# IMPROVING THE STEWART–YU BOUND FOR ALL ABC TRIPLES

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ABSTRACT. We prove that $\log c < AR^{1/3}\exp(-\log R/(3000\log\log R))$ for coprime positive integers $a+b=c$, where $R=\mathrm{rad}(abc)$ and $A$ is absolute and effectively computable. The proof combines estimates at maximal prime powers with a growing number of auxiliary primes in a rational logarithmic form.

## 1. INTRODUCTION

For positive coprime integers $a+b=c$, put $R=\mathrm{rad}(abc)$. The *abc* conjecture predicts $c < K_\varepsilon R^{1+\varepsilon}$ for every $\varepsilon>0$. It remains open. Stewart–Yu [5, p. 226] proved $\log c < R^{2/3+C/\log\log R}$ for $c>2$. Their later bound [6, Theorem 1, p. 170] is $\log c \ll R^{1/3}(\log R)^3$, effectively. We prove the following.

**Theorem 1.** *There is an absolute effectively computable $A>0$ such that*

$$\log c < AR^{1/3}\exp\left(-\frac{\log R}{3000\log\log R}\right) \tag{1.1}$$

*for all positive integers $a+b=c$ with $\gcd(a,b)=1$.*

Equivalently, $\log c < AR^{1/3-1/(3000\log\log R)}$, where the exponent tends to $1/3$ from below as $R\to\infty$. Except for $(1,1,2)$, one has $R\ge 6$ and $\log\log R>0$. In particular, $\log c \ll R^{1/3}$ effectively. For $R\ge 6$, Pasten [3, Theorem 1.4] bounds $\log c$ by $P_{\min}\exp(\kappa_{\mathrm{P}}\sqrt{\log R\log\log R})$. Here $P_{\min}=\min\{P(a),P(b),P(c)\}$, $P(n)$ is the largest prime factor and $P(1)=1$. For $a\le c^{1-\vartheta}$ and fixed $\vartheta>0$, he bounds it by $\vartheta^{-1}\exp(\kappa_{\mathrm{P}}\sqrt{\log R\log\log R})$, with an absolute $\kappa_{\mathrm{P}}>0$. These improve the bound in restricted regions. Neither displayed estimate gives an improvement over the Stewart–Yu bound $\log c\ll R^{1/3}(\log R)^3$ for all triples in terms of $R$ alone.

Put

$$H=\log c,\qquad W=\log R,\qquad L=1+W,\qquad X=R^{1/3}. \tag{1.2}$$

Let $M_i, s_i, \Delta$ be as in Section 3. Let $C_T$ be the effective constant in (3.5) and put $C_P=100000C_V$, with $C_V$ defined in (4.1). On $\mathcal{N}$ in (3.1), we use

$$H<C_TXL^2\Delta^{-2/9},\qquad H<C_PXL^{10}\exp\left(-\frac{W}{156\log W}+\frac{11}{3}\log\Delta\right). \tag{1.3}$$

The second holds for $W\ge W_0$ when $\log\Delta\le(3/1820)W/\log W$. At this boundary both rates equal $1/2730>1/3000$. Section 6 gives an effective common threshold. Stewart [4, Section 4, (4.13)–(4.14)] introduced the growing auxiliary-prime

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construction. Section 4 bounds the cofactors $u_i$ used in applying this construction to additive triples. For earlier $p$-adic estimates, see [7]. The use of AI tools is described at the end of the paper.

## 2. Known estimates

All logarithms are natural. Write $\omega(n)$ for the number of distinct prime divisors of $n$, with $\omega(1) = 0$, and $\pi(x)$ for the number of primes at most $x$. For reduced $x/y$, put $h(x/y) = \log\max(|x|, |y|)$. Write $T(z) = \max(1, h(z))$ for rational $z$. The valuation satisfies $v_p(p) = 1$ and $|x|_p = p^{-v_p(x)}$.

**Fact F1** ([6, Theorem 1, p. 170, (5)])**.** *There is an absolute effectively computable $S \geq 1$ such that*

$$H \leq SXW^3. \tag{2.1}$$

**Fact F2** ([6, Lemma 2, p. 173, and (10), p. 172])**.** *For rational $p$-units $\alpha_1, \dots, \alpha_n$, integer exponents $|b_i| \leq B_0$, $B_0 \geq 3$, and $\Theta = \prod_i \alpha_i^{b_i} - 1 \neq 0$,*

$$v_p(\Theta) < \frac{p}{\log p}(c_9 n)^{2n} \prod_i \frac{\max(h(\alpha_i), \log p)}{\log p} \log B_0, \tag{2.2}$$

*where $c_9 > 0$ is absolute and effectively computable.*

In $K = \mathbb{Q}(\alpha_0)$ of [6, Section 2, (10)], the ramification index at $p$ is 1, including $p = 2$; hence $\mathrm{ord}_{\mathfrak{p}}$ on $\mathbb{Q}$ agrees with $v_p$. Independence is not required, and a base $-1$ is allowed.

**Fact F3** ([1, Proposition 5, p. 517], corrected in [2])**.** *Put $K_0 = 300(16e)^8$. Let $m \geq 1$ and let $\Gamma$ be generated by distinct primes $q_1, \dots, q_m$ and $-1$. The primes form a basis for $\Gamma/\Gamma_{\mathrm{tors}}$. If $\xi \in \Gamma$ and $\xi \neq 1, -1$, then*

$$-\log|1 - \xi| < K_0^m \log\max(e, h(\xi)) \prod_j \log q_j, \tag{2.3}$$

$$-\log|1 - \xi|_p < K_0^m \frac{p}{\log p} \log\max(e, ph(\xi)) \prod_j \log q_j. \tag{2.4}$$

**Lemma 2.1** (Specialisation of the group estimate)**.** *Proposition 5 of* [1]*, with the sign corrected in* [2]*, gives* (2.3)–(2.4) *under the hypotheses of Fact F3.*

*Proof.* Apply Proposition 5 of [1], with the sign corrected in [2], with $K = \mathbb{Q}$ and the parameter $\alpha = 1$ in [1]. The group has positive rank, and $h(\xi) \geq \log 2 > 0$. Its constant is $48(16e)^8$ for $m = 1$. For $m \geq 2$, the specialized coefficient $2(m + 1)\log\max(e, m)(16e)^{3m+5}$ is at most $4^m(16e)^{8m} \leq \big(8(16e)^8\big)^m$, since $m + 1 \leq 2^m$, $\log\max(e, m) \leq 2^{m-1}$ and $3m + 5 \leq 8m$. The factor for the real place is at most $F = 4/\log 2 > 1$. For $m \geq 2$, $F\big(8(16e)^8\big)^m \leq \big((32/\log 2)(16e)^8\big)^m < K_0^m$. For $m = 1$, $48F < 300$; both comparisons follow from $\log 2 > 2/3$. □

**Fact F4** ([4, Lemma 3.1, pp. 301–302]; [8, Main theorem, (1.18), p. 321])**.** *Let $p \geq 5$, $n \geq 7$, and $\ell = \log p$. For independent rational $p$-units $\alpha_i$ and integers $b_i$, not all zero, put $B_e = \max(2, |b_1|, \dots, |b_n|)$. If $\prod_i \alpha_i^{b_i} \neq 1$, then*

$$\begin{aligned} v_p\left(\prod_i \alpha_i^{b_i} - 1\right) < {} & 376\sqrt{n+1}\,[4 + \log(n+1)]\left(7e\frac{p-1}{p-2}\right)^n \prod_i h(\alpha_i) \\ & \cdot \max\left(p(n/\ell)^n, e^n\ell\right)\max\big(\log B_e, 5.4n(n+1)\big). \end{aligned} \tag{2.5}$$

**Lemma 2.2** (Specialisation of the growing-rank estimate)**.** *Under the hypotheses of Fact F4, estimate* (2.5) *follows from its cited sources.*

*Proof.* Stewart's derivation uses the constants 1794 and 5.25. Yu's published rational case has 1790 and 5.84; with these constants, the bound $G_1 \le 5.4n(n+1)$ used here holds for $n \ge 7$. We derive this range directly from Yu [8]. Use [8, (1.9), (1.11), (1.13), (1.28), (1.30)–(1.31)] with $d = f = e_{\mathfrak{p}} = 1$ and Yu's parameters $q_{\mathrm{Y}} = 2$, $q_{\mathrm{Y}}^{u_{\mathrm{Y}}} = 2$. Relabel a minimum-valuation exponent last, as (1.17) requires. We retain 5.84 throughout and use the upper bound $1790 \le 1794$. Put $Q_n = 4 + \log(n+1)$ and $M_e = \max(\log B_e, 5.4n(n+1))$. Then

$$\frac{G_1}{n+1} < 4n + 6 + \log(4n+6) \le 5.4n \quad (n \ge 7).$$

The last inequality holds at 7, and its difference increases. Since $h(\alpha_i) \ge \log 2$ and the auxiliary quantity in [8, (1.15)] equals $\log 2 \log 3/\log 6$, the first term of $h^{(1)}$ is at most

$$\log B_e + \log(2\log 3/\log 6) < \log B_e + \tfrac{1}{4} < 1.001 M_e.$$

Yu's additional term $\log B^\circ$, where $B^\circ = \min\{|b_i| : b_i \ne 0\}$, satisfies $\log B^\circ \le \log B_e \le M_e$. Hence $h^{(1)} \le \max(1.001M_e, (n+1)\ell)$. The saturated residue-group index $\delta(\boldsymbol{\alpha})$ is at least 1. Yu's main theorem uses the heights of the original independent tuple in both Kummer cases; we only enlarge $p/\delta(\boldsymbol{\alpha})$ to $p$. Yu's first bound is consequently at most

$$1794\left(7\frac{p-1}{p-2}\right)^n \frac{(n+1)^{n+1}}{2n!\ell^2}\prod_i h(\alpha_i) \cdot \max\left(p(n/\ell)^n, e^n\ell\right)\max(Q_n,\ell)\max(1.001M_e,(n+1)\ell).$$

Use Stirling's inequality $(n+1)^{n+1}/n! \le e^{n+1}\sqrt{n+1}/\sqrt{2\pi}$. Also $\max(Q_n,\ell)/\ell \le Q_n/\log 5$ and $\max(1.001M_e,(n+1)\ell)/\ell \le 1.001M_e/\log 5$. Finally,

$$\frac{1.001 \cdot 1794e}{2\sqrt{2\pi}(\log 5)^2} < 376.$$

For this comparison one may use $e < 2.718282$, $\log 5 > 1.609437$, and $\sqrt{2\pi} > 2.50662$. This gives (2.5) from the published constants for $n \ge 7$. □

Put $C_H = 6 + \log S$. By (2.1), if $h(\xi) \le H$ and $1 \le y \le R$, then

$$\log\max(e, yh(\xi)) \le C_H L, \tag{2.6}$$

since $\log(RH) \le \log S + 4W/3 + 3\log W \le C_H L$ and $C_H L \ge 6$. For fixed $\gamma > 0$, define the effective constants

$$U_\gamma = \left\lceil \max(1, 4/\gamma^2, 2\log K_0/\gamma) \right\rceil, \qquad C_\gamma = \prod_{\substack{q < e^{U_\gamma} \\ q \text{ prime}}} \max\left(1, \frac{K_0 \max(1,\log q)}{q^\gamma}\right). \tag{2.7}$$

For every squarefree $J$,

$$G(J) := K_0^{\omega(J)} \prod_{q \mid J} \log q \le C_\gamma J^\gamma, \qquad G(1) = 1. \tag{2.8}$$

For $\zeta = \log q \ge U_\gamma$, $\log K_0 \le \gamma\zeta/2$ and $\log\zeta \le \sqrt{\zeta} \le \gamma\zeta/2$. Multiply over the primes dividing $J$.

## 3. Maximal prime powers

Arrange $a \le b$. For each nonunit side $i \in \{a, b, c\}$, put

$$M_i = \max_{q|i} v_q(i), \qquad s_i = \min\{q : v_q(i) = M_i\}, \qquad u_i = i/s_i^{M_i}, \qquad E_i = \operatorname{rad}(u_i).$$

By definition, $s_i \nmid u_i$. On $a^2 > c$, all three sides exceed $\sqrt{c}$. Put

$$x_i = \log s_i, \quad H_i = M_i x_i, \quad \Delta = E_a E_b E_c, \quad J_i = \Delta/E_i, \quad \Omega = \log \Delta.$$

Consider the range

$$\mathcal{N} = \{a^2 > c,\ s_i^{39} > R^{10} \text{ for every } i\}. \tag{3.1}$$

On this range,

$$R = s_a s_b s_c \Delta, \qquad J_a J_b J_c = \Delta^2, \tag{3.2}$$

$$\frac{10}{39}W < x_i < \frac{19}{39}W, \quad \Omega < \frac{9}{39}W < x_i, \quad \log \operatorname{rad}(i) < \frac{19}{39}W, \quad H_i > \frac{5}{19}H. \tag{3.3}$$

We have $x_i \le W - x_j - x_k < 19W/39$ and $\log \operatorname{rad}(i) \le W - x_j - x_k$; summing the three lower bounds gives $\Omega = W - x_a - x_b - x_c < 9W/39$. Maximality gives $H_i \ge x_i \log i / \log \operatorname{rad}(i) > (10/19) \log i > 5H/19$. Also $R \ge 30$, $W > 3$ and $W > 3L/4$. The actual ratios

$$\xi_a = b/c, \qquad \xi_b = a/c, \qquad \xi_c = -a/b \tag{3.4}$$

have height at most $H$ and satisfy $v_q(1 - \xi_i) = v_q(i)$ for $q \mid i$.

Set

$$C_U = 2C_H C_{1/6}, \qquad C_S = \frac{2C_H K_0^2 C_{1/26}}{(\log 2)^2}, \qquad C_T = \frac{741}{50} C_H K_0^2 C_{1/6}. \tag{3.5}$$

If $a^2 \le c$, apply (F3) at infinity to $\xi = b/c$. Here $-\log(1 - \xi) \ge H/2$. Equations (2.6)–(2.8) give

$$H < C_U X L e^{-W/6}. \tag{3.6}$$

This includes unit sides and $a^2 = c$. If $a^2 > c$ and some $s_i^{39} \le R^{10}$, choose $p = s_i = \min\{s_a, s_b, s_c\}$; then $p^{39} \le R^{10}$. Since $\log i \le M_i \log \operatorname{rad}(i) \le M_i W$ and $\log i > H/2$, maximality gives $H < 2WH_i/\log p$. Apply (F3) to the full off-side prime basis in (3.4):

$$H_i < C_H K_0^2 C_{1/26} \frac{p}{\log p} L x_j x_k J_i^{1/26} \le C_H K_0^2 C_{1/26} \frac{p}{\log p} L^3 R^{1/26}.$$

Since $p \le R^{10/39} = X R^{-1/13}$,

$$H < C_S X L^4 e^{-W/26}. \tag{3.7}$$

On all of $\mathcal{N}$, (F3) together with (2.8), with $\gamma = 1/6$, gives

$$\frac{x_j x_k}{x_i} \le \frac{39}{10} W, \qquad C_* = \frac{39}{10} C_H K_0^2 C_{1/6},$$
$$H_i < C_H K_0^2 C_{1/6} L \frac{s_i}{x_i} x_j x_k J_i^{1/6} < C_* s_i L^2 J_i^{1/6}.$$

Multiply and use (3.2)–(3.3):

$$\prod_i H_i < C_*^3 R L^6 \Delta^{-2/3}, \qquad H < C_T X L^2 \Delta^{-2/9}. \tag{3.8}$$

There is no restriction on $\Delta$ in this estimate.

## 4. Cofactors of the maximal prime powers

Assume the triple lies in $\mathcal{N}$. Put $\kappa = \max(1, c_9)$ and $F(n) = (\kappa n)^{2n}$. Choose an integer $m_0 \geq 7$ with $\log(m_0 + 1) \geq 8 \log \kappa + 10$. Define

$$C_F = \max\left(1, \max_{0 \leq m < m_0} \frac{F(m+3)}{((m+1)!)^3}\right), \qquad C_V = \frac{(C_H + 2)C_F}{4(\log 2)^3}. \tag{4.1}$$

Then $C_V \geq 1$, since $C_F \geq F(3) \geq 729$. For squarefree $J$ with $m = \omega(J)$, one has $J \geq (m+1)!$. Integration gives $\log((m+1)!) \geq (m+1)\log(m+1) - m$. It follows that

$$\begin{aligned} \log \frac{F(m+3)}{((m+1)!)^3} &\leq (3-m)\log(m+1) + 2(m+3)\log\kappa + 3m + 4 + \frac{8}{m+1} \\ &\leq m\left[-\tfrac{1}{2}\log(m+1) + 4\log\kappa + 5\right] \quad (m \geq 7). \end{aligned}$$

The last expression is nonpositive for $m \geq m_0$. The finite maximum handles $m < m_0$, including 0. Consequently,

$$F(3 + \omega(J)) \leq C_F J^3. \tag{4.2}$$

Let $l \mid E_i$. Represent $\xi_i$ using $s_j, s_k$, every prime dividing $J_i$, and the sign. These bases are $l$-units. Their coefficients are at most $B_0 = \max(3, H/\log 2)$, with $\log B_0 \leq (C_H + 2)L$. Since $\log l \leq \Omega < x_j, x_k$, the two factors for $s_j, s_k$ in (2.2) have product at most $L^2/[4(\log 2)^2]$. Each remaining prime contributes

$$\max(1, \log q/\log l) \leq \max(1, \log q/\log 2) < q.$$

The sign contributes 1. Apply (F2), (4.2) and $l/\log l \leq E_i/\log 2$:

$$1 \leq v_l(i) < C_V L^3 E_i J_i^4 \leq C_V L^3 \Delta^4. \tag{4.3}$$

Every residual exponent is included, also at $l = 2$. For distinct sides $j, k$, this yields

$$T(u_j/u_k) \leq C_V L^4 \Delta^4. \tag{4.4}$$

If $u_j > 1$, $\log u_j \leq \max_{l \mid u_j} v_l(j)\,\Omega$ and $\Omega < W < L$; if $u_j = 1$, $\log u_j = 0$. Likewise for $u_k$. Coprimality gives $h(u_j/u_k) = \max(\log u_j, \log u_k)$; use $C_V \geq 1$ for $T(u_j/u_k)$.

## 5. Auxiliary primes in the local estimate

**Lemma 5.1.** *Let $p, q, r$ be distinct primes, $\ell = \log p$, and $\Lambda = \log \ell \geq 10^6$. Suppose $\ell \leq \log q, \log r \leq 2\ell$. Let $w_q, w_r$ be coprime positive integers with $\gcd(w_q w_r, pqr) = 1$ and $\omega(w_q w_r) \leq \ell$. For $1 \leq N_q, N_r \leq p^2$, $\sigma \in \{1, -1\}$, and a nonzero difference,*

$$v_p(w_q q^{N_q} - \sigma w_r r^{N_r}) < 25200 p \ell^5 T(w_q/w_r) \exp\left(-\frac{\ell}{52 \log \ell}\right). \tag{5.1}$$

*Proof.* Write $\beta$ for the valuation and $z = (w_q/w_r) q^{N_q} r^{-N_r} > 0$. If $\beta = 0$, the result follows. For $\sigma = 1$, take $\nu = 1$. For $\sigma = -1$, take $\nu = 2$; oddness gives $v_p(z^2 - 1) = v_p(z + 1) = \beta$. Hence $\beta = v_p(z^\nu - 1)$. The denominator is a $p$-unit, and $v_q(z) = N_q > 0$ ensures nonvanishing.

Put $n = \lfloor \ell/(52\Lambda) \rfloor$. Then $n \geq 7$ and $n \geq \ell/(104\Lambda)$. Choose auxiliary primes outside $pqrw_qw_r$. For $\theta = w_q/w_r \neq 1$, use $n-3$ of them and set

$$\begin{aligned} \alpha_1 &= q/(\varpi_4 \cdots \varpi_n), & b_1 &= \nu N_q, \\ \alpha_2 &= 1/r, & b_2 &= \nu N_r, \\ \alpha_3 &= \theta, & b_3 &= \nu, \\ \alpha_i &= \varpi_i \ (i \geq 4), & b_i &= \nu N_q. \end{aligned} \tag{5.2}$$

If $w_q = w_r = 1$, omit $\theta$ and use $n-2$ auxiliaries. The product is $z^\nu$. Take an integer relation $\prod_i \alpha_i^{z_i} = 1$. The valuations at $q$ and $r$ give $z_1 = z_2 = 0$. For $\theta \neq 1$, a prime in its support then gives $z_3 = 0$, and each $\varpi_i$ gives $z_i = 0$ for $i \geq 4$. The tuple without $\theta$ is treated in the same way. All bases are rational $p$-units.

For $m = \lfloor x/2 \rfloor$, Legendre's formula gives

$$\frac{2^{2m}}{2m+1} \leq \binom{2m}{m} \leq (2m)^{\pi(2m)}.$$

Each prime contributes at most $2m$ to the binomial coefficient. Hence $\pi(x)\log x \geq (x-2)\log 2 - \log(x+1)$. For $x \geq 1000$, use $\log 2 > 2/3$ and $x/15 \geq 4/3 + \log(x+1)$ to obtain $\pi(x) \geq 3x/(5\log x)$. At $x = 6\ell\log(6\ell)$,

$$\log x \leq 1.001\Lambda, \qquad \pi(x) \geq \frac{3600}{1001}\ell > 3\ell. \tag{5.3}$$

Here $\log 6 + \log(\Lambda + \log 6) \leq 3 + \log\Lambda \leq 3 + \sqrt{\Lambda}/2 \leq 0.001\Lambda$. There are at most $3 + \ell$ forbidden primes, and $n + 3 < 2\ell$. Therefore $h(\varpi_i) \leq 1.001\Lambda$ is possible.

The auxiliary denominator has logarithm at most $1.001n\Lambda < \ell/2$. Since $1.001\Lambda \geq 1$, we have $(1.001\Lambda)^{n-3} \leq (1.001\Lambda)^n$; the same comparison holds for $n-2$ auxiliaries. Therefore,

$$\prod_i h(\alpha_i) \leq 4\ell^2 T(w_q/w_r)(1.001\Lambda)^n, \qquad B_e \leq 2p^2, \tag{5.4}$$

$$\max(\log B_e, 5.4n(n+1)) \leq 3\ell^2. \tag{5.5}$$

The cofactor-free tuple satisfies the same bounds. Both maximum terms in (F4) are at most $p(n/\ell)^n$, since

$$\log \frac{p(n/\ell)^n}{e^n \ell} \geq \ell - \frac{\ell}{52\Lambda}[1 + \log(104\Lambda)] - \Lambda \geq 0.$$

Here $[1 + \log(104\Lambda)]/(52\Lambda) < 1/2$ and $\ell/2 > \Lambda$. For $p \geq 1000$,

$$\frac{7e^2(1.001)(p-1)}{52(p-2)} \leq \frac{7(68/25)^2(1001/1000)(999/998)}{52} < 1.$$

That upper bound equals $155615229/155937500 < 0.998$. Consequently,

$$\left[7e\frac{p-1}{p-2}\frac{n}{\ell}(1.001\Lambda)\right]^n \leq e^{-n} \leq e\exp\left(-\frac{\ell}{52\Lambda}\right). \tag{5.6}$$

Finally, $\sqrt{n+1}[4 + \log(n+1)] \leq 2\ell$. Insert (5.4)–(5.6) into (F4), and use $752e \cdot 4 \cdot 3 < 25200$. □

## 6. Proof of Theorem 1

Set

$$\tau = \frac{3}{1820}, \quad \eta = \frac{1}{3000}, \quad \rho = \frac{1}{2730}, \quad \mu = \rho - \eta = \frac{3}{91000} > \frac{1}{40000}. \tag{6.1}$$

Define the effective threshold

$$t_0 = \left\lceil \max\left(1000002, \log(100000C_V), \log C_T, \log C_U, \log C_S, \log\frac{2S}{\log 2}\right)\right\rceil, \tag{6.2}$$

$$W_0 = e^{t_0}, \qquad R_0 = e^{W_0}.$$

Assume $W \geq W_0$ and put $t = \log W$. Then

$$t \geq t_0, \qquad \log L \leq t+1, \qquad W \geq t^4, \qquad W/t \geq t^3. \tag{6.3}$$

First suppose the triple lies in $\mathcal{N}$ and $\Omega \leq \tau W/t$. Choose the least $s_i = p$, with the other primes denoted by $q, r$. The equation is $c - b = a$, $c - a = b$, or $a + b = c$. The local difference in Lemma 5.1 has valuation $\beta = M_i$, with cofactors $u_j, u_k$ and the appropriate sign. Their supports are coprime and avoid $pqr$. By (3.3),

$$\frac{10W}{39} < \ell \leq \frac{W-\Omega}{3}, \qquad \ell \leq \log q, \log r < \frac{19}{10}\ell < 2\ell. \tag{6.4}$$

In particular, $t - 2 < \log \ell \leq t$, so $\log \ell \geq 10^6$. The upper bound for $\ell$ uses $p^3 \leq s_a s_b s_c = R/\Delta$, since $p$ is the least of the three primes. Also

$$\omega(u_j u_k) \leq \Omega/\log 2 \leq 2\tau W/t < \ell.$$

Every complete exponent is at most $(S/\log 2)e^{W/3}W^3$. Its logarithm is at most $W/3 + 4t < 20W/39 < 2\ell$. The middle comparison follows from $4t < 7W/39$ in (6.3). Lemma 5.1 applies with exponents at most $p^2$. Using (4.4) gives

$$H < \frac{19}{5}\ell\beta < 100000C_V L^{10}\Delta^4 \exp\left(\ell - \frac{\ell}{52\log\ell}\right). \tag{6.5}$$

The function $g(x) = x - x/(52\log x)$ increases for $\log x \geq 1$. Put $Y = (W-\Omega)/3$ and $\chi = \Omega/W$. Since $t\chi \leq \tau < \log 3$,

$$\frac{3Yt}{W} - \log Y = \log 3 - t\chi - \log(1-\chi) > 0, \qquad \frac{Y}{\log Y} \geq \frac{W}{3t}.$$

Apply this to (6.5):

$$\begin{aligned} H &< 100000C_V X L^{10}\exp\left(-\frac{W}{156t} + \frac{11}{3}\Omega\right) \\ &\leq 100000C_V X L^{10} e^{-\rho W/t}, \end{aligned} \tag{6.6}$$

because

$$\frac{1}{156} - \frac{11}{3}\tau = \frac{2\tau}{9} = \rho. \tag{6.7}$$

If $\Omega > \tau W/t$, (3.8) gives $H < C_T X L^2 e^{-\rho W/t}$ on $\mathcal{N}$. The equality $\Omega = \tau W/t$ was included in (6.6).

The polynomial factors satisfy

$$\log(100000C_V) + 10\log L \leq 11t + 10 \leq \mu t^3 \leq \mu W/t, \tag{6.8}$$

$$\log C_T + 2\log L \leq 3t + 2 \leq \mu t^3 \leq \mu W/t. \tag{6.9}$$

Here $\mu > 1/40000$ and $t^2/40000 > 12$. This covers the cofactor polynomial, $\ell^5$, the factor $\ell$, and all constants in (6.5). For $a^2 \le c$, or $a^2 > c$ with some $s_i^{39} \le R^{10}$, (6.3) gives

$$W/6 - \eta W/t \ge W/12 \ge 2t + 1 \ge \log C_U + \log L, \tag{6.10}$$

$$W/26 - \eta W/t \ge W/52 \ge 5t + 4 \ge \log C_S + 4\log L. \tag{6.11}$$

Here $\eta/t \le 1/52$ and $\eta/t \le 1/12$. Equations (3.6), (3.7), and (6.8)–(6.11) prove $H < Xe^{-\eta W/\log W}$ for $W \ge W_0$.

It remains to cover the finite range. Apart from $(1,1,2)$, at least two distinct primes divide $abc$. Hence $R \ge 6$. Define

$$E_0 = \max\left(\frac{\log 6}{\log\log 6}, \frac{W_0}{t_0}\right), \qquad A = 2\max\left(1, SW_0^3 e^{\eta E_0}\right). \tag{6.12}$$

The function $W/\log W$ has a unique minimum at $W = e$. Its maximum on $[\log 6, W_0]$ is therefore $E_0$. For $6 \le R < R_0$, (F1) yields

$$H \le SXW^3 \le SW_0^3 e^{\eta E_0} Xe^{-\eta W/\log W} < AXe^{-\eta W/\log W}.$$

For $(1,1,2)$, $H < 1 < X$ and $\log\log 2 < 0$. Hence (1.1) also holds there. □

After omitting the cofactor-height and polynomial factors, the formal endpoint $\ell = W/3$ in (5.1) gives

$$X\exp\left(-\frac{W}{156\log(W/3)}\right) = X\exp\left(-\left(\frac{1}{156} + o(1)\right)\frac{W}{\log W}\right) \qquad (W \to \infty).$$

Its leading coefficient is $1/156$, whereas Theorem 1 uses $1/3000$.

## Use of AI tools

The author devised the method of proof. Codex (OpenAI, GPT-6 Astra) and Antigravity (Google, Gemini 4 Carbon) carried out numerical checks and ran blind reviews of the manuscript. Claude Code (Anthropic, Claude Fable 5) checked the proofs and the references. The author checked every step of the proof and every reference, and takes full responsibility for the content of the paper.

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