta, 2\Delta]$
B_0	bound on multiplicities in $\mathscr{P}_\Delta$
$\mathscr{P} = \bigsqcup_d \mathscr{C}_d$	the full Galois CM multiset
$C(\mathscr{P})$	number of ordered pairs of $\mathscr{P}$ within $2r_1$
H	number of discriminants produced by Pasten's sieve
$x = N^{2/3+\delta}$, $\vartheta = \delta/5$	sieve parameters
$f(z)\,dz$	holomorphic differential on X_N, $f = f_{D,M}$ in §5
$\|f\|_{\mathrm{Pas}}$	its non-normalized Petersson norm
F_N	the unitary normalization $\sqrt{N}\,yf/\|f\|_{\mathrm{Pas}}$
$T_d(F_N)$	first absolute moment of $\lvert F_N\rvert$ over $\mathscr{C}_d$
$\mu(z)$	sampling multiplicity of the balls $B(P, r_1)$
r_0, r_1	fixed hyperbolic radii
τ, q_δ	moment exponent and cell exponent
Δ_E	discriminant of the Frey curve E

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